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SOME NOVEL GENERATING FUNCTIONS OF EXTENDED
JACOBI POLYNOMIALS BY GROUP THEORETIC METHOD

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1. INTRODUCTION

The extended Jacobi polynomials defined by Patil and Thakare [1]

$$(1.1) \quad F_n(\alpha, \beta; x) = \frac{(-1)^n}{n!} (x-a)^{-\alpha} (b-x)^{-\beta} \left(\frac{\lambda}{b-a} \right)^n \\ \times D^n [(x-a)^{n+\alpha} (b-x)^{n+\beta}],$$

where $D = \frac{d}{dx}$ and λ is a number such that $\frac{\lambda}{b-a} > 0$, satisfy the ordinary differential equation [2]

$$(1.2) \quad [(x-a)(b-x)D^2 + \{(\alpha+1)(b-x) - (\beta+1)(x-a)\}D + n(1+\alpha+\beta+n)]y = 0.$$

Very recently, attempts have been made [2, 3] in connection with the derivation of generating functions of the extended Jacobi polynomials from the Lie-group viewpoint.

The aim of the present paper is to investigate some novel generating relations of the extended Jacobi polynomial $F_n(\alpha, \beta; x)$ by the application of L. Weisner's group-theoretic method [4] which is vividly presented in the monograph by E.B. McBride [5]. It may be of interest to remark that in course of constructing a Lie algebra we obtain a pair of linear partial differential operators which simultaneously raise (lower) and lower (raise) the parameters α and β of the polynomial under consideration. We would like to mention that our results differ from the traditional concept of a generating function for orthogonal polynomials.

2. GROUP-THEORETIC METHOD

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, α by $y\frac{\partial}{\partial y}$, β by $z\frac{\partial}{\partial z}$ and y by $u(x, y, z)$ in (1.2), we get the partial differential equation

$$(2.1) \quad \left[(x-a)(b-x)\frac{\partial^2}{\partial x^2} + \left\{ \left(y\frac{\partial}{\partial y} + 1 \right) (b-x) - \left(z\frac{\partial}{\partial z} + 1 \right) (x-a) \right\} \frac{\partial}{\partial x} \right. \\ \left. + n \left(1 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + n \right) \right] u(x, y, z) = 0.$$

Thus we see that $u(x, y, z) = F_n(\alpha, \beta; x)y^\alpha z^\beta$ is a solution of (2.1), since $F_n(\alpha, \beta; x)$ is a solution of (1.2).

We now define linear partial differential operators

$$(2.2) \quad \begin{aligned} A_1 &= y\frac{\partial}{\partial y}, \\ A_2 &= z\frac{\partial}{\partial z}, \\ A_3 &= (x-b)yz^{-1}\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \\ A_4 &= (x-a)y^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} \end{aligned}$$

such that

$$(2.3) \quad \begin{aligned} A_1[F_n(\alpha, \beta; x)y^\alpha z^\beta] &= \alpha F_n(\alpha, \beta; x)y^\alpha z^\beta, \\ A_2[F_n(\alpha, \beta; x)y^\alpha z^\beta] &= \beta F_n(\alpha, \beta; x)y^\alpha z^\beta, \\ A_3[F_n(\alpha, \beta; x)y^\alpha z^\beta] &= (\beta + n)F_n(\alpha + 1, \beta - 1; x)y^{\alpha+1}z^{\beta-1}, \\ A_4[F_n(\alpha, \beta; x)y^\alpha z^\beta] &= (n + \alpha)F_n(\alpha - 1, \beta + 1; x)y^{\alpha-1}z^{\beta+1}. \end{aligned}$$

The commutator relations satisfied by A_i ($i = 1, 2, 3, 4$) are

$$(2.4) \quad \begin{aligned} [A_1, A_2] &= 0, & [A_2, A_3] &= -A_3, \\ [A_1, A_3] &= A_3, & [A_2, A_4] &= A_4, \\ [A_1, A_4] &= -A_4, & [A_3, A_4] &= A_1 - A_2 \end{aligned}$$

where $[A, B]u = (AB - BA)u$.

Thus we arrive at the following theorem:

Theorem. *The set of operators $\{1, A_i$ ($i = 1, 2, 3, 4$) where 1 stands for the identity operator, generates a Lie algebra \mathcal{L} .*

It can be shown that the partial differential operator L ,

$$(2.5) \quad L = (x-a)(b-x)\frac{\partial^2}{\partial x^2} + \left[\left(y\frac{\partial}{\partial y} + 1 \right)(b-x) - \left(z\frac{\partial}{\partial z} + 1 \right)(x-a) \right] \frac{\partial}{\partial x} \\ + n \left(1 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + n \right)$$

can be related to A_i ($i = 1, 2, 3, 4$) as follows:

$$(2.6) \quad L = -A_3A_4 + A_1A_2 + A_1 + n(1 + A_1 + A_2 + n).$$

Now one can easily verify that L commutes with each A_i ($i = 1, 2, 3, 4$), i.e.,

$$(2.7) \quad [L, A_i] = 0.$$

The extended form of the groups generated by A_i ($i = 1, 2, 3, 4$) are

$$(2.8) \quad e^{a_1A_1}f(x, y, z) = f(x, e^{a_1}y, z), \\ e^{a_2A_2}f(x, y, z) = f(x, y, e^{a_2}z), \\ e^{a_3A_3}f(x, y, z) = f\left(x + a_3\frac{(x-b)y}{z}, y, z + a_3y\right), \\ e^{a_4A_4}f(x, y, z) = f\left(x + a_4\frac{(x-a)y}{z}, y + a_4z, z\right).$$

From the above we get

$$(2.9) \quad e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}f(x, y, z) \\ = f\left(\left(x + a_4\frac{(x-a)z}{y}\right)\left[1 + a_3\left(a_4 + \frac{y}{z}\right)\right] - a_3b\left(a_4 + \frac{y}{z}\right), \\ e^{a_1}y\left(1 + \frac{a_4}{y}z\right), e^{a_2}z\left\{1 + a_3\left(\frac{y}{z} + a_4\right)\right\}\right).$$

3. GENERATING FUNCTIONS

Now it follows from (2.1) that $u_1(x, y, z) = F_n(\alpha, \beta; x)y^\alpha z^\beta$ is a solution of the system

$$Lu = 0, \\ (A_1 - \alpha)u = 0; \\ Lu = 0, \\ (A_2 - \beta)u = 0; \\ Lu = 0, \\ (A_1 + A_2 - \alpha - \beta)u = 0.$$

From (2.7) one can easily verify that

$$S(L(F_n(\alpha, \beta; x)y^\alpha z^\beta)) = L(S(F_n(\alpha, \beta; x)y^\alpha z^\beta)) = 0,$$

where

$$S = e^{a_4 A_4} e^{a_3 A_3} e^{a_1 A_2} e^{a_1 A_1}.$$

Therefore $S(F_n(\alpha, \beta; x)y^\alpha z^\beta)$ is annihilated by L .

Putting $a_1 = a_2 = 0$ and replacing $f(x, y, z)$ by $F_n(\alpha, \beta; x)y^\alpha z^\beta$ in (2.9), we get

$$(3.1) \quad e^{a_4 A_4} e^{a_3 A_3} [F_n(\alpha, \beta; x)y^\alpha z^\beta] \\ = F_n\left(\alpha, \beta; \left\{x + a_4 \frac{(x-a)z}{y}\right\} \left\{1 + a_3\left(a_4 + \frac{y}{z}\right)\right\} - a_3 b\left(a_4 + \frac{y}{z}\right)\right) \\ \times y^\alpha \left(1 + \frac{a_4}{y} z\right)^\alpha \times z^\beta \left\{1 + a_3\left(a_4 + \frac{y}{z}\right)\right\}^\beta.$$

However,

$$(3.2) \quad e^{a_4 A_4} e^{a_3 A_3} [F_n(\alpha, \beta; x)y^\alpha z^\beta] \\ = \sum_{p=0}^{\infty} \frac{(-a_4)^p}{p!} (-n - \alpha - k)_p \sum_{k=0}^{\infty} \frac{(-a_3)^k}{k!} (-\beta - n)_k \\ \times F_n(\alpha + k - p, \beta - k + p; x)y^{\alpha+k-p} z^{\beta-k+p}.$$

Equating (3.1) and (3.2), we get

$$(3.3) \quad \left(1 + a_4 \frac{y}{z}\right)^\alpha \left\{1 + a_3\left(a_4 + \frac{y}{z}\right)\right\}^\beta F_n\left(\alpha, \beta; \left\{x + a_4 \frac{(x-a)z}{y}\right\} \right. \\ \left. \times \left\{1 + a_3\left(a_4 + \frac{y}{z}\right)\right\} - a_3 b\left(a_4 + \frac{y}{z}\right)\right) \\ = \sum_{p=0}^{\infty} \frac{(-a_4)^p}{p!} (-n - \alpha - k)_p \sum_{k=0}^{\infty} \frac{(-a_3)^k}{k!} (-\beta - n)_k \\ \times F_n(\alpha + k - p, \beta - k + p; x)y^{k-p} z^{-k+p}.$$

We now consider the following cases:

Case 1. Putting $a_3 = 1, a_4 = 0$ and replacing $\frac{a_3 y}{z}$ by $-t$, we get

$$(3.4) \quad (1-t)^\beta F_n(\alpha, \beta; x - (x-b)t) = \sum_{k=0}^{\infty} \frac{(-\beta - n)_k}{k!} F_n(\alpha + k, \beta - k; x)t^k.$$

Case 2. Putting $a_3 = 0$, $a_4 = 1$ and substituting $\frac{a_4 z}{y} = t$, we get

$$(3.5) \quad (1+t)^\alpha F_n(\alpha, \beta; x + (x-a)t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} (-n-\alpha)_p F_n(\alpha-p, \beta+p; x) t^p.$$

Case 3. Putting $a_3 = \frac{1}{w}$, $a_4 = 1$ and $\frac{z}{y} = t$, we get

$$(3.6) \quad (1+t)^\alpha \left(1 + \frac{1}{w} \left(1 + \frac{1}{t}\right)\right)^\beta \\ \times F_n\left(\alpha, \beta; \{x + (x-a)t\} \left\{1 + \frac{1}{w} \left(1 + \frac{1}{t}\right)\right\} - \frac{b}{w} \left(1 + \frac{1}{t}\right)\right) \\ = \sum_{p=0}^{\infty} \frac{(-t)^p}{p!} (-n-\alpha-k)_p \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{wt}\right)^k}{k!} (-\beta-n)_k \\ \times F_n(\alpha+k-p, \beta-k+p; x).$$

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