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ON CLOSED 4-MANIFOLDS ADMITTING A MORSE FUNCTION WITH 4 CRITICAL POINTS

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1. INTRODUCTION

It is not hard to see that both $S^2 \times S^2$ and $S^1 \times S^3$ admit a Morse function with exactly 4 critical points (see our Lemma 2’s proof); we can also find a topological $S^4$ having this property (see §4). The main concern here is the inverse problem. Similar topics have been well investigated which can be found in [1], [2], [4], etc.

By $\mathbb{Z}$, $E^k$, $S^i$ we denote the groups of integers, $k$—Euclidean space and $i$—Euclidean sphere, respectively.

Our main result is as follows

**Theorem.** Let $M$ with $\chi(M) \neq 0$ be a closed connected $C^\infty$ 4-manifold which admits a Morse function with exactly 4 critical points, where $\chi(M)$ is the Euler characteristic of $M$. Then either $M$ is a topological $S^4$ or $M$ is simply connected and

\begin{equation}
H_*(M; \mathbb{Z}) \cong H_*(S^2 \times S^2; \mathbb{Z}),
\end{equation}

i.e., $M$ has the integral homology groups of $S^2 \times S^2$. Furthermore,

a) If (1) holds and such isomorphisms can be geometrically realized (i.e., if there exists a continuous mapping

$$h: S^2 \times S^2 \to M$$

such that its induced homomorphisms

$$h_i: H_i(S^2 \times S^2; \mathbb{Z}) \to H_i(M; \mathbb{Z}), \quad i = 0, 2, 4$$
are isomorphisms), \( M \) has the homotopy type of \( S^2 \times S^2 \);

b) if \( M \) is of a \( C^\infty \) product structure, \( M \) is diffeomorphic with \( S^2 \times S^2 \);

c) if \( M \) admits a Riemannian metric of positive curvature and with this metric \( M \) can be isometrically immersed into \( E^6 \), \( M \) is a topological \( S^4 \); if \( M \) admits a Riemannian structure of non-negative curvature and with this structure \( M \) can be isometrically embedded into \( E^6 \), either \( M \) is a topological \( S^4 \) or \( M \) is diffeomorphic with \( S^2 \times S^2 \).

Remarks. 1. (See §3) We have two alternate versions for the hypothesis “\( \chi(M) \neq 0 \)” in the theorem.

2. I wonder whether the condition in a) is superfluous or not. The realization for general simply connected spaces is not always possible. (See [15, p. 183])

The proof of the theorem will be given in §3. We shall present some preliminaries in §2 and discuss the case “\( c_2(f) = 1 \)” of the theorem in §4.

2. Preliminaries

Unless otherwise specified, all manifolds involved in this paper are closed, connected, smooth and finite dimensional. \( M^n \) means manifold \( M \) is \( n \)-dimensional, \( e^n \) an \( n \)-cell, \( \mathbb{F} \) an arbitrary field, \( \chi(M) \) the Euler characteristic of \( M \),

\[
\beta_i(M^n; \mathbb{Z}) = \text{rank} \, H_i(M^n; \mathbb{Z}); \\
\beta_i(M^n; \mathbb{F}) = \dim H_i(M^n; \mathbb{F}); \\
\beta(M^n; \mathbb{F}) = \sum_{i=0}^{n} \beta_i(M^n; \mathbb{F}).
\]

Given a Morse function \( f \) defined on a smooth manifold \( M^n \), by \( c(f), c_i(f) \) we denote the number of critical points of \( f \) and that of index \( i \), respectively. The Morse number of \( M^n \) is denoted by \( \gamma(M^n) \), i.e.

\[
\gamma(M^n) = \min \{ c(\varphi) \mid \varphi : M^n \to \mathbb{R} \text{ is a Morse function} \}.
\]

Similarly,

\[
\gamma_i(M^n) = \min \{ c_i(\varphi) \mid \varphi : M^n \to \mathbb{R} \text{ is a Morse function} \}.
\]

Clearly, for any Morse function \( \varphi \) defined on \( M \), we have

\[
\beta_i(M^n; \mathbb{F}) \leq c_i(\varphi), \\
\beta(M^n; \mathbb{F}) \leq \sum_{i=0}^{n} \gamma_i(M^n) \leq c(\varphi).
\]
In particular, if $\beta(M^n; F) = c(\varphi)$, all inequalities above become equalities. Using Kunneth’s formula, we slightly modify the result in [5, p. 217–218] as follows

\textbf{Lemma 1.} Given two Morse functions

$$\varphi: N^n \to \mathbb{R}, \quad \psi: Q^q \to \mathbb{R},$$

then the function $\varphi + \psi: N^n \times Q^q \to \mathbb{R}$ defined by

$$(x, y) \in N^n \times Q^q \mapsto \varphi(x) + \psi(y) \in \mathbb{R}$$

is a Morse function and

$$c_i(\varphi + \psi) = \sum_{j+k=i} c_j(\varphi)c_k(\psi).$$

In particular, if both $\varphi$ and $\psi$ are tight (i.e., $c(\varphi) = \gamma(N)$ and $c(\psi) = \gamma(Q)$),

$$c(\varphi + \psi) = \gamma(N^n)\gamma(Q^q) \geq \gamma(N^n \times Q^q).$$

If there exists a field $\mathbb{F}$ such that

$$\gamma(N^n) = \beta(N^n; F) \text{ and } \gamma(Q^q) = \beta(Q^q; F),$$

then $\gamma(N^n \times Q^q) = \gamma(N^n)\gamma(Q^q)$.

\textbf{3. The Proof of the Theorem}

To prove the theorem and study the general case, we establish first the following main lemma.

\textbf{Lemma 2.} Let $f$ be a Morse function defined on a closed connected smooth 4-manifold $M$ and $c(f) = 4$. Then

(a) TFAE

(a)$_1$ $M$ is a topological $S^4$;

(a)$_2$ $c_0(f) + c_4(f) = 3$ or $c_2(f) = 1$;

(a)$_3$ $\chi(M) = 2$.

(b) TFAE

(b)$_1$ $M$ is simply connected and has the integral homology groups of $S^2 \times S^2$;

(b)$_2$ $c_2(f) = 2$;

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(b) \( \chi(M) = 4 \);
(b) there exist CW-complexes \( K \) and \( L \) with the same collection of cells such that \( M \) and \( S^2 \times S^2 \) have the homotopy type of \( K \) and \( L \) respectively.
(c) TFAE
(c) \( M \) has the mod 2 homology groups of \( S^1 \times S^3 \) and the “mod 2” is replaced by “integral” when \( M \) is orientable;
(c) \( c_1(f) = c_3(f) = 1 \);
(c) \( \chi(M) = 0 \);
(c) there exist CW-complexes \( S \) and \( T \) with the same collection of cells such that \( M \) and \( S^1 \times S^3 \) have the homotopy type of \( S \) and \( T \) respectively;
(c) \( M \) is non-simply connected.
(d) TFAE
(d) \( \chi(M) \neq 0 \);
(d) \( M \) is simply connected;
(d) \( c_1(f) + c_3(f) \leq 1 \).

P r o o f. The condition \( c(f) = 4 \) and Theorem 12.1 in [10, p. 383] imply

\[ 2 \leq c_0(f) + c_4(f) \leq 3. \]

1) When \( c_0(f) + c_4(f) = 3 \), \( M \) is a topological \( S^4 \) by Theorem 12.1 in [10, p. 383] and Reeb theorem.

If \( c_2(f) = 1 \), then \( c_0(f) = 1 = c_4(f) \), otherwise we can set \( c_4(f) = 2 \), then by Theorem 12.1 in [10, p. 383] we have \( c_3(f) \geq 1 \) that implies \( c(f) \geq 5 \), contradicting the hypothesis \( c(f) = 4 \). Therefore we can set

\[ c_1(f) = 1, \quad c_3(f) = 0. \]

By the improved Morse inequalities by Pitcher that

\[ c_i(f) \geq \beta_i(M; \mathbb{Z}) + t_i(M; \mathbb{Z}) + t_{i-1}(M; \mathbb{Z}), \]

where \( t_i(M; \mathbb{Z}) \) is the torsion number of \( H_i(M; \mathbb{Z}) \), we have

\[ \beta_i(M; \mathbb{Z}) = \beta_i(S^4; \mathbb{Z}), \quad i = 0, 1, 2, 3, 4, \]

\[ H_*(S^4; \mathbb{Z}) \cong H_*(M; \mathbb{Z}). \]

Since \( c_1(-f) = 0 \), \( M \) is simply connected by Cor. 10.18 in [13, p. 225], it follows that \( M \) is a homotopy \( S^4 \), i.e. \( M \) is a topological \( S^4 \) by Freedman’s theorem in [3, p. 371].
2) If $c_2(f) = 2$, then $c_0(f) = c_4(f) = 1$ and $c_1(f) = c_3(f) = 0$. It follows that $M$ has the integral homology groups of $S^2 \times S^2$. Since $c_1(f) = 0$, $M$ is simply connected by Cor. 10.18 in [13, p. 225].

3) If $c_1(f) = c_3(f) = 1, c_0(f) = c_4(f) = 1$ and $c_2(f) = 0$. By Morse inequalities, we have

$$\beta_2(M; \mathbb{F}) = 0, \quad \beta_1(M; \mathbb{F}) = \beta_0(M; \mathbb{F}), \quad \beta_4(M; \mathbb{F}) = \beta_3(M; \mathbb{F})$$

hold for any field $\mathbb{F}$. Therefore:

$$H_*(M; \mathbb{Z}_2) \cong H_*(S^1 \times S^3; \mathbb{Z}_2);$$

and so

$$\beta_i(M; \mathbb{Z}_2) = \beta_i(S^1 \times S^3; \mathbb{Z}).$$

In particular, if $M$ is orientable, then by the homology duality and the improved Morse inequalities, we have

$$H_*(M; \mathbb{Z}) \cong H_*(S^1 \times S^3; \mathbb{Z}).$$

4) We claim $c_1(f) \neq 2$ (equivalently $c_3(f) \neq 2$). Otherwise $c_0(f) = c_4(f) = 1$ and $c_2(f) = c_3(f) = 0$, and then

$$\beta_i(M; \mathbb{F}) = 0, \quad i = 1, 2, 3; \quad \beta_0(M; \mathbb{F}) = 1,$$

resulting in

$$-1 = c_2(f) - c_1(f) + c_0(f) \geq \beta_2(M; \mathbb{F}) - \beta_1(M; \mathbb{F}) + \beta_0(M; \mathbb{F}) = 1,$$

which is absurd.

We have exhibited all possible values of $c_i(f)$ and proved that (a) $\Rightarrow$ (a)\textsubscript{1}, (b) $\Rightarrow$ (b)\textsubscript{1} and (c) $\Rightarrow$ (c)\textsubscript{1}. That (a)\textsubscript{1} $\Rightarrow$ (a)\textsubscript{3}, (a) $\Rightarrow$ (a)\textsubscript{3}, (b) $\Rightarrow$ (b)\textsubscript{3} and (c) $\Rightarrow$ (c)\textsubscript{3} are trivially true.

Our conclusion (b)\textsubscript{3} $\Rightarrow$ (b)\textsubscript{2} follows from the facts that 1) implies $\chi(M) = 2$ and that 3) implies $\chi(M) = 0$.

The proof of (b)\textsubscript{2} $\Rightarrow$ (b)\textsubscript{4}: Given $c_2(f) = 2$, then $c_0(f) = c_4(f) = 1$ and $c_1(f) = c_3(f) = 0$. By Theorem 3.5 in [7, p. 20], $M$ has the homotopy type of a $CW$-complex with a collection of one $e^0$, two $e^2$'s and one $e^4$.

On the other hand, given a natural embedding $S^n \hookrightarrow E^{n+1}$, then for any unit vector $p \in E^{n+1}$, the linear height function $l_p : S^n \to \mathbb{R}$ defined by $x \in S^n \mapsto \langle p, x \rangle$ (where $\langle , , \rangle$ denotes the usual inner product in $E^{n+1}$) is a Morse function with only
2 critical points, so $l_p$ is tight and $\beta(S^n; \mathbb{F}) = 2 = \gamma(S^n)$. From Lemma 1, we know that
\[ \varphi = l_p + l_q : S^2 \times S^2 \to \mathbb{R} \]
satisfies
\[ c(\varphi) = \gamma(S^2)\gamma(S^2) = \gamma(S^2 \times S^2) \]
and
\[ c_0(\varphi) = c_4(\varphi) = 1, \ c_2(\varphi) = 2, \ c_1(\varphi) = c_3(\varphi) = 0, \]
so $S^2 \times S^2$ like $M$ has the homotopy type of a $CW$-complex with a collection of one $e^0$, two $e^2$'s and one $e^4$.

The proof of (b)$_4 \Rightarrow$ (b)$_3$: The Euler characteristic of a manifold is a homotopy type invariant and $K$ and $L$ have the same collection of cells, so
\[ \chi(M) = \chi(K) = \chi(L) = \chi(S^2 \times S^2) = 4. \]

The proofs of (c)$_1 \Rightarrow$ (c)$_3 \Rightarrow$ (c)$_2 \Rightarrow$ (c)$_4 \Rightarrow$ (c)$_3$ are the analogues of that of (b). The proof of (c)$_5 \Rightarrow$ (c)$_2$: $M$ is non-simply connected if and only if $c_1(f) = c_3(f) = 1$ holds according to 1) and 2).

(c)$_1 \Rightarrow$ (c)$_5$ is trivial.

The proof of (a)$_3 \Rightarrow$ (a)$_2$: When $\chi(M) = 2$, only $c_0(f) + c_4(f) = 3$ or $c_2(f) = 1$ holds by 2) and 3).

Now the (d) follows immediately from (a), (b) and (c).

This concludes the proof of the lemma. \hfill \Box

Now we are in a position to prove the theorem.

Proof of Theorem. Since $\chi(M) \neq 0$, from the proof of Theorem 2 we know that $M$ is simply connected and furthermore either $M$ is a topological $S^4$ or $M$ has the integral homology groups of $S^2 \times S^2$.

a) If $M$ is the latter and there exists a continuous map
\[ h : S^2 \times S^2 \to M \]
for which its induced homomorphisms
\[ h_* : H_i(S^2 \times S^2; \mathbb{Z}) \to H_i(M; \mathbb{Z}), \ i = 0, 2, 4 \]
are isomorphisms, then since both $M$ and $S^2 \times S^2$ are simply connected $CW$-complexes, using Theorem 25 in [14, p. 406], we conclude that $M$ is homotopically equivalent to $S^2 \times S^2$. 

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b) Let $M$ have a $C^\infty$ product structure, i.e. $M = N \times Q$, then $\dim N = 2$ or 1.

If $\dim N = 1$, $N \approx S^1$ (here we denote “diffeomorphic to” by $\approx$), which contradicts the simply-connectedness of $M$. It follows that

$$\dim N = 2 = \dim Q.$$ 

But $\gamma(N) = 4 - \chi(N), \gamma(Q) = 4 - \chi(Q)$, thus

$$\beta(N; \mathbb{Z}_2) = \gamma(N), \beta(Q; \mathbb{Z}_2) = \gamma(Q).$$

Applying Lemma 1 to $N$ and $Q$, we get

$$\gamma(M) = 4, \quad \gamma(N) = 2 = \gamma(Q).$$

Therefore

$$M = N \times Q \approx S^2 \times S^2.$$ 

c) Let $M$ be an orientable Riemannian manifold of positive curvature and let $I : M \to E^6$ be an isometrical immersing. By Moore’s theorem (e.g. see [6, p. 116]) or [8, p. 72]), $I(M)$ is a topological $S^4$ and so $I$ is an embedding, $M$ is therefore a topological $S^4$.

Let $M$ with a Riemannian structure of non-negative curvature be isometrically embedded into $E^6$ and $I : M \to E^6$ such an embedding. Since $M$ is simply connected, by Baldin and Mercuri’s result (see [6, p. 116]), we conclude that either $M$ is a homotopy $S^4$ and hence a topological $S^4$ or $M \approx S^2 \times S^2$. This completes the proof of the theorem.

**Remark.** Under the hypothesis of Lemma 2, if $M$ is a non-simply connected product manifold, $M \approx S^1 \times Q^3$; if $Q^3$ satisfies $\gamma(Q^3) = \beta(Q^3; \mathbb{Z}_2)$, $M \approx S^1 \times S^3$. Because product manifold $M$ is non-simply connected, $M = N^1 \times Q^3 \approx S^1 \times S^3$ by the proof of the Theorem. Therefore

$$H_*(M; \mathbb{F}) \cong H_*(S^1 \times S^3; \mathbb{F})$$

holds for any field $\mathbb{F}$ and therefore

$$\beta(M; \mathbb{F}) = 4 = \gamma(M).$$

By Kunneth’s formula, we know that

$$\beta(Q^3; \mathbb{F}) = 2$$
holds for any field $F$, so
\[ H_*(Q^3; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z}). \]

If, in addition, $\gamma(Q^3) = \beta(Q^3; \mathbb{Z}_2)$, $Q^3 \approx S^3$. Hence
\[ M \approx S^1 \times S^3. \]

4. ON CASE $c_2(f) = 1$

For $(M, f)$ satisfying
\[
\begin{align*}
(2) \quad & c_0(f) + c_4(f) = 3 \\
(3) \quad & c_2(f) = 2 \\
(4) \quad & c_1(f) = c_3(f) = 1
\end{align*}
\]

we have its corresponding models. In fact, models $(M, f)$ satisfying (3) and (4) have been shown in the proof of Lemma 2; the model $(M, f)$ for (2) can be realized by a hypersurface of $E^5$, which is similar to a U-shape tube with two smooth caps on its two ends, and a linear height function defined on the hypersurface. We show the model as follows:

A subset $T$ of $E^5 = \{(x, y, z, u, v) \mid x, y, z, u, v \in E^1\}$ is defined by the equation
\[
(\sqrt{u^2 + v^2 - a})^2 + x^2 + y^2 + z^2 = b^2, \quad a > b > 0.
\]

Obviously, $T$ can be obtained by “revolving a 3-sphere in $E^5$
\[
\left\{ \begin{array}{l}
(v - a)^2 + x^2 + y^2 + z^2 = b^2, \\
u = 0
\end{array} \right.
\]
around subspace $Oxyz$; so $T$ is connected and closed. If hyperplane $v = 0$ is regarded as a “level surface” and $v$-axis as the “vertical” axis, then sublevel set $T_- : v < 0$ can be given by
\[
v = -\sqrt{(a \pm \sqrt{b^2 - x^2 - y^2 - z^2})^2 - u^2}.
\]
Similar to the case of a "vertical" torus in $E^3$, the linear height function on $T_-$ can be expressed as

$$f(x, y, z, u, v) = -\sqrt{(a \pm \sqrt{b^2 - x^2 - y^2 - z^2})^2 - u^2 + a + b}.$$ 

Then

$$df = 0 \iff x = y = z = u = 0, \quad v = \pm b - a,$$

i.e. $f$ has just 2 critical points $(0, 0, 0, 0, \pm b - a)$ on $T_-$ and of which $(0, 0, 0, 0, -b - a)$ is the minimum point of $f$. It is easily verified that the Hessian matrices of $f$ at the 2 critical points are nondegenerate, so $f$ is a Morse function on $T_-$. 

Since level surface $v = 0$, which is the boundary of $T_-$, consists of two 3-spheres in $E^5$

$$\begin{cases} 
(u \pm a)^2 + x^2 + y^2 + z^2 = b^2, \\
v = 0,
\end{cases}$$

$T_-$ is a “U-shape tube” with two upward ends. We cover its each end with a “cap”, i.e., a smooth $4-$disc and then obtain the required hypersurface $M$ of $E^5$. Meanwhile, we extend $f$ naturally onto the two “caps”. We still denote the extension of $f$, which is a linear height function defined on $M$, by $f$, then the two tops of the caps are critical points of $f$ of index 4 and hence $f$ has exactly 4 critical points on $M$, and

$$c_0(f) + c_4(f) = 3.$$ 

Then $M$ is a topological $S^4$ by the Theorem. This concludes the construction of the required model.

Our main purpose of this section is to probe into (just!) the probability of the existence of $(M, f)$ satisfying

\begin{equation}
(5) \quad c_0(f) = c_2(f) = c_3(f) = c_4(f) = 1, \quad c_1(f) = 0.
\end{equation}

To the end, we assume that (5) holds and under the assumption we determine the types of the 4 critical points of $f$ and calculate the homology groups of the sublevels $f_t$ and level manifolds $f^t$. We need some preliminaries.

Morse introduced the following notions and results in his [11, p. 257–258] and [12, p. 259–260]:

For a Morse function $f$ defined on an orientable manifold $M^n$ and a real number $t$, we denote the sublevel set $\{x \in M \mid f(x) \leq t\}$ by $f_t$, and $\beta_k(f_t; \mathbb{F})$ by $\beta_k(d)$. Suppose open interval $(a, b)$ contains just one critical value $c$ of $f$ and $f$ take its critical value $c$ only at one critical point $p_c$ of index $k$. Set

$$\Delta \beta_q(c) = \beta_q(b) - \beta_q(a), \quad q = 0, 1, \ldots, n.$$
Then $\Delta \beta_k(c) = 1$ or $\Delta \beta_{k-1} = -1$. If the former (resp. latter) holds, the critical point $p_c$ is said to be of increasing (resp. decreasing) type or linking (resp. nonlinking) type. Notice that

$$\beta_i(b) = \beta_i(c), \quad \beta_i(a) = \beta_i(f_c - \{p_c\}),$$

$$\Delta \beta_q(c) = \beta_q(c) - \beta_q(f_c - \{p_c\}).$$

By Theorem 29.2 in [12, p. 260],

$$\Delta \beta_q(c) = \begin{cases} 1, & \text{if } q = k \text{ and } p_c \text{ is of linking type}, \\ -1, & \text{if } q = k - 1 \text{ and } p_c \text{ is of nonlinking type}, \\ 0, & \text{in other cases}. \end{cases}$$

Thus the notions of linking and nonlinking types are mutually exclusive and complementary.

We write

$$\Delta B_i(c) = \beta_i(f^{c+r}) - \beta_i(f^{c-\varepsilon})$$

for any regular value $c$ of $f$ and any sufficiently small real number $\varepsilon$.

Let $(M, f)$ satisfy the conditions of Theorem 2 and (5), then $M$ is a topological $S^4$. Applying Corollary 39.1 of [12, p. 361] to this $(M, f)$, we choose $f$ such that

$$f(p_i) = i, \quad i = 0, 2, 3, 4$$

for which $p_i$ is a critical point of $f$ of index $i$. Take regular values $a, b, c, d, e$ of $f$ such that

$$a < 0 < b < 2 < c < 3 < d < 4 < e.$$

Then we have

**Proposition 3.** For $f$ chosen above, the critical points $p_0$, $p_2$ and $p_4$ are of linking type and $p_3$ nonlinking type. Moreover,

$$\beta_q(i) = \begin{cases} 1, & \text{if } (q, i) = (2, 2), (4, 4) \text{ or } (0, i), \text{ where } i = 0, 2, 3, 4, \\ 0, & \text{in other cases}. \end{cases}$$

**Proof.** Clearly, $p_0$, $p_4$ are of linking type and as is $p_2$, since by applying Morse inequalities to $f$, $f_c$, we have

$$\beta_2(c) = 1,$$
thus
\[ \Delta \beta_2(2) = 1. \]

It is easily checked that
\[ \Delta \beta_3(3) = 0, \]
and so
\[ \Delta \beta_2(3) = -1, \quad \beta_2(d) = 0. \]

Hence \( p_3 \) is of nonlinking type. \( \square \)

**Remarks.** 1. An analogous argument shows that to \(-f\), its critical points \( p_4, p_3, \) and \( p_0 \), with indeces 0, 1, 4, resp., are of linking type but \( p_2 \), with index 2, nonlinking type.

2. [1, p. 8–9] indicates: Let \( f \) be a Morse function defined on a closed \( C^\infty \) manifold \( M^n \), then the following three conditions are equivalent
   a) For any field \( \mathbb{F} \), \( c_k(f|t) = \beta_k(f; \mathbb{F}) \) holds for any \( t \in \mathbb{R} \) and \( k = 0, 1, 2, \ldots, n \);
   b) The homomorphisms between homology groups
   \[ H_i(f; \mathbb{F}) \to H_i(M^n; \mathbb{F}), \quad i = 0, 1, 2, \ldots, n \]
   induced by inclusion \( f_t \hookrightarrow M^n \) are injective;
   c) Every critical point of \( f \) is of linking type.

Now our Proposition 3 implies that for \((M, f)\) satisfying (5),
   a)' \( 1 = c_k(f) > \beta_k(M; \mathbb{F}) = 0, \quad k = 2, 3; \)
   b)' The induced homomorphism
   \[ (\mathbb{F} \cong) \quad H_2(f_c; \mathbb{F}) \to H_2(M; \mathbb{F}) \quad (= 0) \]
   is not injective;
   c)' the critical point \( p_3 \) of \( f \) is of nonlinking type.

It follows that our Proposition 3 does not contradict the results in [1, p. 8–9]. It can be verified that (5) is compatible with Morse inequalities, the theorem on a character of homology \( S^4 \) in [11, p. 259] and Corollary 1.2 as well as (7.11) in [9, p. 256–257]. Besides, by Lemma 1.1 in [10, p. 352], the critical point \( p_3 \) of \(-f\) of index 1 is of linking type, which is consistent with our Proposition 3. All these facts seem to be to a great extent in favor of the existence of \((M, f)\) satisfying (5).

3. In his [16, p. 100], Willmore said that recent work by Cerf made it appear that (which has not been proved! cf. V. V. Sharko’s work, say, MR1989f, 57038)
\[ \gamma(M) = \sum_{i=0}^{n} \gamma_i(M) \]
holds for any closed $C^\infty$ manifold. If the equality holds, our $f$ satisfying (5) has two superfluous saddle points, i.e., there exists a Morse function $f^*: M \to \mathbb{R}$ with only 2 critical points.

As the end of this paper, we study the topology of the level hypersurfaces of $M$ with respect to any regular value of $f$ and obtain

**Proposition 4.** Under the hypothesis of Proposition 3, $f^b \approx S^3 \approx f^d$, $f^c$ has the integral homology groups of $S^1 \times S^2$.

**Proof.** We denote set $\{ x \in M \mid f(x) \geq t \}$ by $f_t^+$. Since the points $p_0$ in $f_b$ and $p_4$ in $f_d^+$ are extreme points of $f$ and $f_b$ (resp. $f_d^+$) contains no critical points of $f$ other than $p_0$ (resp. $p_4$), and $f^b$ (resp. $f^d$) is the boundary of $f_b$ (resp. $f_d^+$). Then by Morse lemma,

$$f^b \approx S^3 \approx f^d.$$  

For $f^c$, since $M$ is a homology $S^4$, $p_0$ and $p_2$ as critical points of $f$ are of increasing type rel. $f^0$ and $f^2$, resp., but $p_3$ decreasing type rel. $f^3$ by Corollary 7.2 in [9, p. 256]. Thus by Theorem 5.1 in [9, p. 252],

$$1 = \Delta B_2(2) = \beta_2(f^c),$$
$$1 = \Delta B_1(2) = \beta_1(f^c),$$
$$0 = \Delta B_i(2) = \beta_i(f^c) - 1, \quad i = 0, 3.$$  

that is,

$$\beta_i(f^c; \mathbb{F}) = \beta_i(S^1 \times S^2; \mathbb{F}) = \beta_i(S^1 \times S^2; \mathbb{Z}), \quad i = 0, 1, 2, 3$$

hold for any field $\mathbb{F}$. Then by the improved Morse inequalities, we have

$$H_*(f^c; \mathbb{Z}) \cong H_*(S^1 \times S^2; \mathbb{Z}).$$

It follows, moreover, that $f^b$, $f^c$ and $f^d$ are connected closed orientable hypersurfaces of $M$. This proves the proposition. \qed

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References


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