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FREE ALMOST-P-LATTICES

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1. INTRODUCTION

This work is the result of trying to describe the free almost-p-lattices going along the lines with the paper [1] of Berman and Dwinger in which the finitely generated free distributive p-lattices are described. In doing so we find that the varieties of almost-p-lattices generated by L_{nn}, $n \geqslant 1$ (see definitions next section) are defined by the same equations used by Lee in [2] in order to describe the subvarieties of the variety of distributive p-algebras. This is accomplished in Section 4. In Section 5 we use this result to describe the join irreducible elements of a free almost p-lattices with n generators generalizing in this way to almost-p-lattices the results of Berman and Dwinger for distributive p-lattices. Section 2 is devoted to give the necessary definition and preliminares and in Section 3 some facts related to atoms of finitely generated almost-p-lattices, needed in the sequel, are studied.

2. DEFINITIONS AND PRELIMINARIES

An *almost-p-lattice* (abbreviated in the sequel to *apl*) is an algebra $\langle L; +, \cdot, ', 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ where $\langle L; +, \cdot, 0, 1\rangle$ is a distributive lattice with greatest and least elements and the unary operation $'$ satisfies:

- $0' = 1$ and $1' = 0$.
- $(x + y)' = x'y'$.
- $(xy)'' = x''y''$.
- $x''' = x'$.
- $xx' = 0$.

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The class of *apl*'s is a variety which will be denoted APL. This variety is a subvariety of the variety of Semi De Morgan algebras introduced by Sankappanavar in [5]. The well known variety of distributive p-algebras (*pdl* for short) is a subvariety of APL. For a $L \in APL$ define:

$$B(L) = \{x' \colon x \in L\};$$
$$D(L) = \{x \in L \colon x' = 0\};$$
$$pdl(L) = \{x \in L \colon x \leqslant x''\};$$
$$S(L) = \{x \in L \colon x \nleqslant x''\} = L \setminus pdl(L).$$

An element of $D(L)$ is called *dense.* $\langle B(L); \dot{+}, \cdot, ', 0, 1\rangle$, where $x \dot{+} y = (x'y')'$, is a Boolean algebra [5 , Theorem 2.4 and Corollary 2.7]. $pdl(L)$ is a *pdl* and it is a subalgebra of L. The subdirectly irreducible (s.i. for short) *apl*'s are characterized in [6, Theorem 5.5] by being those having just 2 dense elements. If $L \in APL$ is s.i., its dense element different from 1 will be denoted generically by d. In [3] the finite s.i. *apl*'s are described. They are denoted by L_{nk}, $n = 1, 2, \cdots$; $k = 1, 2, \cdots, n$. Their main properties are:

(i) $B(L_{nk}) \cong \mathbf{2}^n$, the n-atom Boolean algebra.
(ii) L_{nk} has k coatoms one of them being d.
(iii) $B(L_{nk}) \setminus \{1\} = [0, d)$.
(iv) $\langle [0, d]; +, ., ^\sim, 0, d\rangle$, where $x^\sim = x'$ if $x \neq 0$ and $0^\sim = d$, is a Boolean algebra.
(v) $S(L_{nk}) = \{x \in L \colon x'' < x\} = L \setminus ([0, d] \cup \{1\})$.
(vi) $pdl(L_{nk}) = [0, d] \cup \{1\} \cong L_{n1}$.
(vii) No element of $S(L_{nk})$ can be an atom of L_{nk}. In other words, d covers all the atoms of L_{nk}.
(viii) $\langle S(L_{nk}), +, \cdot\rangle$ is a sublattice of L_{nk}. Moreover, it is isomorphic to a sublattice of $[0, d]$, an embedding being $x \mapsto xd$.
(ix) If $b \in S(L_{nk})$ is a coatom of L_{nk} then $bd = b''$ is a coatom of $[0, d]$ or equivalently, b' is an atom of L_{nk}.
(x) There exists a unique atom a of L_{nn} such that $(\Pi S(L_{nn}))d = a$.
(xi) $L_{mk} \in V(L_{nn})$, the variety generated by L_{nn}, $k \leqslant m \leqslant n$.

The following rules of computation will be used frequently. The first two are valid in any *apl*. The last two are valid just in any *pdl*.

- $x \leqslant y$ implies $y' \leqslant x'$.
- $x'' = 0$ iff $x = 0$ [6, Theorem 2.2]
- $xy = 0$ implies $y \leqslant x'$.
- $x \leqslant x''$.

In this section Lemma 2.1 of [1] (in any finite *pdl* the unary operation $'$ is determined by the atoms), is extended to *apl*'s. This result is the key fact in the description of the join irreducible elements of a free finitely generated *apl*. In what follows, L will stand for a finite *apl*. For $x \in L$ define

$$A_x = \{a \in L \colon a \leqslant x, a \text{ atom of } L\}.$$

By $L \leqslant_{SD} \prod_{i \in I} L_i$ we mean that L is a subdirect product of the family $\{L_i \colon i \in I\}$. If the L_i's are s.i. then it is said that $\prod_{i \in I} L_i$ is a subdirect representation of L and the L_i's are called the components of L in such a subdirect representation. If $(x_i)_{i \in I}$ corresponds to $x \in L$ then x_i is called the i-coordinate of x.

Fact 1. *Let a be an atom of L. Then the coordinates of a different from 0 in any subdirect representation of L are necessarily atoms in the respective component.*

Proof. Let $\prod_{i \in I} L_i$ be a subdirect representation of L and suppose that $a_i \neq 0_i$. If a_i is not an atom of L_i then there exists $c_i \in L_i$ with $0_i < c_i < a_i$. As $L \leqslant_{SD} \prod_{i \in I} L_i$ there is $b \in L$ such that $b_i = c_i$. So, $0 < ba < a$ (because $(ba)_i = b_i < a_i$), a contradiction. □

Fact 2. *Let $x, y \in L$. Then $x' = y'$ implies $A_x = A_y$.*

Proof. Let $a \in A_x$. Then $ax = a$ and therefore $ay' = 0$. If $ay = 0$, then $a(y + y') = 0$. Consider a subdirect representation of L. Let i be such that $a_i \neq 0_i$. Clearly $y_i + y_i' \in D(L_i) = \{d_i, 1_i\}$. Then by property (vii) and Fact 1, $y_i + {y'}_i \geqslant a_i$. As i was arbitrary the only condition being $a_i \neq 0_i$, it follows that $y + y' \geqslant a$, a contradiction. So, $ay = a$ and $a \in A_y$. □

Fact 3. *Suppose that L is s.i. Then $A_x = A_y$ implies $x' = y'$.*

Proof. From property (vii) it follows that $A_{xd} = A_{yd}$ and since xd and yd are in $pdl(L)$ which is a *pdl*, then by [1, Lemma 2.1], $x' = (xd)' = (yd)' = y'$. □

Fact 4. *Let $L = \prod_{i \in I} L_i$ where the L_i's are s.i. and $x, y \in L$. Then $A_x = A_y$ implies $x' = y'$.*

Proof. Select $x_i \neq 0_i$ and let $z_i \in A_{x_i}$. Call z the element of L whose i-coordinate is z_i and all the others are zeros. Then $z \in A_x = A_y$. It follows that

$z_i \leqslant y_i$, i.e., $z_i \in A_{y_i}$. So, $A_{x_i} \subseteq A_{y_i}$. Clearly, $y_i \neq 0_i$. Then, by symmetry, $A_{y_i} \subseteq A_{x_i}$. So $A_{x_i} = A_{y_i}$. Now apply Fact 3 to get $x_i' = y_i'$. Remains to prove that if $x_i = 0_i$ then $y_i = 0_i$. But this can be seen in the argument above. □

Lemma 3.1. *Let L be a finite apl and let $x, y \in L$. Then $x' = y'$ if and only if $A_x = A_y$.*

Proof. One direction is Fact 2. For the other direction let $\bar{L} = \prod_{i \in I} L_i$ be a subdirect representation of L. For $z \in L$ define

$$\overline{A_z} = \{a \in \bar{L} \colon a \text{ atom of } \bar{L},\ a \leqslant z\}.$$

Let $a \in \overline{A_y}$. We claim that there exists a_0, atom of L, such that $a \leqslant a_0$. For if this were not true then $z = \Sigma\{\text{atoms of } L\} \in L$ would be such that $za = 0$. As $z' = 0$ in L, $z' = 0$ in $\bar{L}$ which means that z would covered all the atoms of $\bar{L}$. This would implied $za = a$, a contradiction. Now, if $a_0 \notin A_y$ then $a_0 y = 0$. But $a_0 y \geqslant a$. So, $a_0 \in A_y = A_x$ and therefore $a \leqslant a_0 \leqslant x$, i.e., $a \in \bar{A}_x$. It has been proved that $\bar{A}_y \subseteq \bar{A}_x$. Similarly one get the reverse inclusion and the desired result is received now by applying Fact 4. □

Fact 5. *If a is an atom of L then $a \in pdl(L)$.*

Proof. Let $\prod_{i \in I} L_i$ be a subdirect representation of L. By Fact 1, for each i, either $a_i = 0_i$ or a_i is an atom of L_i. By properties (v), (vi), and (vii), $a_i \in [0_i, d_i] \cup \{1_i\} = pdl(L_i)$. ($a_i = 1_i$ implies $L_i = \{0_i, 1_i\}$). So $a_i \leqslant a_i''$. As i was arbitrary we have $a \leqslant a''$. □

Fact 6. *Let c' be a coatom of $B(L)$. Then c covers exactly one atom a of L.*

Proof. Suppose that c covers the atoms a_1 and a_2 of L. Then $a_i' \geqslant c'$, $i = 1, 2$. By Fact 2, $a_i' \neq 1 = 0'$. Then, as c' is coatom of $B(L)$, $a_i' = c'$, $i = 1, 2$. So, $A_{a_1} = \{a_1\} = A_{a_2} = \{a_2\}$, i.e., $a_1 = a_2$. □

Fact 7. *a atom and $ab = 0$ implies $a \leqslant b'$.*

Proof. It is an easy consequence of Fact 1 and the fact that $b + b'$ is dense. □

Fact 8. *If a is atom of L then a' is a coatom of $B(L)$.*

Proof. Suppose that $b' \geqslant a'$, As a is atom, either $ab'' = a$ or $ab'' = 0$. In the former case, $a \leqslant b''$ and consequently $a' \geqslant b'$. So $a' = b'$. In the later case, $(ab'')'' = a''b'' = (ab)'' = 0$. From [6, Theorem 2,2] it follows $ab = 0$ and since a is atom then $a \leqslant b'$ (Fact 7). Now we have $a + a' \leqslant b'$ implies $0 = (a + a')' \geqslant b''$ so that $b' = 1$. □

In [2] Lee consider the family of equations

$$(E_n) \qquad (x_1 \cdots x_n)' + \sum_{i=1}^{n} (x_1 \cdots x_i{}' \cdots x_n)' = 1, \qquad n \geqslant 1$$

for *pdl*'s. There it is proved that if L is a *pdl* then $L \in V(L_{n1})$ if and only if $L \models E_n$. Here we consider the equation (E_n) for *apl*'s. The main result is the following:

Theorem 4.1. *Let $L \in APL$. Then the following are equivalent:*

(1) $L \models (E_n)$.
(2) $L \in V(L_{nn})$.

The comparison between the number of maximal filters of L that contain a given prime ideal P of L and that one of the maximal filters of *pdl*(L) that contain $P \cap$ *pdl*(L) allows us to approach the proof of this result in the same way as in [2]. The following Lemma will be used very often in this section.

Lemma 4.2. *Let M be a maximal filter of L and let $c \in L$. Then $c \notin M \Leftrightarrow c' \in M$.*

Proof. ($\Rightarrow$) Suppose on the contrary that $c' \notin M$. Then $[M \cup \{c\}) = [M \cup \{c'\}) = L$ from which it follows that there exist $x, y \in M$ such that $c' \geqslant xc$ and $c \geqslant yc'$. Putting $z = xy$ one has:

$$0 = (z(c + c'))'' = z''(c + c')'' = z''(c'c'')' = z''.$$

Now invoke [6, Theorem 2.2] to get $z = xy = 0 \in M$, a contradiction. The other implication is obvious. □

The next proposition is one direction of [2, Theorem 2] extended to *apl*'s. The proof is exactly the same if the previous lemma is used.

Proposition 4.3. *Suppose $L \models E_n$. Then for each prime filter P of L, there are at most n distinct maximal filters that contain P.*

For a prime filter P of L define:

$\hat{P} = P \cap$ *pdl*(L);
$\mathscr{M}_P$ = maximal filters of L that contain P ;
$\mathscr{M}_{\hat{P}}$ = maximal filters of L that contain $\hat{P}$;
$\hat{\mathscr{M}}_{\hat{P}}$ = maximal filters of *pdl*(L) that contain $\hat{P}$.

Notice that $\mathscr{M}_P \subseteq \mathscr{M}_{\hat{P}}$ and consequently $|\mathscr{M}_P| \leqslant |\mathscr{M}_{\hat{P}}|$.

Proposition 4.4. $|\mathscr{M}_{\hat{P}}| = |\hat{\mathscr{M}}_{\hat{P}}|$. *So,* $|\mathscr{M}_P| \leqslant |\hat{\mathscr{M}}_{\hat{P}}|$.

Proof. One proves first that $|\mathscr{M}_{\hat{P}}| \leqslant |\hat{\mathscr{M}}_{\hat{P}}|$ by proving that the application $\mathscr{M}_{\hat{P}} \longrightarrow \hat{\mathscr{M}}_{\hat{P}}$; $M \mapsto \hat{M}$ is one to one. For suppose that $\hat{M}_1 = \hat{M}_2$. Let $x \in M_1$. If $x \notin M_2$, then by Lemma 4.2, $x' \in M_2$. So $x' \in \hat{M}_2 = \hat{M}_1$, a contradiction because $x \in M_1$. So, $M_1 \subseteq M_2$. Similarly one obtains the reverse inclusion. To prove the reverse inequality consider the application $\hat{\mathscr{M}}_{\hat{P}} \longrightarrow \mathscr{M}_{\hat{P}}$; $M \mapsto [M)$. Notice first that it make sense. Clearly $[M) \supseteq \hat{P}$. To see that $[M)$ is a maximal filter of L, pick $x \in L \backslash [M)$. One wants $[\{x\} \cup [M)) = L$. There are two cases to be consider: $x'' \in M$ and $x'' \notin M$. In the former one, put $y = xx''$. As $y'' = (xx'')'' = x'' \geqslant xx'' = y$, $y \in pdl(L)$. Clearly $y \notin M$ ($y \in M \Rightarrow x \in M$ since $x \geqslant y$) and since M is maximal filter of $pdl(L)$ it follows that $[\{y\} \cup M) = pdl(L)$. Let $\emptyset \neq T \subseteq M$, T finite, such that $0 = y\Pi T$. Then $0 = x\Pi S$ where $S = T \cup \{x''\} \subseteq M$. This means that $0 \in [\{x\} \cup [M))$, i.e., $[\{x\} \cup [M)) = L$ as wanted. In the case $x'' \notin M$, one get from Lemma 4.2 that $x' \in M$. Then $0 = xx' \in [\{x\} \cup [M))$, i.e., $[\{x\} \cup [M)) = L$. This finish the proof that $[M)$ is a maximal filter of L. Now we show that the map is one to one. Let $M_1, M_2 \in \hat{\mathscr{M}}_{\hat{P}}$, $M_1 \neq M_2$. If $[M_1) = [M_2)$, pick $x \in M_1 \setminus M_2$. As $x \in M_1 \subseteq [M_1) = [M_2)$ one may pick $\emptyset \neq T \subseteq M_2$, T finite, such that $x \geqslant \Pi T$. Since M_2 is filter and $x \in pdl(L)$ then $x \in M_2$, a contradiction. Therefore, $[M_1) \neq [M_2)$. This ends the proof. □

Lemma 4.5. *Let P be a prime filter of L such that $|\mathscr{M}_{\hat{P}}| = n$ and $|\mathscr{M}_P| = k$ Then L is a homomorphic image of $L_{n,n-k+1}$.*

This makes sense since as it was observed, $k \leqslant n$. Notice that if L is a *pdl* then $k = n$ and the conclusion of the lemma is that L is a homomorphic image of $L_{n,1}$ which is [2, Lemma 1].

Proof of Lemma 4.5. Let $a_1, \cdots, a_n$ be the atoms of $L_{n,n-k+1}$ and $b_{k+1}, \cdots, b_n$ its coatoms distinct from d. Here the coatoms are numbered in such a way that $b_i d = \sum_{j \neq i} a_j$; in other words, $(b_i d)' = a_i$, $k+1 \leqslant i \leqslant n$. Observe that $a_i + b_i = 1$. Let $\mathscr{M}_{\hat{P}} = \{M_1, M_2, \cdots, M_n\}$ and $\mathscr{M}_P = \{M_1, \cdots, M_k\}$. Define φ: $L \longrightarrow L_{n,n-k+1}$ by the formula

$$\varphi(x) = \begin{cases} \Pi\{b_i\colon x \notin M_i, k+1 \leqslant i \leqslant n\}, & \text{if } x \in P; \\ \Sigma\{a_i\colon x \in M_i, 1 \leqslant i \leqslant n\}, & \text{otherwise.} \end{cases}$$

It can be verified, in the same way as in [2, Lemma 1], that φ is an epimorphism. □

Proof of Theorem 4.1. (1) $\Rightarrow$ (2). $L \models (E_n)$ implies $pdl(L) \models (E_n)$. Then by Proposition 4.3 (or [2, Theorem 2]), $|\hat{\mathcal{M}}_{\hat{P}}| \leqslant n$. So by Proposition 4.4, $|\mathcal{M}_P| \leqslant |\mathcal{M}_{\hat{P}}| \leqslant n$. Now repeat the proof of [2, Theorem 3] verbatim using of course Lemma 4.5 instead of [2, Lemma 1]. For (2) $\Rightarrow$ (1) it will be enough to prove that $L_{nn} \models (E_n)$. Suppose on the contrary that there exist $c_1, \cdots, c_n \in L_{nn}$ such that

$$e_1' + \cdots + e_n' + e_{n+1}' < 1$$

where $e_j = (\Pi_{i \neq j} c_i) c_j'$, $1 \leqslant j \leqslant n$, $e_{n+1} = \Pi c_i$. It is clear that $e_j \leqslant c_j'$, $1 \leqslant j \leqslant n$, and that the left hand side of the inequality above is dense, i.e., is precisely d. Thus $0 < e_i = \Sigma A_i$, $1 \leqslant i \leqslant n$, where $A_i \neq \emptyset$ is some set of atoms of $[0, d]$. See Section 2, property (iv). If $1 \leqslant i \neq j \leqslant n$, then $e_i e_j = 0$ and consequently $A_i \cap A_j = \emptyset$. Hence $\sum_{i=1}^{n} |A_i| = n$ and $|A_i| = 1$, $1 \leqslant i \leqslant n$. So, the e_i's are the atoms of $[0, d]$ which are exactly those of L_{nn}. Since $e_i e_{n+1} = 0$, $1 \leqslant i \leqslant n$, it follows that $e_{n+1} = 0$; but then $e_{n+1}' = 1$, a contradiction. □

Corollary 4.6 ([4, Lemma 8]). *The following are equivalent:*

(1) $L \in V(L_{nn})$.
(2) *L satisfies the following property: let $x_0, \cdots, x_n \in L$ such that $x_i x_j = 0$, $i \neq j$, $1 \leqslant i, j \leqslant n$. Then $x_0' + \cdots + x_n' = 1$.*

Proof. (2) $\Rightarrow$ (1) is the same as in [4]. (1) $\Rightarrow$ (2). $x_i x_j = 0$ implies $x_i'' x_j'' = 0$. As $x_i'', x_j'' \in pdl(L)$, $x_i'' \leqslant x_j''' = x_j'$. Thus, $x_0'' \leqslant x_1' x_2' \cdots x_n'$. So,

$$\begin{aligned} x_0' + x_1' + \cdots + x_n' &\geqslant (x_1' \cdots x_n')' + (x_1'' x_2' \cdots x_n')' \\ &\quad + \cdots + (x_1' \cdots x_{n-1}' x_n'')' = 1, \end{aligned}$$

later equality due to Theorem 4.1. □

5. Finitely generated *apl*'s

In this section, unless stated otherwise, L will stand for a *apl* generated by the set $X = \{x_1, x_2, \cdots, x_n\}$. For $1 \leqslant i \leqslant n$ define:

$$x_i^0 = x_i x_i'' \qquad \text{and} \qquad x_i^1 = x_i'.$$

For $1 \leqslant j \leqslant 2^n$ define:

$$a_j = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}, \varepsilon_i \in \{0, 1\}; \quad b_j = a_j'; \qquad (b_j)_i = (a_j)_i = \varepsilon_i.$$

Define also the following sets:

$$A = \{a_j \colon 1 \leqslant j \leqslant 2^n\}; \qquad B = \{b_j \colon 1 \leqslant j \leqslant 2^n\}; \qquad G = X \cup B.$$

This sets coincide with those defined in [1] in the case L is a *pdl*.

Lemma 5.1 ([1, Lemma 2.3]).

(1) *If $a_j \neq 0$ then a_j is an atom.*
(2) *If for $i \neq j$, a_i and a_j are atoms then $a_i \neq a_j$.*
(3) *Each atom of L is in A.*

Proof. (1) It is easy to see that $a_j x_i \in \{0, a_j\}$. Also, if $a_j y$ and $a_j z$ are in $\{0, a_j\}$ so are $a_j yz$ and $a_j(y + z)$. Suppose now that $a_j z = 0$. Observe that $a_j \in pdl(L)$. Since $z + z'$ is dense then $a_j \leqslant z + z'$. So, $a_j \leqslant z'$. Clearly, if $a_j \leqslant z$ then $a_j z' = 0$. So, $a_j z \in \{0, a_j\}$ implies $a_j z' \in \{0, a_j\}$.

(2) Suppose on the contrary that $a_i = a_j$. Let

$$a_i = x_1{}^{\varepsilon_1} \cdots x_n{}^{\varepsilon_n}; \quad a_j = x_1{}^{\eta_1} \cdots x_n{}^{\eta_n}.$$

Since $i \neq j$, we may assume that there is a k such that $\varepsilon_k = 0$ and $\eta_k = 1$. Thus $a_i \leqslant x_k{}^0 = x_k x_k{}''$ and $a_i = a_j \leqslant x_k{}^1 = x_k{}'$. Hence $a_i \leqslant x_k x_k{}'' x_k{}' = 0$, a contradiction.

(3) Let $s = \sum_{j=1}^{2^n} a_j$. Thus $s = \Pi_{i=1}^n (x_i{}^0 + x_i{}^1)$. Now verify $s' = 0$. So, by Lemma 3.1, $A_1 = A_s$. □

From the previous lemma, Fact 6 and Fact 8, it follows that the b_j's different from 1 are distinct and exhaust all the coatoms of $B(L)$. If L is freely generated by X then no $b_j = 1$.

Proposition 5.2. *Let R and T be non-empty subsets of G and consider the following statements:*

(0) $\Pi R \leqslant \Sigma T$;
(1) $R \cap T \neq \emptyset$;
(2) $R \supseteq \{b_j \colon (b_j)_i = 0$ *for each* $x_i \in R\}$;
(3) $|T \cap B| > m$;
(4) $|B \setminus R \cap B| > m$.

Then,

(i) *in $F_{APL}(X)$, (0) iff either (1) or (2).*
(ii) *In $L = F_{V(L_{m,m})}(X)$, (0) implies either (1) or (2) or (4) and either (1) or (2) or (3) implies (0).*

Proof. It is an adaptation of the proof of [1, Theorem 2.8].

(i): ($\Leftarrow$) (1) suffices in any lattice. (2) implies that $\Pi R = 0$.

($\Rightarrow$) Suppose that neither (1) nor (2) are satisfied. With out loss of generality, we may add to T those b's that are not in R. Let $|T \cap B| = t$. If $t = 0$ then all b's are in R and therefore $\Pi R \leqslant \Pi B = 0$. Assume that $t > 0$. Let $g\colon L \longrightarrow \mathbf{2}^{2^n}$ be the epimorphism obtained by composition of the canonical epimorphism $L \longrightarrow L/\Phi$ ($\Phi = \{(z, w) \in L \times L\colon z' = w'\}$) and some isomorphism $L/\Phi \longrightarrow \mathbf{2}^{2^n}$. Notice that $t \leqslant 2^n$. Let now $h\colon \mathbf{2}^{2^n} \longrightarrow \mathbf{2}^t$ be an epimorphism such that

$$h(g(a_i)) = \begin{cases} \text{atom of } \mathbf{2}^t, & \text{if } b_i \in T; \\ 0, & \text{otherwise.} \end{cases}$$

Define now $f\colon L \longrightarrow \mathbf{2}^t$ by $f = h \circ g$. With out loss of generality we may assume that $x_1, \cdots, x_k$ are all the x's in R. We claim that $f(x_1), \cdots, f(x_k)$ cover a common atom of $\mathbf{2}^t$. For if all the atoms of L of the form $(x_1{}^0 \cdots x_k{}^0 \cdots)$ go to 0 by f then all the b's of the form $(x_1{}^0 \cdots x_k{}^0 \cdots)'$ are in R, (because (1) is not satisfied and the additional assumption $R \cup T \supseteq B$). This is against the assumption that (2) is not satisfied. Thus the claim is proved. Now select an atom of $\mathbf{2}^t$ covered by $f(x_1), \cdots, f(x_k)$, say $a = f(x_1) \cdots f(x_k) \cdots$. Consider the *apl* L_{tt} and identify $(B(L_{tt}) \setminus \{1\}) \cup \{d\}$ with $\mathbf{2}^t$ in such a way that $d\Pi S(L_{tt}) = a$ (property (x) Section 2). Call u_i the element of $S(L_{tt})$ such that $u_i d = f(x_i)$, $1 \leqslant i \leqslant k$. Define $\gamma\colon X \longrightarrow L_{tt}$ by:

$$\gamma(x_i) = \begin{cases} u_i & \text{if } x_i \in R; \\ f(x_i) & \text{if } x_i \notin R. \end{cases}$$

The definition of γ is based on property (viii). Let $\bar{\gamma}$ be the extension of γ to L. It is easy to verify that

$$\bar{\gamma}(\Pi R) = \Pi\{u_i\colon x_i \in R\} \qquad \text{and} \qquad \bar{\gamma}(\Sigma T) \in [0, d].$$

Now by property (viii), $\bar{\gamma}(\Pi R) \in S(L_{tt})$. So, by property (v), $\Pi R \nleqslant \Sigma T$.

(ii): Assume (3). Then, by Lemma 5.1, $a_i a_j = 0$ if $i \neq j$. So, by Corollary 4.6, $\Sigma T = 1$. Suppose now that neither (1) nor (2) nor (4) are satisfied. Then $t = |T \cap B| \leqslant m$ and since $L_{tt} \in V(L_{mm})$ for $t \leqslant m$ and the negation of (4), the argument above can be used again to conclude that $\Pi R \nleqslant \Sigma T$. □

Lemma 5.3. *Let $z \in L$. Then $z' = \Pi\{b_i\colon a_i \leqslant z\}$.*

Proof. Observe first that $\Pi\{b_i\colon a_i \leqslant z\} = (\Pi\{a_j{}'\colon a_j \leqslant z\})''$. Let $w = (\Pi\{a_j{}'\colon a_j \leqslant z\})'$. We shall prove that $A_w = A_z$. The desired result will follow then from

Lemma 3.1. Let $a \in A_z$. Then $a\Pi\{a_i'\colon a_i \leqslant z\} = 0$. It follows from Fact 7 that $a \leqslant (\Pi\{a_i'\colon a_i \leqslant z\})' = w$, i.e., $a \in A_w$. Conversely, let $a \in A_w$. If $az = 0$ then $aa_i = 0$ for all i such that $a_i \leqslant z$ and again from Fact 7 $a \leqslant a_i' = b_i$. Thus, $a \leqslant \Pi\{b_i\colon a_i \leqslant z\}$ and therefore $aw = 0$, a contradiction. Hence $a \leqslant z$. □

Corollary 5.4 ([1, Theorem 2.4]). *L, as bounded distributive lattice, is generated by $G = X \cup B$.*

Define $\overline{G} = \{\Pi T\colon T \subseteq G\}$. Observe that $\overline{G}$ contains the join irreducible elements of L. For $z \in \overline{G}$ define:

$$\beta(z) = \{b_i \in B\colon b_i \geqslant z\}, \qquad \chi(z) = \{x_i \in X\colon x_i \geqslant z\}.$$

Clearly, $z = \Pi\beta(z)\Pi\chi(z)$.

Theorem 5.5 ([1, Theorem 3.3]). *The join irreducible elements of $F_{V(L_{mm})}(X)$ are the non-zero elements z of $\overline{G}$ for which $2^n - m \leqslant |\beta(z)| < 2^n$.*

Proof. If $z \in \overline{G}$ with $|\beta(z)| = 2^n$ then $\beta(z) = B$ and since $\Pi B = 0$ it follows that $z = 0$. If $|\beta(z)| < 2^n - m$ then there exist say $b_0, \cdots, b_m \in B$ with $b_i \ngeqslant z$, $0 \leqslant i \leqslant m$. As $a_i a_j = 0$ then by Corollary 4.6, $z = z1 = z(b_0 + \cdots + b_m) = zb_0 + \cdots + zb_m$. Finally, suppose that $2^n - m \leqslant |\beta(z)| < 2^n$ and that $z = \Pi T_1 + \cdots + \Pi T_r$ where $T_i \subseteq G$, $1 \leqslant i \leqslant r$. If $\Pi T_i \neq z$ for all i, then for each i there is a $t_i \in T_i$ such that $t_i \ngeqslant z$. Thus $0 \neq z = \Pi\beta(z)\Pi\chi(z) \leqslant t_1 + \cdots + t_r$ in contradiction with Proposition 5.2 with $R = \beta(z) \cup \chi(z)$ and $T = \{t_1, \cdots, t_r\}$. □

Corollary 5.6. *$\overline{G} \setminus \{0\}$ is the set of join irreducible elements of $L = F_{APL}(X)$.*

Proof. Just observe that for $m \geqslant 2^n$, $F_{APL}(X) \in V(L_{mm})$ from which it follows that $F_{APL}(X) \cong F_{V(L_{mm})}(X)$. □

Let us give now formulas to compute the number of join irreducible elements. Denote the number of such elements z with $|\chi(z)| = k$ and $|\beta(z)| = j$ by η_{kj}. Bear in mind that

$$(\Pi\chi(z))' = \Pi\{b_j\colon (b_j)_i = 0 \text{ for } x_i \in \chi(z)\}.$$

Then, for $L = F_{APL}(X)$ we have

$$\eta_{kj} = \begin{cases} \binom{n}{k}\binom{2^n}{j}, & \text{if } k = 0 \text{ and } 1 \leqslant j < 2^n \\ & \text{or } k \geqslant 1 \text{ and } 0 \leqslant j < 2^{n-k}; \\ \binom{n}{k}\left(\binom{2^n}{j} - \binom{2^n - 2^{n-k}}{j - 2^{n-k}}\right), & \text{if } k \geqslant 1 \text{ and } 2^{n-k} \leqslant j < 2^n. \end{cases}$$

Consequently, the total number of join irreducible elements of $L = F_{APL}(X)$ is given by

$$2^{2^n+n} - \sum_{k=0}^{n} \binom{n}{k} 2^{2^n-2^{n-k}}.$$

For the case $L = F_{V(L_{mm})}(X)$ we distinguish two cases:
(i) $m \leqslant 2^{n-1}$. Then

$$\eta_{kj} = \begin{cases} \binom{2^n}{j}, & \text{if } k = 0 \text{ and } 2^n - m \leqslant j < 2^n; \\ \binom{n}{k}\left(\binom{2^n}{j} - \binom{2^n-2^{n-k}}{j-2^{n-k}}\right), & \text{if } k \geqslant 1 \text{ and } 2^{n-k} \leqslant j < 2^n. \end{cases}$$

The total number of join irreducible elements for this case is given by

$$(\dagger) \qquad 2^n \sum_{j=1}^{m} \binom{2^n}{j} - \sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{m} \binom{2^n - 2^{n-k}}{j}.$$

(ii) $m > 2^{n-1}$. Let $2 \leqslant k_0 \leqslant n-1$ such that $2^n - 2^{n-k_0} < m < 2^n - 2^{n-(k_0+1)}$. Then

$$\eta_{kj} = \begin{cases} \binom{n}{k}\binom{2^n}{j}, & \text{if } 0 \leqslant k \leqslant k_0 \text{ and } 2^n - m \leqslant j \\ & < 2^{n-k}; \\ \binom{n}{k}\left(\binom{2^n}{j} - \binom{2^n-2^{n-k}}{j-2^{n-k}}\right), & \text{if } 1 \leqslant k \leqslant k_0 \text{ and } 2^{n-k} \leqslant j < 2^n \\ & \text{or } k_0 < k \text{ and } 2^n - m \leqslant j < 2^n. \end{cases}$$

The total number of join irreducible elements for this case is given by

$$(\ddagger) \quad 2^n \sum_{j=1}^{m} \binom{2^n}{j} - \sum_{k=1}^{k_0} \binom{n}{k} (2^{2^n-2^{n-k}} - 1) - \sum_{k=k_0+1}^{n} \binom{n}{k} \sum_{j=1}^{m} \binom{2^n - 2^{n-k}}{j}.$$

For instance, if $n = 2$ and $m = 3$, as $3 > 2^{2-1}$, apply $(\ddagger)$ with $k_0 = 1$ to get 43. If $n = 2$ and $m = 2$ then apply $(\dagger)$ to get 28.

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