Vítězslav Novák
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TERNARY STRUCTURES AND PARTIAL SEMIGROUPS

VITEZSLAV NOVÁK, Brno

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Transitive ternary structures and, especially, cyclically ordered sets can be transformed into other structures: into quasi-ordered sets ([3]), double binary structures ([4]), E-systems ([5]) etc. In this paper we describe a relation between transitive ternary structures and partial semigroups.

1. C-SEMIGROUPS

1.1. Let \( G \neq \emptyset \) be a set, let \( \cdot \) be a partial binary operation on \( G \) which has the following property:

let \( x, y, z \in G \); if one of products \( (x \cdot y) \cdot z, x \cdot (y \cdot z) \) or both products \( x \cdot y, y \cdot z \) are defined then both products \( (x \cdot y) \cdot z, x \cdot (y \cdot z) \) are defined and \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \).

Then the structure \( G = (G, \cdot) \) is called a partial semigroup.

1.2. A homomorphism of partial semigroups is defined in the obvious way. Thus, if \( G = (G, \cdot) \), \( H = (H, \cdot) \) are partial semigroups and \( f : G \to H \), then \( f \) is a homomorphism of \( G \) into \( H \) if

\[
\begin{align*}
  x, y \in G \text{ and } x \cdot y \text{ is defined} & \implies f(x) \cdot f(y) \text{ is defined in } H \text{ and } f(x \cdot y) = \\
  = f(x) \cdot f(y)
\end{align*}
\]

A bijective homomorphism of \( G \) onto \( H \) such that \( f^{-1} \) is a homomorphism of \( H \) onto \( G \) is an isomorphism of \( G \) onto \( H \); \( G \) and \( H \) are isomorphic if there exists an isomorphism of \( G \) onto \( H \).

Let us note that a bijective homomorphism \( f \) of \( G \) onto \( H \) is an isomorphism iff \( x, y \in G \), \( f(x) \cdot f(y) \) is defined \( \implies x \cdot y \) is defined.

1.3. Let \( G = (G, \cdot) \) be a partial semigroup, \( e \in G \). The element \( e \) is a unit in \( G \) if the following is satisfied:
if \(e \cdot x\) is defined for some \(x \in G\) then \(e \cdot x = x\), if \(y \cdot e\) is defined for some \(y \in G\) then \(y \cdot e = y\).

Let us denote by \(E(G)\) the set of all units of a partial semigroup \(G\). In the sequel we shall deal with partial semigroups \(G = (G, \cdot)\) with the following property:

(*) for any \(x \in G\) there are units \(e, e' \in E(G)\) such that \(e \cdot x\) is defined and \(x \cdot e'\) is defined.

We shall need some trivial and well known properties of partial semigroups: we present them with proofs as the proofs are very simple.

1.4. Lemma. Let \(G = (G, \cdot)\) be a partial semigroup satisfying (*). Then for any \(x \in G\) there exists just one unit \(e \in E(G)\) such that \(e \cdot x\) is defined and there exists just one unit \(e' \in E(G)\) such that \(x \cdot e'\) is defined.

Proof. Let \(e_1, e_2 \in E(G)\) and \(e_1 \cdot x, e_2 \cdot x\) be defined. Then \(e_2 \cdot x = x\) so that \(e_1 \cdot (e_2 \cdot x)\) is defined. Hence \((e_1 \cdot e_2) \cdot x\), thus \(e_1 \cdot e_2\) is defined and then \(e_1 \cdot e_2 = e_1 = e_2\).

Similarly the second assertion. \(\square\)

1.5. Let \(G = (G, \cdot)\) be a partial semigroup satisfying (*) and \(x \in G\). We denote by \(e_L(x)\) the unit \(e \in E(G)\) for which \(e \cdot x\) is defined and by \(e_R(x)\) the unit \(e' \in E(G)\) for which \(x \cdot e'\) is defined. \(e_L(x)\) will be called the left unit of \(x\), \(e_R(x)\) the right unit of \(x\).

Thus \(e_L, e_R\) are mappings \(G \rightarrow E(G)\).

1.6. Lemma. Let \(G\) be a partial semigroup satisfying (*) and \(e \in E(G)\). Then \(e_L(e) = e_R(e) = e\).

Proof. We have \(e_L(e) \cdot e = e = e_L(e)\) and similarly \(e = e_R(e)\). \(\square\)

1.7. Lemma. Let \(G = (G, \cdot)\) be a partial semigroup satisfying (*), let \(x, y \in G\) and let \(x \cdot y\) be defined. Then \(e_L(x \cdot y) = e_L(x), e_R(x \cdot y) = e_R(y)\).

Proof. Denote \(e_L(x \cdot y) = e\). As \(e \cdot (x \cdot y)\) is defined, \((e \cdot x) \cdot y\) and therefore \(e \cdot x\) is defined. Then \(e = e_L(x)\). Similarly for the right unit. \(\square\)

1.8. Lemma. Let \(G = (G, \cdot)\) be a partial semigroup satisfying (*) and \(x, y \in G\). Then \(x \cdot y\) is defined iff \(e_R(x) = e_L(y)\).

Proof. If \(x \cdot y\) is defined then \((x \cdot e_R(x)) \cdot y\) is defined, thus \(x \cdot (e_R(x) \cdot y)\) and also \(e_R(x) \cdot y\) is defined which implies \(e_R(x) = e_L(y)\). Conversely, let \(e_R(x) = e_L(y) = e\). Then both \(x \cdot e\) and \(e \cdot y\) are defined, thus \((x \cdot e) \cdot y = x \cdot y\) is defined. \(\square\)
We shall study partial semigroups $G = (G, \cdot)$ satisfying $(\ast)$ with the further property:
$(\ast\ast)$ the pair of mappings $\{e_L, e_R\}$ distinguishes elements of $G$, i.e.
\[ x, y \in G, \ e_L(x) = e_L(y), \ e_R(x) = e_R(y) \implies x = y. \]
Partial semigroups in which $(\ast), (\ast\ast)$ hold will be called c-semigroups.

2. Ternary structures

2.1. Let $G \neq \emptyset$ be a set, let $t$ be a ternary relation on $G$. The pair $G = (G, t)$ will be called a ternary structure. A ternary relation $t$ on $G$ (and the structure $(G, t)$) is called transitive if
\[ x, y, z, u \in G, \ (x, y, z) \in t, \ (z, y, u) \in t \implies (x, y, u) \in t. \]

Let $(G, t)$ be a ternary structure and $x \in G$. We say that $x$ is an isolated element if neither $(x, y, z) \in t$ nor $(y, x, z) \in t$ nor $(z, x, y) \in t$ for any $y, z \in G$.

2.2. Let $G = (G, t), H = (H, t')$ be ternary structures and $f : G \to H$. $f$ is a homomorphism of $G$ into $H$ if
\[ x, y, z \in G, \ (x, y, z) \in t \implies (f(x), f(y), f(z)) \in t'. \]

A homomorphism $f$ of $G$ into $H$ is strong if it is surjective and
\[ u, v, w \in H, \ (u, v, w) \in t' \implies \text{there exist} \ x \in f^{-1}(u), \ y \in f^{-1}(v), \ z \in f^{-1}(w) \]
with $(x, y, z) \in t$.

A bijective strong homomorphism of $G$ onto $H$ is an isomorphism. Ternary structures $G, H$ are isomorphic if there is an isomorphism of $G$ onto $H$.

2.3. Let $(G, t)$ be a ternary structure. We put
\[ r(t) = \{(x, y, x) \in G^3; \ \text{there is} \ z \in G \ \text{with} \ (x, y, z) \in t \text{ or} \ (z, y, x) \in t \} \]
and denote $c(t) = t \cup r(t)$

2.4. Lemma. Let $(G, t)$ be a transitive ternary structure. Then the structure $(G, c(t))$ is transitive, as well.

Proof. Let $(x, y, z) \in c(t), (z, y, u) \in c(t)$. If $(x, y, z) \in t$, $(z, y, u) \in t$ then $(x, y, u) \in t \subset c(t)$. If $(x, y, z) \in c(t) - t$ then $z = x$ and thus $(x, y, u) \in c(t)$. Similarly in the case $(z, y, u) \in c(t) - t$. Hence $c(t)$ is a transitive relation. \[\square\]
2.5. Let \((G,t)\) be a transitive ternary structure. We define a partial binary operation \(\cdot\) on the set \(c(t)\) as follows:

for \(m_1 = (x_1, y_1, z_1) \in c(t)\), \(m_2 = (x_2, y_2, z_2) \in c(t)\) the product \(m_1 \cdot m_2\) is defined iff \(x_2 = z_1\), \(y_2 = y_1\); in that case \(m_1 \cdot m_2 = (x_1, y_1, z_2)\).

In other words, we put 

\[(x, y, z) \cdot (z, y, u) = (x, y, u).\]

2.6. **Theorem.** Let \((G,t)\) be a transitive ternary structure. Then \(G = (c(t), \cdot)\) is a c-semigroup in which \(E(G) = r(t)\) and \(e_L(m) = (x, y, x)\), \(e_R(m) = (z, y, z)\) for any \(m = (x, y, z) \in c(t)\).

**Proof.** Let \(m_1, m_2, m_3 \in c(t)\) and suppose that \((m_1 \cdot m_2) \cdot m_3\) is defined. Then \(m_1 = (x, y, z), m_2 = (z, y, u)\) for suitable \(x, y, z, u \in G\) and \(m_1 \cdot m_2 = (x, y, u)\). Thus \(m_3 = (u, y, v)\) for a suitable \(v \in G\) so that \((m_1 \cdot m_2) \cdot m_3 = (x, y, v)\). We see that \(m_2 \cdot m_3\) is defined and \(m_2 \cdot m_3 = (z, y, v)\) so that \((m_1 \cdot m_2) \cdot m_3\) is defined and \(m_1 \cdot (m_2 \cdot m_3) = (x, y, v) = (m_1 \cdot m_2) \cdot m_3\). Similarly in the case when \(m_1 \cdot (m_2 \cdot m_3)\) is defined. Let both \(m_1 \cdot m_2\) and \(m_2 \cdot m_3\) be defined. Then \(m_1 = (x, y, z), m_2 = (z, y, u), m_3 = (u, y, v)\); thus \(m_1 \cdot m_2 = (x, y, u)\) and \((m_1 \cdot m_2) \cdot m_3\) is defined. Hence \((c(t), \cdot)\) is a partial semigroup. If \(e \in r(t)\) then \(e = (x, y, x)\) so that if \(e \cdot m\) is defined for some \(m \in c(t)\) then \(m = (x, y, z)\) and \(e \cdot m = (x, y, z) = m\). Similarly if \(m \cdot e\) is defined for some \(m \in c(t)\). Thus \(e \in E(G)\) and \(r(t) \subset E(G)\).

Let \(m = (x, y, z) \in c(t)\). Then \(e = (x, y, x) \in r(t)\), thus \(e \in E(G)\) and \(e \cdot m = (x, y, x) \cdot (x, y, z) = (x, y, z) = m\). We see that \(e = e_L(m)\); similarly \(e' = (z, y, z) = e_R(m)\). Thus the partial semigroup \(G = (c(t), \cdot)\) satisfies (*) and \(e_L(m) = (x, y, x), e_R(m) = (z, y, z)\) for any \(m = (x, y, z) \in c(t)\).

We show \(E(G) = r(t)\). If \(e \in E(G)\) then \(e_L(e) = e\) by 1.6 so that \(e \cdot e\) is defined and \(e \cdot e = e\). If \(e = (x, y, z)\) then necessarily \(e = (z, y, u)\) so that \(z = x\) and \(e = (x, y, x) \in r(t)\). Thus \(E(G) \subset r(t)\), which implies \(E(G) = r(t)\).

Let \(m_1 = (x_1, y_1, z_1) \in c(t), m_2 = (x_2, y_2, z_2) \in c(t)\) and \(e_L(m_1) = e_L(m_2)\), \(e_R(m_1) = e_R(m_2)\). Then \((x_1, y_1, x_1) = (x_2, y_2, x_2)\) so that \(x_1 = x_2, y_1 = y_2\) and \((z_1, y_1, z_1) = (z_2, y_2, z_2)\) so that \(z_1 = z_2\). Hence \(m_1 = m_2\) and the pair of mappings \(\{e_L, e_R\}\) distinguishes elements of \(c(t)\), i.e. \((c(t), \cdot)\) is a c-semigroup. \(\square\)
3. MAPPINGS S AND T

3.1. Let $G = (G, t)$ be a transitive ternary structure. Denote by $S(G) = (c(t), \cdot)$ the $c$-semigroup constructed in 2.5. If $T$ is the class of all ternary structures and $C$ is the class of all $c$-semigroups then $S$ is a mapping of $T$ into $C$:

$$S: T \to C.$$ 

3.2. Let $M = (M, \cdot)$ be a $c$-semigroup. Let us define a binary relation $\varrho(M)$ on the set $E(M)$ as follows:

$$(e, e') \in \varrho(M) \iff \text{there is } m \in M \text{ with } e = e_L(m), \ e' = e_R(m).$$

3.3. **Lemma.** Let $M = (M, \cdot)$ be a $c$-semigroup. Then the relation $\varrho(M)$ on $E(M)$ is reflexive and transitive.

**Proof.** If $e \in E(M)$ then $e_L(e) = e_R(e) = e$ by 1.6 and $(e, e) \in \varrho(M)$ by definition. Let $e_1, e_2, e_3 \in E(M)$, $(e_1, e_2) \in \varrho(M)$, $(e_2, e_3) \in \varrho(M)$. Then there exist $m, n \in M$ with $e_1 = e_L(m), e_2 = e_R(m), e_2 = e_L(n), e_3 = e_R(n)$. By 1.8 the product $m \cdot n$ is defined and by 1.7 $e_L(m \cdot n) = e_L(m) = e_1, e_R(m \cdot n) = e_R(n) = e_3$. Thus $(e_1, e_3) \in \varrho(M)$. \qed

3.4. The relation $\varrho(M)$ on $E(M)$ need not be symmetric so that it is not an equivalence relation in general. Let $\Theta(M)$ be the equivalence relation on $E(M)$ generated by $\varrho(M)$. Thus $(e, e') \in \Theta(M)$ iff there exist a positive integer $n$ and elements $e_1, \ldots, e_n \in E(M)$ such that $e_1 = e, e_n = e'$ and $(e_i, e_{i+1}) \in \varrho(M) \cup \varrho(M)^{-1}$ for all $i = 1, \ldots, n - 1$.

3.5. Let $M = (M, \cdot)$ be a $c$-semigroup, $\varrho(M)$ the binary relation on $E(M)$ defined in 3.2 and $\Theta(M)$ the equivalence relation on $E(M)$ generated by $\varrho(M)$. Put

$$G = E(M) \cup E(M)|_{\Theta(M)}$$

and define a ternary relation $t$ on $G$:

$$(x, y, z) \in t \iff x, z \in E(M), y \in E(M)|_{\Theta(M)}, (x, z) \in \varrho(M) \text{ and } x, z \in y.$$ 

We denote by $T(M)$ the ternary structure $(G, t)$.

3.6. **Theorem.** Let $M = (M, \cdot)$ be a $c$-semigroup. Then $T(M) = (G, t)$ is a transitive ternary structure in which $t = c(t)$. 

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Proof. Let \( x, y, z, u \in G \), \((x, y, z) \in t\), \((z, y, u) \in t\). Then \( x, z, u \in E(M) \), \( y \in E(M) \mid _{\Theta(M)} \), \((x, z) \in \varrho(M)\), \(x, z \in y\) and \((z, u) \in \varrho(M)\), \(z, u \in y\). By 3.3 \((x, u) \in \varrho(M)\) and \(x, u \in y\). Thus \((x, y, u) \in t\) and \(t\) is transitive. Let \(x, y, z \in G\), \((x, y, z) \in t\) so that \(x, z \in E(M)\), \(y \in E(M) \mid _{\Theta(M)} \), \((x, z) \in \varrho(M)\), \(x, z \in y\). By 3.3 \((x, x) \in \varrho(M)\) and thus \((x, y, x) \in t\); similarly \((z, y, z) \in t\). Hence \(c(t) = t\).

3.6 implies that \(T\) is a mapping of \(C\) into \(T\), i.e.

\[ T : C \to T. \]

3.7. Theorem. Let \(M = (M, \cdot)\) be a \(c\)-semigroup. Then \(M\) is isomorphic to \((S \circ T)(M)\).

Proof. Denote \(T(M) = (G, t)\) where \(G = E(M) \cup E(M) \mid _{\Theta(M)}\) and \((S \circ T)(M) = S(G, t) = (c(t), \cdot)\). By 3.6 we have \(c(t) = t\). Let us define a mapping \(f : M \to c(t)\): \(m \in M \implies f(m) = (e_L(m), y, e_R(m))\) where \(y \in E(M) \mid _{\Theta(M)}\) is such an element that \(e_L(m) \in y\), \(e_R(m) \in y\). By the definition of the relation \(t\) we have \(f(m) \in t = c(t)\) so that \(f\) is really a mapping of \(M\) into \(c(t)\). Let \((x, y, z) \in c(t)\). Then \(x, z \in E(M)\), \(y \in E(M) \mid _{\Theta(M)} \), \((x, z) \in \varrho(M)\), \(x, z \in y\) and \((x, z) \in \varrho(M)\), which means that there exists \(m \in M\) with \(x = e_L(m)\), \(z = e_R(m)\). Then by definition \((x, y, z) = f(m)\) and the mapping \(f\) is surjective.

Let \(m, n \in M\) and \(f(m) = f(n)\). Then \((e_L(m), y, e_R(m)) = (e_L(n), y, e_R(n))\) where \(e_L(m) \in y\), \(e_L(n) \in z\), thus \(e_L(m) = e_L(n)\), \(e_R(m) = e_R(n)\). Hence \(m = n\), \(M\) being a \(c\)-semigroup. Thus \(f : M \to c(t)\) is injective and also bijective.

Let \(m, n \in M\) and let \(m \cdot n\) be defined. By definition \(f(m) = (e_L(m), y, e_R(m))\) where \(e_L(m) \in y\), \(e_L(n) \in z\) and \(f(n) = (e_L(n), y, e_R(n))\) where \(e_L(n) \in y\), \(e_R(n) \in z\). As \(m \cdot n\) is defined, by 1.8 we have \(e_R(m) = e_L(n)\). This implies \(y = z\) so that \(f(n) = (e_L(m), y, e_R(n))\). Hence the product \(f(m) \cdot f(n)\) is defined in \((c(t), \cdot)\) and \(f(m) \cdot f(n) = (e_L(m), y, e_R(n))\). By 1.7 we have \(e_L(m \cdot n) = e_L(m)\), \(e_R(m \cdot n) = e_R(n)\) and further \(e_L(m \cdot n) = e_L(m) \in y\), \(e_R(m \cdot n) = e_R(n) \in z = y\). Thus \(f(m \cdot n) = (e_L(m \cdot n), y, e_R(m \cdot n)) = (e_L(m), y, e_R(n)) = f(m) \cdot f(n)\) and \(f\) is a homomorphism of \(M\) onto \((S \circ T)(M)\).

Let \(m, n \in M\) and let the product \(f(m) \cdot f(n)\) be defined in \((S \circ T)(M) = (c(t), \cdot)\). As \(f(m) = (e_L(m), y, e_R(m))\) with \(e_L(m), e_R(m) \in y\), \(f(n) = (e_L(n), z, e_R(n))\) with \(e_L(n), e_R(n) \in z\), we necessarily have \(y = z\), \(e_R(m) = e_L(n)\). By 1.8 we see that \(m \cdot n\) is defined in \(M\) and thus \(f : M \to c(t)\) is an isomorphism of \(M\) onto \((S \circ T)(M)\).

3.8. Theorem. Let \(G = (G, t)\) be a transitive ternary structure without isolated elements and such that \(c(t) = t\). Then there exists a strong homomorphism of the structure \((T \circ S)(G)\) onto the structure \(G\).
Proof. By definition we have $S(G) = (c(t), \cdot) = (t, \cdot)$; let us denote by $M$ this $c$-semigroup. Then $(T \circ S)(G) = T(M) = (E(M) \cup E(M)|_{\Theta(M)},')$ where $(u, v, w) \in t' \iff u, w \in E(M), v \in E(M)|_{\Theta(M)}$, $u, w \in v$ and there exists $m \in t$ with $u = e_L(m), \ w = e_R(m)$. If $m = (x, y, z)$ then by 2.6 we have $e_L(m) = u = (x, y, x), e_R(m) = w = (z, y, z)$. Let us define a mapping $f: E(M) \cup E(M)|_{\Theta(M)} \rightarrow G$

then $f(u) = y$.

We must show that the definition of $f$ is correct, i.e. the following implication holds:

if $u \in E(M)|_{\Theta(M)}$, $(x_1, y_1, x_1) \in u, (x_2, y_2, x_2) \in u$ then $y_1 = y_2$.

Assume $(x_1, y_1, x_1) \in u, (x_2, y_2, x_2) \in u$. Then either $(x_1, y_1, x_1) = (x_2, y_2, x_2)$ which implies $y_1 = y_2$ or there exists a finite sequence $(p_1, q_1, p_1), (p_2, q_2, p_2), \ldots, (p_n, q_n, p_n)$ of elements in $E(M)$ such that $(p_1, q_1, p_1) = (x_1, y_1, x_1), (p_n, q_n, p_n) = (x_2, y_2, x_2)$ and $((p_i, q_i, p_i), (p_{i+1}, q_{i+1}, p_{i+1})) \in \mathcal{E}(M) \cup \mathcal{E}(M)^{-1}$ for $i = 1, \ldots, n - 1$. It suffices to show that in this case $q_i = q_{i+1}$ for $i = 1, \ldots, n - 1$. If $((p_i, q_i, p_i), (p_{i+1}, q_{i+1}, p_{i+1})) \in \mathcal{E}(M)$ then there exists $m = (p, q, r) \in t$ with $(p_i, q_i, p_i) = e_L(m), (p_{i+1}, q_{i+1}, p_{i+1}) = e_R(m)$. Then by 2.6 $(p_i, q_i, p_i) = (p, q, p), (p_{i+1}, q_{i+1}, p_{i+1}) = (r, q, r)$ and $q_i = q = q_{i+1}$.

If $((p_i, q_i, p_i), (p_{i+1}, q_{i+1}, p_{i+1})) \in \mathcal{E}(M)^{-1}$ then $((p_{i+1}, q_{i+1}, p_{i+1}), (p_i, q_i, p_i)) \in \mathcal{E}(M)$ and $q_{i+1} = q_i$ as well. Thus $q_1 = \ldots = q_n$, i.e. $y_1 = y_2$.

Let $x \in G$. As $G$ has no isolated elements there are $y, z \in G$ such that $(x, y, z) \in t$ or $(z, y, x) \in t$ or $(y, x, z) \in t$. In the first and second cases we have $(x, y, x) \in r(t) \subseteq t$ and by 2.6 $(x, y, x) \in E(M)$. Then by definition $f(x, y, x) = x$. In the third case $(y, x, y) \in r(t) \subseteq t$ and $(y, x, y) \in E(M)$. If $u \in E(M)|_{\Theta(M)}$ is such an element that $(y, x, y) \in u$ then $f(u) = x$ by the definition of $f$. Thus $f: E(M) \cup E(M)|_{\Theta(M)} \rightarrow G$ is surjective.

Let $u, v, w \in E(M) \cup E(M)|_{\Theta(M)}, (u, v, w) \in t'$. Then $u, w \in E(M), v \in E(M)|_{\Theta(M)}$, $u, w \in v$ and there exists $m = (x, y, z) \in t$ such that $u = e_L(m), w = e_R(m)$. Thus $u = (x, y, x), w = (z, y, z)$ and $f(u) = x, f(w) = z, f(v) = y$ by definition of $f$. Hence $(f(u), f(v), f(w)) \in t$ and $f$ is a surjective homomorphism of $(T \circ S)(G)$ onto $G$.

Let $x, y, z \in G, (x, y, z) \in t$. Then $(x, y, x) \in t, (z, y, z) \in t$ and $(x, y, x) \in E(M), (z, y, z) \in E(M)$. If we denote $(x, y, z) = m, (x, y, x) = u, (z, y, z) = w$ and if $v \in E(M)|_{\Theta(M)}$ is such an element that $u \in v$ then $u = e_L(m), w = e_R(m)$ and $(u, w) \in \mathcal{E}(M), u, w \in v$. Then $(u, v, w) \in t'$ by the definition of $t'$ and at the same
time \( f(u) = x, f(v) = y, f(w) = z \). Hence the homomorphism \( f \) of \((T \circ S)(G)\) onto \( G \) is strong. \( \square \)

4. Examples

4.1 Let \( G = \{x, y, z, u\}, t = \{(x, y, z), (z, y, u), (x, y, u), (x, y, x), (z, y, z), (u, y, u)\}, G = (G, t) \). We construct \((T \circ S)(G)\).

Clearly \( c(t) = t \) and \( G \) contains no isolated elements. Let us denote \( m_1 = (x, y, z), m_2 = (z, y, u), m_3 = (x, y, u), e_1 = (x, y, x), e_2 = (z, y, z), e_3 = (u, y, u) \). By 2.5 and 2.6 in the c-semigroup \( S(G) = M \) we have:

\[
\begin{align*}
m_1 \cdot m_2 &= m_3, \\
e_1 &= e_L(m_1) = e_L(m_3), \\
e_2 &= e_R(m_1) = e_L(m_2), \\
e_3 &= e_R(m_2) = e_R(m_3).
\end{align*}
\]

Thus \( E(M) = \{e_1, e_2, e_3\} \) and by 3.2 \( (e_1, e_2) \in g(M), (e_2, e_3) \in g(M), (e_1, e_3) \in g(M) \) so that \( \Theta(M) = E(M)^2, E(M)|_{\Theta(M)} = \{\{e_1, e_2, e_3\}\} \) and \((T \circ S)(G) = (\{e_1, e_2, e_3, \{e_1, e_2, e_3\}\}, t')\), where by 3.5

\[
\begin{align*}
(e_1, \{e_1, e_2, e_3\}, e_2) &\in t', \\
(e_2, \{e_1, e_2, e_3\}, e_3) &\in t', \\
(e_1, \{e_1, e_2, e_3\}, e_3) &\in t', \\
(e_1, \{e_1, e_2, e_3\}, e_1) &\in t', \\
(e_2, \{e_1, e_2, e_3\}, e_2) &\in t', \\
(e_3, \{e_1, e_2, e_3\}, e_3) &\in t'.
\end{align*}
\]

The mapping \( f: E(M) \cup E(M)|_{\Theta(M)} \to G \) constructed in the proof of Theorem 3.8 is

\[
f(e_1) = x, \ f(e_2) = z, \ f(e_3) = u, \ f(\{e_1, e_2, e_3\}) = y
\]

and it is an isomorphism of \((T \circ S)(G)\) onto \( G \).

4.2. Let \( G = \{x, y, z\}, t = \{(x, y, z), (y, z, x), (z, x, y), (x, y, x), (z, y, z), (y, z, y), (x, z, x), (z, x, z), (y, x, y)\}, G = (G, t) \); we find \((T \circ S)(G)\).

As in 4.1, we have \( c(t) = t \) and \( G \) contains no isolated elements. Put \( m_1 = (x, y, z), m_2 = (y, z, x), m_3 = (z, x, y), e_1 = (x, y, x), e_2 = (z, y, z), e_3 = (y, z, y), e_4 = (x, z, x), e_5 = (z, x, z), e_6 = (y, x, y) \).
In the $c$-semigroup $S(G) = M$ we have
\[ e_1 = e_L(m_1), \ e_2 = e_R(m_1), \ e_3 = e_L(m_2), \ e_4 = e_R(m_2), \ e_5 = e_L(m_3), \ e_6 = e_R(m_3) \]

and the product in $M$ is defined only with the corresponding units. Further we have

\[ (e_1, e_2) \in g(M), \ (e_3, e_4) \in g(M), \ (e_5, e_6) \in g(M) \]

so that

\[ E(M)|_{\Theta(M)} = \{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\} \]

and

\[ (T \circ S)(G) = T(M) = \{e_1, e_2, e_3, e_4, e_5, e_6, \{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}\}, t' \]

where

\[ (e_1, \{e_1, e_2\}, e_2) \in t', \]
\[ (e_3, \{e_3, e_4\}, e_4) \in t', \]
\[ (e_5, \{e_5, e_6\}, e_6) \in t', \]
\[ (e_1, \{e_1, e_2\}, e_1) \in t', \]
\[ (e_2, \{e_1, e_2\}, e_2) \in t', \]
\[ (e_3, \{e_3, e_4\}, e_3) \in t', \]
\[ (e_4, \{e_3, e_4\}, e_4) \in t', \]
\[ (e_5, \{e_5, e_6\}, e_5) \in t', \]
\[ (e_6, \{e_5, e_6\}, e_6) \in t'. \]

As $G$ has three elements and the carrier of the structure $(T \circ S)(G)$ has nine elements, the structures $G$ and $(T \circ S)(G)$ cannot be isomorphic. The strong homomorphism $f$ of $(T \circ S)(G)$ onto $G$ constructed in the proof of Theorem 3.8 has the form

\[ f(e_1) = x, \ f(e_2) = z, \ f(e_3) = y, \ f(e_4) = x, \ f(e_5) = z, \ f(e_6) = y, \]
\[ f(\{e_1, e_2\}) = y, \ f(\{e_3, e_4\}) = z, \ f(\{e_5, e_6\}) = x. \]

4.3. **Problem.** Find necessary and sufficient conditions for a transitive ternary structure $G = (G, t)$ to be isomorphic to $(T \circ S)(G)$. 

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References


Author’s address: Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic.