

Jiří Rachůnek

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ON SOME VARIETIES OF WEAKLY ASSOCIATIVE
LATTICE GROUPS

JIŘÍ RACHŮNEK, Olomouc

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1. BASIC NOTIONS

A *weakly associative lattice* (*wa-lattice*) is an algebra $A = (A, \vee, \wedge)$ with two binary operations satisfying the identities

$$\begin{array}{ll} \text{(I)} & a \vee a = a; & a \wedge a = a. \\ \text{(C)} & a \vee b = b \vee a; & a \wedge b = b \wedge a. \\ \text{(Abs)} & a \vee (a \wedge b) = a; & a \wedge (a \vee b) = a. \\ \text{(WA)} & ((a \wedge c) \vee (b \wedge c)) \vee c = c; & ((a \vee c) \wedge (b \vee c)) \wedge c = 0. \end{array}$$

This notion has been introduced by E. Fried in [3] and by H. L. Skala in [11] and [12]. The notion of a *wa-lattice* is then a generalization of that of a lattice because the identities of associativity of the operations \vee and \wedge are replaced by weaker conditions of weak associativity (WA). Nevertheless, similarly as for lattices, we can define also for *wa-lattices* a binary relation \leq on A such that

$$\forall a, b \in A; a \leq b \iff_{df} a \wedge b = a.$$

This relation is reflexive and antisymmetric and every two-element subset $\{a, b\} \subseteq A$ has the join $\sup\{a, b\} = a \vee b$ and the meet $\inf\{a, b\} = a \wedge b$ in A . Moreover (also as for lattices), each such binary relation defines on A a structure of a *wa-lattice*. (So, we can equivalently view any *wa-lattice* as a set with a binary relation.)

A *tournament*, i.e. a set $A \neq \emptyset$ with a reflexive and antisymmetric binary relation \leq such that

$$\forall a, b \in A; a \leq b \text{ or } b \leq a,$$

is a special case of a *wa-lattice*.

If $(G, +)$ is a group and $(G, \vee, \wedge) = (G, \leq)$ is a *wa*-lattice and if for any $a, b, c, d \in A$

$$(D_{\vee}) \quad a + (b \vee c) + d = (a + b + d) \vee (a + c + d),$$

then the system $G = (G, +, \vee, \wedge)$ is called a *weakly associative lattice group* (*wal-group*). (See [8], [9], [10]. In [12] a *wal-group* is called a *trellis-group*.)

It is evident that in a *wal-group* the conditions

$$(D_{\wedge}) \quad a + (b \wedge c) + d = (a + b + d) \wedge (a + c + d),$$

$$(M) \quad a \leq b \implies c + a + d \leq c + b + d$$

are satisfied for any $a, b, c, d \in A$ and that every of these conditions is equivalent to (D_{\vee}) .

If for a *wal-group* G the *wa*-lattice (G, \leq) is a tournament, then G is called a *totally semi-ordered group* (a *to-group*).

In contrast to the situation for lattice ordered groups (*l-groups*) and linearly ordered groups (*o-groups*) that are torsion free, there are many non-trivial finite *wal-groups* and *to-groups*.

2. THE LATTICE OF *wal*-IDEALS

The kernels of homomorphisms of *wal-groups* (i.e. *wal-homomorphisms*) will be called *wal-ideals*. The *wal-ideals* are special cases of *wal-subgroups*, that means of such subgroups which are both subgroups and *wa*-sublattices.

More precisely:

Proposition 1. ([8, Theorems 9 and 11], [10, Lemma 2.1].) *For a normal convex wal-subgroup H of a wal-group G the following conditions are equivalent:*

(a) H is a *wal-ideal* of G .

(b) $\forall a, b \in H, x, y \in G (x \leq a, y \leq b \implies \exists c \in H. x \vee y \leq c)$.

(c) $\forall a, b, c \in H, x, y \in G; x \leq a, y \leq b \implies (x \vee y) \vee c \in H$.

Denote by $\mathcal{L}(G)$ the set of *wal-ideals* of a *wal-group* G . It is evident that $\mathcal{L}(G)$ ordered by set inclusion forms a complete lattice with the least element $\{0\}$ and the greatest element G .

Proposition 2. *If G is a wal-group, then $\mathcal{L}(G)$ is a complete sublattice of the lattice of subgroups of the group G .*

Proof. It is obvious that the intersection of any system of *wal-ideals* of G is also a *wal-ideal* of G . Moreover, *wal-groups* are Ω -groups in the sense of Kurosch.

hence by [6, III.2.4], the *wal*-ideal generated by a system of *wal*-ideals of a *wal*-group G coincides with the subgroup of the additive group of G generated by these ideals as subgroups. \square

Let us show that the lattice $\mathcal{L}(G)$ is distributive. For this, we will use known properties of varieties of algebras. The class of all *wal*-groups is by definition a variety of algebras of type $\langle +, 0, -(\cdot), \vee, \wedge \rangle$ of signature $\langle 2, 0, 1, 2, 2 \rangle$. Recall that a variety of algebras is called *arithmetical* if it is congruence distributive and permutable. (See [2].)

Theorem 3. *The variety of all wal-groups is arithmetical.*

Proof. By [2, Theorem II.12.5], the variety \mathcal{V} is arithmetical if and only if there is a ternary Mal'cev term $m(x, y, z)$ such that

$$m(x, y, x) = m(x, y, y) = m(y, y, x) = x$$

in \mathcal{V} .

For the variety of *wal*-groups we can use the term

$$m(x, y, z) = x - (((x \vee y) \wedge (x \vee z)) \wedge (y \vee z)) + z,$$

which is in the case of *l*-groups equivalent to the term used in [4, p. 231]. \square

We get, as a direct corollary, the following theorem.

Theorem 4. *The lattice of wal-ideals of any wal-group is distributive.*

3. THE LATTICE OF VARIETIES OF *wal*-GROUPS

It is well known (see e.g. [4], [5], [7]) that the varieties of *l*-groups (considered in the language $\mathcal{L} = (+, 0, -(\cdot), \vee, \wedge)$) form a complete dually Brouwerian lattice \mathbf{L} in which the variety of abelian *l*-groups \mathcal{Ab}_l is the least non-zero element. The variety of representable *l*-groups \mathcal{R}_l is another important element of \mathbf{L} because it is the variety generated by all linearly ordered groups. (The elements of \mathcal{R}_l are precisely all subdirect sums of *o*-groups.) Recall that \mathcal{R}_l is characterized by any of the following identities:

- (1) $(x \wedge (-y - x + y)) \vee 0 = 0,$
- (2) $2(x \wedge y) = 2x \wedge 2y.$

It is clear that the varieties of *wal*-groups considered also in the language \mathcal{L} form a complete lattice **WAL**, too.

Theorem 5. *The lattice **WAL** is distributive and contains the lattice **L** as a complete \wedge -subsemilattice.*

Proof. In general, if \mathcal{V} is an arbitrary variety of algebras, X is an infinite countable set, and F_X is the free algebra on X in the variety \mathcal{V} , then the lattice of subvarieties of \mathcal{V} is anti-isomorphic to the lattice $FI(F_X)$ of fully invariant congruences on F_X . Since, by Theorem 3, the lattice $\text{Con}(F_X)$ of all congruences on F_X is distributive and since the fully invariant congruences form a (complete) sublattice of $\text{Con}(F_X)$, the lattice **WAL** is distributive.

Further, by [9, Proposition 1.10] a *wal*-group G is an l -group if and only if the identity

$$(L) \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

is satisfied in G , therefore $\mathbf{L} \subseteq \mathbf{WAL}$. □

Denote by Ab_{wal} the variety of abelian *wal*-groups and by \mathcal{R}_{wal} the class of representable *wal*-groups (i.e. the class of *wal*-groups that are subdirect sums of *to*-groups). Recall that in the case of l -groups the representable l -groups form the variety \mathcal{R}_l and that $\text{Ab}_l \subset \mathcal{R}_l$. Hence there are questions what are the relations between the classes Ab_{wal} and \mathcal{R}_{wal} and whether \mathcal{R}_{wal} forms a variety of *wal*-groups.

To answer the first question, we will recall some notions and results of [9]. A convex *wal*-subgroup H of a *wal*-group G is called *solid* if it satisfies the condition (c) from Proposition 1. If H is a convex *wal*-subgroup, then we can define the structure of a *wa*-lattice on the set $G/_l H$ of left cosets of G by H by

$$x + H \leq y + H \iff_{df} \exists a \in H; x + a \leq y.$$

A solid subgroup H is called *straightening*, if it satisfies the following mutually equivalent conditions:

- (a) $x, y \in G, 0 \leq x \wedge y \in H \implies x \in H \text{ or } y \in H$.
- (b) $x, y \in G, x \wedge y = 0 \implies x \in H \text{ or } y \in H$.
- (c) $G/_l H$ is a tournament.

Note that for *wal*-groups, the notion of a straightening subgroup is not equivalent, in contrast to l -groups, to the notion of a prime subgroup, i.e. finitely irreducible element of the lattice of solid subgroups of G . (See [9, Remark 2.2].)

By ([9, Theorem 2.6]) we have that a *wal*-group is representable if and only if the intersection of all its straightening *wal*-ideals is equal to $\{0\}$.

Theorem 6. *The classes $\mathcal{A}b_{wal}$ and \mathcal{R}_{wal} are non-comparable.*

Proof. It is obvious that if G is an l -group, then $G \in \mathcal{R}_l$ if and only if $G \in \mathcal{R}_{wal}$, hence $\mathcal{R}_{wal} \not\subseteq \mathcal{A}b_{wal}$.

Conversely, consider the abelian wal -group $G = (\mathbb{Z}, +, \leq)$, with the positive cone $G^+ = \{x \in G; 0 \leq x\} = \{0, 1, 2, 4, \dots, 2n, \dots\}$. Since G has no straightening subgroup different from G , we have by [9, Theorem 2.6] that $G \notin \mathcal{R}_{wal}$, thus $\mathcal{A}b_{wal} \not\subseteq \mathcal{R}_{wal}$. \square

However, neither of the identities (1) and (2) characterizing \mathcal{R}_l in \mathbf{L} gives an answer to the question whether \mathcal{R}_{wal} is a variety of wal -groups.

For instance, let $G = (\mathbb{Z}_3, +, \leq)$, where $G^+ = \{0, 1\}$. Then

$$\begin{aligned} (2 \wedge -2) \vee 0 &= 1 \neq 0, \\ 2(1 \wedge 2) &= 2 \neq 1 = 2 \cdot 1 \wedge 2 \cdot 2, \end{aligned}$$

hence G satisfies neither (1) nor (2). But G is a to -group, therefore $G \in \mathcal{R}_{wal}$.

Nevertheless, we have

Proposition 7. *The class \mathcal{R}_{wal} is a variety of wal -groups.*

Proof. We will use Birkhoff's characterization of varieties as classes of algebras of a given type closed under products, subalgebras and homomorphic images. For this, if H is a subgroup of a wal -group G , then $H \leq G$ will mean that H is a wal -subgroup of G , and $H \trianglelefteq G$ will mean that H is a normal wal -subgroup of G .

a) Obviously, the product (i.e. the cardinal sum) of representable wal -groups is a representable wal -group, too.

b) Let G be a subdirect sum of to -groups G_i , $i \in I$, and let $H \leq G$. Let us consider any straightening wal -ideal S_j of G and denote $H_j = H \cap S_j$. It is evident that $H_j \trianglelefteq H$. Let $a, b \in H_j$, $x \in H$, $a \leq x$, $x \leq b$ (in short: $a \leq x \leq b$, although $a \leq b$ need not be true). Because $a, b \in S_j$, we have $x \in S_j \cap H = H_j$. Hence H_j is convex.

Let $a, b, c \in H_j$, $x, y \in H$, $x \leq a$, $y \leq b$. Then $(x \vee y) \vee c \in S_j \cap H = H_j$, and thus H_j satisfies condition (c) of Proposition 1. That means H_j is a wal -ideal of H .

Let $x, y \in H$, $x \wedge y = 0$. Then $x \in S_j$ or $y \in S_j$, hence $x \in H_j$ or $y \in H_j$. Therefore H_j is straightening.

Now, let $(S_j; j \in J)$ be the system of all straightening wal -ideals of G . Then

$$\bigcap_{j \in J} H_j = \bigcap_{j \in J} (H \cap S_j) \subseteq \bigcap_{j \in J} S_j = \{0\},$$

and so, by [9, Theorem 2.6], H is a representable wal -group.

c) Let f be a *wal*-homomorphism of a *wal*-group G onto a *wal*-group G' , let H_i be a straightening *wal*-ideal of G , and let $H'_i = f(H_i)$. Since *wal*-groups are Ω -groups, H'_i is, by [6, III.2.12], a *wal*-ideal of G' .

Consider $x' + H'_i, y' + H'_i \in G'/H'_i$. Let $x, y \in G, f(x) = x', f(y) = y'$. We can suppose that $x + H_i \leq y + H_i$. Then there is $a \in H_i$ such that $x + a \leq y$, and hence $x' + f(a) \leq y'$. Because $f(a) \in H'_i$, we have $x' + H'_i \leq y' + H'_i$, therefore H'_i is straightening.

Suppose that G is representable and that $(H_i, i \in I)$ is the system of all straightening *wal*-ideals of G . If there is $j \in I$ such that $f(H_j) = \{0'\}$, then $\{0'\}$ is a straightening *wal*-ideal of G' , hence G' is a *to*-group. Let $H'_i = f(H_i) \neq \{0'\}$ for each $i \in I$. Because f induces a bijection of the set of *wal*-ideals of G which are not contained in $\text{Ker } f$ onto the set of all *wal*-ideals of G' , and because the *wal*-lattices G/H_i and $G'/f(H_i)$ are isomorphic, f induces also a bijection of the set of straightening *wal*-ideals of G onto the set of straightening *wal*-ideals of G' . If $H' = \bigcap_{i \in I} H'_i \neq \{0'\}$, then $H = f^{-1}(H')$ is a *wal*-ideal of G which is contained in all straightening *wal*-ideals of G , and thus $H = \{0\}$, a contradiction. Therefore $H' = \{0\}$, which means G' is representable. \square

Let us return to the identities (1) and (2) which characterize the variety of representable l -groups in \mathbf{L} . We have proved that there are representable *wal*-groups not satisfying these conditions. Therefore, there is a natural question whether, in the class of representable *wal*-groups, l -groups are the only ones that satisfy both (1) and (2). However, the answer to this question is negative.

For instance, consider the *wal*-group $G = (\mathbb{Z}, +, \leq)$, where $G^+ = \{2^k; k \geq 0\} \cup -(\mathbb{Z}^+ \setminus \{2^k; k \geq 0\}) = \{0, 1, 2, -3, 4, -5, -6, -7, 8, -9, -10, \dots, -15, 16, -17, \dots\}$. Evidently G is a *to*-group, hence it is representable. Moreover, G is not an *o*-group.

Let $0 \leq k \in \mathbb{Z}$. Then

$$2^k - (-2^k) = 2^{k+1} \in G^+,$$

hence $-2^k \leq 2^k$ and we have $-2^k \leq 0$.

Let $k \geq 1$. Then

$$-(2k+1) - (2k+1) = -2(2k+1) \in G^+,$$

hence $2k+1 < -(2k+1)$, and $2k+1 < 0$.

Let $k \geq 3, k \neq 2^l, \forall l \geq 0$. Then

$$-2k - 2k = -4k \neq -2^m, \forall m \geq 0,$$

hence $2k < -2k$, and $2k < 0$.

Therefore

$$\forall x \in G; (x \wedge -x) \vee 0 = 0,$$

and so the identity (1) is satisfied.

Now, let $x, y \in G$. Since x, y are comparable, we can suppose e.g. $x \leq y$, hence $2(x \wedge y) = 2x$.

Let $y - x = 2^k, k \geq 0$. Then

$$2y - 2x = 2(x + 2^k) - 2x = 2^{k+1} \in G^+,$$

hence $2x \leq 2y$, i.e. $2x \wedge 2y = 2x$.

Let $y - x = -(2k + 1), k \geq 1$. Then $2y - 2x = -2(2k + 1) \in G^+$, so $2x \wedge 2y = 2x$.

Finally, let $y - x = -2k, k \geq 3, k \neq 2^l, \forall l \geq 0$. Then

$$2y - 2x = -2(2k) \neq 2^m, \forall m \geq 0,$$

hence $2x \wedge 2y = 2x$.

Therefore G satisfies also the condition (2), and thus the variety of representable *wal*-groups satisfying both (1) and (2) is larger than the variety \mathcal{R}_l .

Now, let us consider the identity

$$(3) \quad (x \vee 0) \wedge ((-y - x + y) \vee 0) = 0,$$

which is in the case of *l*-groups equivalent to the identity (1).

Let G be a *to*-group, $x \in G$. If $x \geq 0$, then

$$(x \vee 0) \wedge ((-y - x + y) \vee 0) = x \wedge 0 = 0.$$

If $x < 0$, then

$$(x \vee 0) \wedge ((-y - x + y) \vee 0) = 0 \wedge (-y - x + y) = 0.$$

Hence G satisfies (3), and therefore, in contrast to the condition (1), every representable *wal*-group also satisfies (3). But not even the condition (3) is sufficient to the characterization of the variety \mathcal{R}_{wal} , because any abelian *wal*-group also satisfies (3).

Therefore, let us consider the identity

$$(4) \quad (2x \wedge ((y + x) \wedge (2y \wedge (x + y)))) \wedge (2y \wedge ((x + y) \wedge (2x \wedge (y + x)))) = 2x \wedge 2y,$$

which is in the case of *l*-groups equivalent to (2).

Let G be a *to*-group, $x, y \in G$. Let $x \geq y$. Then

$$2x \wedge ((y + x) \wedge (2y \wedge (x + y))) = 2x \wedge ((y + x) \wedge 2y) = 2x \wedge 2y,$$

$$2y \wedge ((x + y) \wedge (2x \wedge (y + x))) = 2y \wedge ((x + y) \wedge (y + x)),$$

and if $x + y \leq y + x$, then

$$2y \wedge ((x + y) \wedge (y + x)) = 2y \wedge (x + y) = 2y.$$

Similarly for $y + x \leq x + y$.

Hence we have

$$\begin{aligned} & (2x \wedge ((y + x) \wedge (2y \wedge (x + y)))) \wedge (2y \wedge ((x + y) \wedge (2x \wedge (y + x)))) \\ & = (2x \wedge 2y) \wedge 2y = 2x \wedge 2y. \end{aligned}$$

Since x and y on both sides of (4) appear symmetrically, the same result is valid also for the case $x < y$.

Thus G satisfies (4), and therefore every representable *wal*-group satisfies (4), too.

At the same time, there are abelian *wal*-groups not satisfying the property (4). For instance, let $G = (\mathbb{Z}, +)$, where $G^+ = \{0, 1, 2, 4, 6, \dots, 2n, \dots\}$. Let us consider $x = 3$, $y = 8$. Then

$$\begin{aligned} & (2 \cdot 3 \wedge ((8 + 3) \wedge (2 \cdot 8 \wedge (3 + 8)))) \wedge (2 \cdot 8 \wedge ((3 + 8) \wedge (2 \cdot 3 \wedge (8 + 3)))) \\ & = (6 \wedge (11 \wedge (16 \wedge 11))) \wedge (16 \wedge (11 \wedge (6 \wedge 11))) \\ & = (6 \wedge (11 \wedge 10)) \wedge (16 \wedge (11 \wedge 5)) \\ & = (6 \wedge 10) \wedge (16 \wedge 5) = 6 \wedge 4 = 4, \end{aligned}$$

but

$$2 \cdot 3 \wedge 2 \cdot 8 = 6 \wedge 16 = 6.$$

Hence the identity (4) separates the varieties \mathcal{Ab}_{wal} and \mathcal{R}_{wal} . The following question remains open.

Question. *Does the identity (4) characterize the variety \mathcal{R}_{wal} ?*

Now, we can draw a fragment of the lattice **WAL**. It is clear that the variety \mathcal{Ab}_l is still an atom of **WAL**. Indeed, let \mathcal{V} be a variety of *wal*-groups such that $\mathcal{Ab}_l \not\subseteq \mathcal{V}$ and let V contain a non-trivial *l*-group G . Then

$$\mathcal{Ab}_l \subseteq \text{Var}_l(G) \subseteq \text{Var}_{wal}(G) \subseteq \mathcal{V},$$

a contradiction. Therefore \mathcal{V} contains no non-trivial l -group.

But Ab_l is not the least non-trivial variety of **WAL** because it is non-comparable, for example, with the variety \mathcal{T}_3 of wal -groups satisfying the identity

$$(T_3) \quad 3x = 0.$$

Let \mathcal{X}_i denote the variety of wal -groups satisfying the identity (i), $i = 1, 2, 3, 4$, and \mathcal{T}_n the variety of wal -groups satisfying the identity

$$(T_n) \quad nx = 0,$$

where $n > 1$, n odd, \mathcal{G}_{wal} the variety of all wal -groups, \mathcal{G}_l the variety of all l -

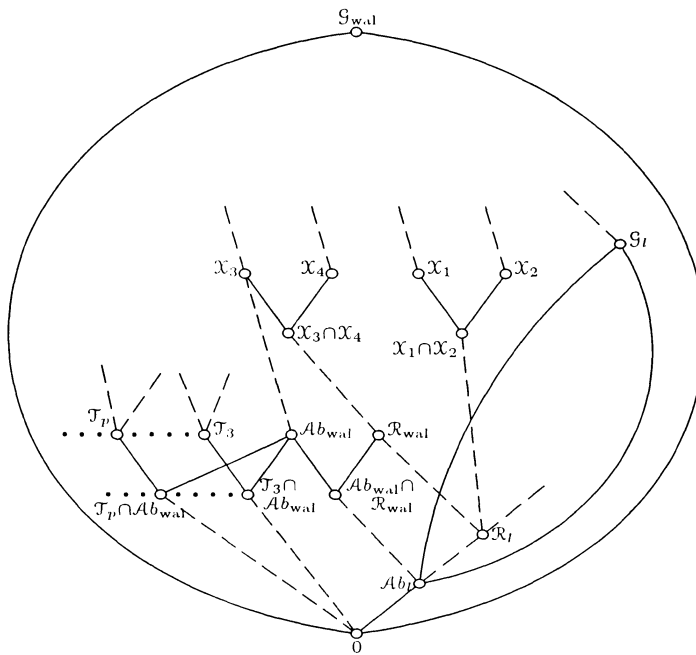


Fig. 1

groups, and \mathcal{O} the trivial variety. Then the connections among these varieties are demonstrated in Figure 1.

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Author's address: Department of Algebra and Geometry, Faculty of Sciences, Palacký University, Tomkova 38, 779 00 Olomouc, Czech Republic.