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ALMOST TOTALLY PROJECTIVE GROUPS

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INTRODUCTION

Let G be an additively written abelian p-group. For each ordinal α, we define $p^\alpha G$ inductively by $p^\alpha G = p(p^{\alpha-1}G)$ when α is isolated and $p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$ when α is a limit. Here, as usual, $pG = \{px : x \in G\}$ and $p^0G = G$.

A subgroup H of G is *isotype* if $p^\alpha G \cap H = p^\alpha H$ for each α. Dually, H is *nice* if $p^\alpha(G/H) = p^\alpha G + H/H$. A subgroup is *balanced* if it is both nice and isotype.

Nice subgroups originally appeared in conjunction with a combinatorial principle, called Axiom 3, which was introduced by one of the authors [H1] in 1967. From the beginning two versions of Axiom 3 quickly emerged. In this connection, consider the following properties for a collection $\mathcal{C}$ of nice subgroups of G.

(0) $0 \in \mathcal{C}$.

(1) $\mathcal{C}$ is closed with respect to unions of ascending chains.

(1^+) $\langle C_i\rangle_{i\in I} \in \mathcal{C}$ if $C_i \in \mathcal{C}$ for each $i \in I$.

(2) If B is a countable subgroup of G, there exists $C \in \mathcal{C}$ such that $B \subseteq C$ and C is countable.

(2^+) If $A \in \mathcal{C}$ and B is a countable subgroup of G, there exists $C \in \mathcal{C}$ such that $A + B \subseteq C$ and C/A is countable.

Each of the following versions of Axiom 3 has proved useful, with one being more relevant or easier to apply to a given situation than the other.

Axiom 3(H): G has a collection $\mathcal{C}$ of nice subgroups that satisfies conditions $(0), (1^+)$ and (2).

Axiom 3(G): G has a collection $\mathcal{C}$ of nice subgroups that satifies conditions $(0), (1)$ and (2^+).

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Note that (2) and (2^+) are equivalent in the presence of (1^+). Therefore, if a group G satisfies Axiom 3(H), then it also satisfies Axiom 3(G). In [H3], it was shown that Axioms 3(H) and 3(G) are actually equivalent. As is well known, groups satisfying these axioms play a central role in the structure theory of infinite abelian groups. In fact, groups satisfying Axiom 3(H) (or the equivalent Axiom 3(G)) coincide with the class of totally projective groups and with the class of simply presented abelian p-groups; see [H1], [H2], [G] and [F]. The significance of Axiom 3 notwithstanding, groups satisfying the weaker version associated with conditions (0), (1) and (2) have not been investigated. It is our purpose here to initiate a study of such groups. We call them *almost totally projective groups.* If a collection $\mathcal{C}$ of nice subgroups of G satisfies conditions $(0), (1)$ and (2), we shall call $\mathcal{C}$ a *weak axiom 3 system* for G. Since a totally projective group of length not exceeding ω_1 is a direct sum of countable groups (i.e. a d.s.c.), we say that an almost totally projective group of length $\leqslant \omega_1$ is *almost a d.s.c.*.

The remainder of the paper is divided into three sections. In the first, we record some general results concerning the class of almost totally projective groups. Next, we specialize to consider the class of almost d.s.c.'s. Finally, in the last section, we demonstrate that the class of almost totally projective groups arises naturally when analyzing the structure of unit groups of commutative modular group algebras.

Almost totally projective groups

Throughout this section and the next, G denotes an additively written abelian p-group for an arbitrary but fixed prime p. If $x \in G$, we denote the height of x in G by $|x|$. Thus, $|x| = \alpha$ means $x \in p^{\alpha}G \setminus p^{\alpha+1}G$. If $x \in p^{\alpha}G$ for all ordinals α, we set $|x| = \infty$ and we write $p^{\infty}G$ for the subgroup of G consisting of the elements of height ∞. Of course $p^{\infty}G$ is simply the maximal divisible subgroup of G. For $x \in G$, we shall employ the more complete notation $|x|_G$ for the height of x in G, if it is necessary for clarity to emphasize in which group G the height of x is to be computed.

In order to state our first result, we recall a notion introduced in [H4] in connection with the study of isotype subgroups of totally projective groups. Namely, a subgroup H of G is a *separable subgroup* of G if for every $g \in G$ there is a corresponding countable subgroup K of H such that

$$\sup\{|g+k|_G \colon k \in K\} = \sup\{|g+h|_G \colon h \in H\}.$$

Since $\sup\{|g+h|_G \colon h \in H\}$ is actually attained by some $|g+h|$ for a nice subgroup H, every nice subgroup of G is separable. Therefore, separability may be viewed as a generalization of niceness.

Proposition 1. *Suppose H is an isotype subgroup of G. In order for H to be almost totally projective, it is necessary that H be separable in G.*

P r o o f. The proof of this proposition is similar to the proof that a completely decomposable torsion-free group is absolutely separable (see [AH]), but nevertheless we include the short proof for completeness.

Let $\mathcal{C}$ be a weak axiom 3 system for H and suppose to the contrary that H is not separable in G. Then, there exists $g \in G$ such that, for each countable subgroup K of H, we can find an element $h^* \in H$ such that $|g + h^*| > |g + x|$ for every $x \in K$.

By a routine back-and-forth argument, there exists an ascending chain

$$0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n \subseteq \ldots (n < \omega_0)$$

of countable subgroups H_n of H such that $H_n \in \mathcal{C}$ for each n and such that the following condition holds.

($\star$) For each $n < \omega_0$, there exists $h_{n+1} \in H_{n+1}$ such that $|g + h_{n+1}| > |g + h|$ for all $h \in H_n$.

Now set $H_\omega = \bigcup\limits_{n<\omega_0} H_n$ and observe that H_ω is a countable subgroup of H belonging to $\mathcal{C}$. Since H_ω is countable, there exists $h^* \in H$ such that $|g + x| < |g + h^*|$ for every $x \in H_\omega$. Since H_ω is nice in H, there exists $h' \in H_\omega$ such that $|h^* - x|_H \leqslant |h^* - h'|_H$ for all $x \in H_\omega$. Moreover, since $h' \in H_n$ for some n, there exists by condition ($\star$) an element $h'' \in H_{n+1}$ such that $|g + h''| > |g + h'|$. Observe that we now have $|h^* - h''|_H \leqslant |h^* - h'|_H$. Also, $|g + h'| < |g + h^*|$ and so $|g + h'| = |(g + h^*) - (g + h')| = |h^* - h'|$. Combining the above with the hypothesis that H is isotype in G, we conclude that $|g + h'| < |(g + h^*) - (g + h'')| = |h^* - h''| = |h^* - h''|_H \leqslant |h^* - h'|_H = |h^* - h'| = |g + h'|$, which is a contradiction. □

We conclude this section with several observations concerning the class of almost totally projective groups. The first of these results is based on the fact that the existence of a weak axiom 3 system for a group G of cardinality $\aleph_1$ implies the existence of an axiom 3 system for G.

Proposition 2. *An almost totally projective group of cardinality not exceeding $\aleph_1$ is totally projective.*

In the next section, we shall see that there exists, for every cardinal $\kappa \geqslant \aleph_2$, an almost totally projective group of cardinality κ which is not totally projective. The reader will recognize the next two propositions as analogs of the corresponding well-known results for totally projective groups.

Proposition 3. *For an ordinal α, if both $p^\alpha G$ and $G/p^\alpha G$ are almost totally projective, then so is G.*

Proof. Recall that N is a nice subgroup of G if and only if $p^\alpha N$ is a nice subgroup of $p^\alpha G$ and $N + p^\alpha G/p^\alpha G$ is a nice subgroup of $G/p^\alpha G$. Hence, weak axiom 3 systems for $p^\alpha G$ and $G/p^\alpha G$ lead to a weak axiom 3 system for G. □

Proposition 4. *The class of almost totally projective groups is closed under the formation of arbitrary direct sums.*

Proof. Suppose $G = \bigoplus_{i \in I} G_i$. As is well known, if $N = \bigoplus_{i \in I} N_i$ with $N_i \subseteq G_i$ for all $i \in I$, then N is a nice subgroup of G if and only if N_i is a nice subgroup of G_i for each $i \in I$. Hence, weak axiom 3 systems for the G_i yield a weak axiom 3 system for G. □

In the next section, we are able to obtain much more definitive results for almost totally projective groups of length not exceeding ω_1.

Almost d.s.c.'s

Recall that two subgroups A and B of G are *compatible*, written $A \parallel B$, if for each pair $(a, b) \in A \times B$ there exists $c \in A \cap B$ such that $|a + b| \leqslant |a + c|$. Observe that compatibility is a symmetric relation and is inductive in the sense that an ascending union of subgroups, each compatible with A, is again compatible with A.

The following lemma has appeared, at least implicitly, in earlier literature. Its proof is straightforward and is omitted here.

Lemma 5. *Suppose H is an isotype subgroup of G. If N is a nice subgroup of G such that $N \parallel H$, then $H + N/N$ is isotype in G/N and $N \cap H$ is a nice subgroup of H.*

As mentioned earlier, we can prove more definitive results for almost totally projective groups of length not exceeding ω_1. Recall again that a totally projective group of length not exceeding ω_1 is a direct sum of countable groups. Such a group is generally called a d.s.c. Hence, by definition, G is almost a d.s.c. if G has length not exceeding ω_1 and if G has a weak axiom 3 system.

Among the isotype subgroups of d.s.c.'s, we can completely characterize those that are almost d.s.c.'s. For an indication of the generality and complexity of isotype subgroups of d.s.c.'s, see [HM].

Theorem 6. *Suppose H is an isotype subgroup of a d.s.c. G. Then, H is almost a d.s.c. if and only if H is separable in G.*

Proof. If H is almost a d.s.c., then H is separable in G by Proposition 1. Conversely, suppose H is separable in $G = \bigoplus_{i \in I} C_i$, where each C_i is countable. If $J \subseteq I$, set $G(J) = \bigoplus_{j \in J} C_j$.

Let $\mathcal{N}$ be the collection of all subgroups N of G such that $N = G(J)$ for some $J \subseteq I$ and $N \parallel H$. It is routine to verify that every countable subgroup of G is contained in some countable subgroup in $\mathcal{N}$. If we set $\mathcal{C} = \{H \cap N \colon N \in \mathcal{N}\}$, then $\mathcal{C}$ consists of nice subgroups of H by Lemma 5. Obviously, $0 \in \mathcal{C}$ and every countable subgroup of H is contained in some countable subgroup in $\mathcal{C}$. To see that $\mathcal{C}$ is a weak axiom 3 system for H, it remains to show that $\mathcal{C}$ is closed under ascending unions. In order to do this, suppose that

$$H \cap G(I_0) \subseteq H \cap G(I_1) \subseteq \ldots \subseteq H \cap G(I_\alpha) \subseteq \ldots (\alpha < \mu)$$

is an ascending chain in $\mathcal{C}$ with $G(I_\alpha) \parallel H$ for each $\alpha < \mu$.

Note that the I_α's need not ascend, so we cannot conclude that the union of the chain is necessarily $H \cap G(\bigcup I_\alpha)$. To circumvent this difficulty, we set $J_\beta = \bigcap_{\beta \leqslant \alpha} I_\alpha$ for each $\beta < \mu$ and define $J = \bigcup_{\beta < \mu} J_\beta$. We claim that $\bigcup_{\alpha < \mu} (H \cap G(I_\alpha)) = H \cap G(J)$.

Observe that $J_0 \subseteq J_1 \subseteq \ldots \subseteq J_\beta \subseteq \ldots (\beta < \mu)$. Thus, if $x \in H \cap G(J)$, there is an ordinal $\beta_0 < \mu$ such that $x \in G(J_{\beta_0})$. Therefore, $x \in H \cap G(I_{\beta_0}) \subseteq \bigcup_{\alpha < \mu} (H \cap G(I_\alpha))$, which shows that $H \cap G(J) \subseteq \bigcup_{\alpha < \mu} (H \cap G(I_\alpha))$.

Conversely, if $x \in \bigcup_{\alpha < \mu} (H \cap G(I_\alpha))$ and $x \neq 0$, there exists $\alpha_0 < \mu$ such that $x \in H \cap G(I_{\alpha_0})$. If $x = x_1 + x_2 + \ldots + x_k$, $0 \neq x_j \in C_{i(j)}$ $(1 \leqslant j \leqslant k)$, is the unique representation of x with respect to the direct sum decomposition $G = \bigoplus_{i \in I} C_i$, we conclude that $\{i(1), \ldots, i(k)\} \subseteq I_\alpha$ for every $\alpha \geqslant \alpha_0$. Thus, $\{i(1), \ldots, i(k)\} \subseteq \bigcap_{\alpha_0 \leqslant \alpha} I_\alpha = J_{\alpha_0}$ and $x \in H \cap G(J_{\alpha_0}) \subseteq H \cap G(J)$. Therefore $\bigcup_{\alpha < \mu} (H \cap G(I_\alpha)) \subseteq H \cap G(J)$, and the claim is established.

Finally, to see that $H \cap G(J) \in \mathcal{C}$, we need to show that $G(J) \parallel H$. If $y \in G(J)$ and $h \in H$, it then follows that $y \in G(I_\alpha)$ for some α. But $G(I_\alpha) \parallel H$ implies that $|y + h| \leqslant |y + z|$ for some $z \in H \cap G(I_\alpha) \subseteq H \cap G(J)$, and $G(J) \parallel H$. □

Although it was an open question for some time, it is now known that a balanced subgroup of a d.s.c. need not be a d.s.c. (see, for example, [H4]). However, we do have the following positive result for balanced subgroups of d.s.c.'s. This result is an immediate corollary of the preceding theorem.

Corollary 7. *Any balanced subgroup of a d.s.c. is almost a d.s.c.*

Using Corollary 7, we can show, in contrast to Proposition 2, the existence of a preponderance of almost d.s.c.'s that are not d.s.c.'s.

Corollary 8. *The class of almost d.s.c.'s properly contains the class of d.s.c.'s. Indeed, for any cardinal $\kappa \geqslant \aleph_2$, there exists a group of cardinality κ which is almost a d.s.c. but is not a d.s.c.*

Proof. We know from [H4] that there are groups of cardinality $\aleph_2$ that appear as balanced subgroups of d.s.c.'s that are not themselves d.s.c.'s. Let B be such a balanced subgroup and take C to be any d.s.c. of cardinality κ. By Corollary 7, B is almost a d.s.c. Thus, $H = B \oplus C$ is almost a d.s.c. by Proposition 4. Moreover, $|H| = \kappa$. However, H cannot be a d.s.c. since any summand of a d.s.c. is again a d.s.c. □

Unfortunately, we have not been able to solve the summand question for almost d.s.c.'s: is a summand of an almost d.s.c. again almost a d.s.c.? We can however answer this question affirmatively for the case that the original group is an isotype subgroup of a d.s.c.

Proposition 9. *Suppose H is an isotype subgroup of a d.s.c. G. If H is almost a d.s.c., then every direct summand of H is almost a d.s.c.*

Proof. If $H = A \oplus B$, then clearly A is isotype in G. Since H is separable in G by Proposition 1, it is routine to verify that A is also separable in G. Therefore, A is almost a d.s.c. by Theorem 6. □

One would naturally be interested in the class of groups that have enough nice subgroups only in the sense that condition (2) of Axiom 3 is satisfied; namely, those groups that have the property that every countable subgroup is contained in a countable nice subgroup. This class seems to be much less tractable than the class of almost totally projective groups. The next result is an example of such a group that is not almost totally projective.

Proposition 10. *There exists an isotype subgroup H of a d.s.c. G such that H is not a separable subgroup (and therefore not almost totally projective), but every countable subgroup of H is contained in a countable nice subgroup of H.*

Proof. By the structure theory of N-groups set forth in [H5], there exists a reduced d.s.c. G and a subgroup H of G such that

(i) H is isotype in G.
(ii) $p^\alpha(G/H) = p^\alpha G + H/H$ for all $\alpha < \omega_1$ and $p^{\omega_1}(G/H) = \langle g + H\rangle$ is cyclic of order p.
(iii) G/H is totally projective.

We remark that (iii) will not be needed in the sequel.

First observe that H is not a separable subgroup of G. To see this, note that (ii) implies that $\sup\{|g+h|: h \in H\} = \omega_1$, which is not cofinal with ω_0. Therefore, in order for H to be separable in G, it is necessary that there exists $h_0 \in H$ such that $|g+h_0| = \omega_1$. But $p^{\omega_1}G = 0$ would then imply that $g \in H$, a contradiction.

Since H is isotype in G, we may assume that $pg = 0$. Set $H^+ = \langle H, g\rangle$. If $N \subseteq G$, we say that N is *almost compatible* with H, if for each pair $(x, h) \in (N \setminus H^+) \times H$, there exists $c \in N \cap H$ such that $|x+h| \leqslant |x+c|$. We claim that the following properties hold.

(I) The property of being almost compatible with H is inductive; that is, if

$$N_1 \subseteq N_2 \subseteq \ldots \subseteq N_\alpha \subseteq \ldots$$

is an ascending chain of subgroups of G with N_α almost compatible with H for all α, then $\bigcup N_\alpha$ is also almost compatible with H.

(II) Every countable subgroup C of G is contained in a countable subgroup N of G such that N is almost compatible with H.

(III) Every countable subgroup C of G is contained in a countable subgroup N of G such that:

(a) $g \in N$,
(b) N is a direct summand of the d.s.c. G,
(c) N is almost compatible with H.

Observe that (I) is clear and that (III) follows from (I) and (II) by a routine back-and-forth argument. Therefore, it suffices to establish (II), which we now do. First note that for every $x \in G \setminus H^+$ and $h \in H$, $|x+h| < \omega_1$. Therefore, for each $x \in C \setminus H^+$ there is a corresponding sequence $\{h_n^x\}_{n<\omega_0}$ of elements of H such that

$$\sup\{|x+h_n^x|: n < \omega_0\} = \sup\{|x+h|: h \in H\}$$

Set $N_0 = C$ and $N_1 = \langle N_0, h_n^x : n < \omega_0, x \in N_0 \setminus H^+\rangle$. Then, N_1 is countable and for each pair $(x, h) \in (N_0 \setminus H^+) \times H$, there exists $n < \omega_0$ such that $|x+h| \leqslant |x+h_n^x|$ and $h_n^x \in N_1 \cap H$. If we replace $N_0 = C$ by N_1 and repeat the process, we obtain $N_2 \supseteq N_1 \supseteq N_0 = C$ such that N_2 is a countable subgroup of G and, for each pair $(x, h) \in (N_1 \setminus H^+) \times H$, there exists $c \in N_2 \cap H$ such that $|x+h| \leqslant |x+c|$. Continuing in this way, we obtain $N = \bigcup_{k<\omega_0} N_k$ which is a countable subgroup of G that contains C and is almost compatible with H. Therefore, (II) is established.

Now let A be an arbitrary countable subgroup of H. Choose a countable subgroup N of G that contains A and satisfies properties (a)–(c) of (III). Set $B = N \cap H$. Since

B is countable, so is $\lambda = \sup\{|b|: b \in B, b \neq 0\} + 1$. Moreover, from property (ii), there exists $h_\lambda \in H[p]$ such that $|g - h_\lambda| \geqslant \lambda$. Set $B^+ = \langle B, h_\lambda \rangle$. Then B^+ is a countable subgroup of H which contains A.

We claim that B^+ is a nice subgroup of H. Suppose to the contrary that B^+ is not nice in H. In this case, there exists $h \in H \setminus B^+$ and a limit ordinal μ such that $\mu = \|h + B^+\|$, where

$$\|h + B^+\| = \sup\{|h + b^+| + 1: b^+ \in B^+\}$$

is the coset valuation. Note that we can compute heights in G since H is isotype. Since B^+ is countable, μ is a countable limit ordinal. Consequently, for each $\sigma < \mu$, there exist a sequence of ordinals $\{\sigma(n): n < \omega_0\}$ and a sequence of elements $\{b_n^+: n < \omega_0\} \subseteq B^+$ such that $\sigma < \sigma(1) < \sigma(2) < \ldots < \sigma(n) < \ldots < \mu$, $\sup\{\sigma(n): n < \omega_0\} = \mu$, and $|h + b_n^+| = \sigma(n)$.

For each $n < \omega_0$, there exist $b_n \in B$ and an integer k_n with $0 \leqslant k_n < p$ and $b_n^+ = b_n + k_n h_\lambda$. By passing to a subsequence if necessary, we may assume that $k_n = k$ for all n. So,

$$\sigma < \sigma(n) = |(h + b_{n+1}^+) - (h + b_n^+)| = |b_{n+1} - b_n|$$

for each n. Since $\sigma < \mu$ was arbitrary, we conclude that $\mu \leqslant \sup\{|b|: b \in B, b \neq 0\} < \lambda$. Moreover, since $|g - h_\lambda| \geqslant \lambda > \mu > \sigma(n)$ for all n, $|h + b_n + kg| = \sigma(n)$ and $\|h + N\| \geqslant \mu$. Thus, $|h + x| \geqslant \mu$ for some $x \in N$, since N is a summand of G and hence nice.

If $x \in N \setminus H^+$, then the fact that N is almost compatible with H implies that there exists $c \in N \cap H = B$ such that $|h + c| \geqslant \mu$ and the contradiction $\|h + B^+\| > \mu$ is obtained. On the other hand, if $x \in N \cap H^+$, there exist $h_0 \in H$ and an integer m such that $0 \leqslant m < p$ and $x = h_0 + mg$. Set $y = h_0 + mh_\lambda$. Since $h_0 = x - mg \in N \cap H = B$, $y \in B^+$ and $|x - y| = |g - h_\lambda| \geqslant \lambda$. We conclude that $|h + y| = |(h + x) - (x - y)| \geqslant \mu$ and $\|h + B^+\| > \mu$, which is again a contradiction. □

Units of modular group algebras

Let G be an abelian p-group and suppose F is a field of characteristic p. As is customary in dealing with the group algebra FG of G over F, we use multiplicative notation for G, even though G is abelian. If $U(G)$ denotes the group of units of FG, it is easily seen that $U(G) = F^* \times V(G)$ is the direct product of the multiplicative group F^* of F with the p-group

$$V(G) = \{c_1 g_1 + \ldots + c_n g_n \in FG: c_i \in F, g_i \in G, \textstyle\sum c_i = 1\}$$

of normalized units. This essentially reduces the study of the unit group $U(G)$ to the study of $V(G)$. It is an open question as to whether G is always a direct factor of $V(G)$; see, for example, [K]. However, we do have the following result if G is not too large.

Theorem 11 ([HU]). *Suppose F is perfect and $G = \coprod_{i \in I} G_i$ is a coproduct of groups G_i indexed by a set I. If $|G_i| \leqslant \aleph_1$ for each $i \in I$, then G is a direct factor of $V(G)$ and $V(G)/G$ is totally projective.*

We remark that Theorem 11 generalizes a theorem of W. May [M2], where G itself is assumed to have cardinality not exceeding $\aleph_1$ and length $\leqslant \omega_1$.

The strategy of the proof of Theorem 11 is to observe first that G is a balanced subgroup of $V(G)$. Then, using techniques developed in [M1], [M2], and [HU], an axiom 3 system for $V(G)/G$ is constructed. Therefore, $V(G)/G$ is totally projective. Since the class of totally projective groups coincides with the class of balanced projective p-groups (see, for example, [F]), the conclusion that $V(G) = G \times T$ for some totally projective T is obtained. This leads us to state the following.

Direct Factor Conjecture. *If F is perfect, then G is a direct factor of $V(G)$ and $V(G)/G$ is totally projective.*

In view of the outline given above for the proof of Theorem 11, to settle the Direct Factor Conjecture in the affirmative it suffices to show that $V(G)/G$ is totally projective for arbitrary G. Unfortunately, we have been unable to do this. However, it is shown in [H7] that if F is perfect, then $V(G)/G$ has a ν-basis for any p-group G. We suppress a discussion of ν-bases here and direct the interested reader to [H6]. Nevertheless, we hasten to add that the results in [H6] show that any p-group with a ν-basis is, in our present terminology, almost totally projective. We summarize this discussion as our final result.

Theorem 12. *Suppose F is a perfect field of characteristic $p \neq 0$ and G is an abelian p-group. Then $V(G)/G$, the group of normalized units of the group algebra FG modulo G, is almost totally projective.*

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