HAUSDORFF COMPLETIONS OF QUASI-UNIFORM SPACES

HERMANN RENDER, Duisburg

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INTRODUCTION

It is an old question in the theory of quasi-uniform spaces which quasi-uniformities have a $T_2$-completion, cf. [6, p. 71]. In [14] the methods of nonstandard analysis have been used to derive necessary and sufficient conditions for the existence of a $T_2$-completion.

In this paper we give a standard proof of the following sufficient condition given in [14]: if $(X, \mathcal{V})$ is a quasi-uniform $T_2$-space containing a compatible uniformity $\mathcal{U}$ then $X$ possesses a $T_2$-completion. More general, we prove here that $\mathcal{V}$ possesses a $T_2$-completion if and only if any compatible quasi-uniformity $\mathcal{W} \supset \mathcal{V}$ possesses a $T_2$-completion. It follows from the methods of proof that the finest uniformity and the finest quasi-uniformity on a completely regular $T_2$-space $X$ have a $T_2$-completion exactly of the cardinality of the Stone-Čech compactification $\beta(X)$. A striking consequence of our main result is that every non-compact uniform $T_2$-space has a $T_2$-completion which is different from the usual uniform completion. All these results are contained in the first section.

It is a matter of fact that the construction of the $T_2$-completion in section 1 does not yield $T_2$-compactifications (except when the remainder is finite). In the second section we investigate a modified construction which is useful for locally compact spaces. In this case we can prove that every topological $T_2$-compactification satisfying a certain natural condition can be considered as a quasi-uniform $T_2$-compactification.
1. **T₂-completions.**

A *completion* of a quasi-uniform space \((X, \mathcal{V})\) is a complete quasi-uniform space \((Y, \mathcal{W})\) that has a dense subspace quasi-isomorphic to \((X, \mathcal{V})\). The induced topology of a quasi-uniform space \((X, \mathcal{V})\) is denoted by \(\tau(\mathcal{V})\). Recall that two quasi-uniformities are *compatible* if they induce the same topology. A quasi-uniformity \(\mathcal{V}\) is *point-symmetric* if for each \(V \in \mathcal{V}\), \(x \in X\) there exists a symmetric \(U \in \mathcal{V}\) such that \(U[x] \subset V[x]\). Throughout the paper we assume the following basic construction:

**1.1 Definition.** Let \((X, \mathcal{V})\) be a quasi-uniform space and \(\mathcal{W}\) be a quasi-uniformity on a larger set \(S\) such that the restriction \(\mathcal{W}|X\) is a compatible weaker quasi-uniformity than \(\mathcal{V}\). We define a filter \(\widehat{\mathcal{W}}\) on \(S \times S\) in the following way: for \(V \in \mathcal{V}\), \(U \in \mathcal{W}\) define

\[
\widehat{V}_U := \bigcup_{x \in X} (\{x\} \times V[x]) \cup \bigcup_{y \in S \setminus X} (\{y\} \times (\{y\} \cup (U[y] \cap X))).
\]

By definition, \(\widehat{\mathcal{W}}\) is the filter generated by \(\widehat{V}_U\) with \(V \in \mathcal{V}\), \(U \in \mathcal{W}\).

**1.2 Proposition.** \(\widehat{\mathcal{W}}\) is a quasi-uniformity on \(S\) finer than \(\mathcal{W}\), in particular \(\tau(\mathcal{W}) \subset \tau(\widehat{\mathcal{W}})\).

**Proof.** It is easy to see that \(\widehat{\mathcal{W}}\) is a quasi-uniformity. For the second statement let \(U \in \mathcal{W}\). Then \(V := U \cap X\) is in \(\mathcal{V}\). Now check that \(\widehat{V}_U \subset U\). \(\square\)

Even in the case that \(\mathcal{W}\) is a uniformity it may occur that \(\tau(\widehat{\mathcal{W}})\) is different from \(\tau(\mathcal{W})\), cf. the proof of Proposition 1.8 or Theorem 1.12. However it is easy to see that \(i: (X, \mathcal{V}) \rightarrow (S, \widehat{\mathcal{W}})\) is a quasi-unimorphism.

The following theorem is our main result. It is a modification of a nonstandard construction of a \(T₂\)-completion given in [14, Theorem 3.3]. In contrast to that result we now *assume* the existence of a larger complete space \((S, \mathcal{W})\).

**1.3 Theorem.** Let \((X, \mathcal{V})\) be a quasi-uniform space. If \(\mathcal{W}\) is a complete quasi-uniformity on a larger space \(S\) such that \(\mathcal{W}|X \subset \mathcal{V}\) are compatible then \(S\) is complete with respect to \(\widehat{\mathcal{W}}\).

**Proof.** Let \(\mathcal{F}\) be a \(\widehat{\mathcal{W}}\)-Cauchy filter on \(S\) and let \(U \in \mathcal{W}\). We now consider two cases: in the first one we assume that \(G_F := F \cap X\) is non-empty for all \(F \in \mathcal{F}\). Then \(\{G_F : F \in \mathcal{F}\}\) generates a filter \(\mathcal{G}\) on \(S\) and we claim that \(\mathcal{G}\) is a \(\mathcal{W}\)-Cauchy filter: for \(U \in \mathcal{W}\) there exists \(V \in \mathcal{V}\) with \(V \subset U \cap (X \times X)\). Since \(\mathcal{F}\) is a \(\widehat{\mathcal{W}}\)-Cauchy filter there exists \(y \in S\) and \(F \in \mathcal{F}\) such that \(F \subset \widehat{V}_U[y] \subset U[y]\) (note that \(V[y] \subset U[y]\) if \(y \in X\)). By \(\mathcal{W}\)-completeness \(\mathcal{G}\) has an adherent point \(z \in S\), i.e., that \(G_F \cap U[z] \neq \emptyset\)

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for all $F \in \mathcal{F}$ and $U \in \mathcal{U}$. In the case of $z \in X$ we obtain $F \cap \hat{V}_U[z] \neq \emptyset$ since $\mathcal{U}|X$ and $\mathcal{V}$ are compatible. If $z \in S \setminus X$ then obviously $F \cap \hat{V}_U[z] \neq \emptyset$.

In the second case there exists $F_0 \in \mathcal{F}$ with $F_0 \cap X = \emptyset$. Since $\mathcal{F}$ is a $\tau_{\mathcal{U}}$-Cauchy filter there exists $F \in \mathcal{F}$ with $F \subseteq \hat{V}_U[y]$ for some $y \in S$. But $y$ cannot be in $X$; otherwise we would have $F \subseteq V[y]$ and therefore $F \cap F_0 \subseteq V[y] \cap F_0 \subseteq X \cap F_0 = \emptyset$, a contradiction. Since $y \notin X$ we obtain $F \subseteq \hat{U}[y] \cap X \cup \{y\}$. Hence we obtain $F \cap F_0 = \{y\}$. Thus $\mathcal{F}$ is the ultrafilter consisting of all subsets $B \subseteq S$ with $y \in B$. Therefore $\mathcal{F}$ converges to $y$ and the proof is complete.

1.4 Remark. A short review of the proof shows that $(S, \tau_{\mathcal{U}})$ is convergence complete if $(S, \mathcal{U})$ is convergence complete.

1.5 Corollary. Let $i = 1$ or $i = 2$. If $(X, \mathcal{V})$ possesses a $T_i$-completion then any compatible quasi-uniformity $\mathcal{U} \supset \mathcal{V}$ possesses a $T_i$-completion.

Proof. Let $(S, \mathcal{U})$ be a $T_i$-completion of $(X, \mathcal{V})$. Since $\mathcal{U}|X = \mathcal{V} \subseteq \mathcal{U}$ induce the same topology $\mathcal{U}|X$ is a complete quasi-uniformity in which $(X, \mathcal{U})$ is embedded. Now consider the closure of that subspace in $S$ with respect to $\mathcal{U}|X$. For the separation property just note that $\tau(\mathcal{U}) \subseteq \tau(\mathcal{U}|X)$ by Proposition 1.2.

1.6 Corollary. Let $(X, \mathcal{V})$ be a quasi-uniform $T_2$-space. If there exists a compatible uniformity $\mathcal{U} \supset \mathcal{V}$ then $(X, \mathcal{V})$ possesses a $T_2$-completion.

Proof. A uniform $T_2$-space $\mathcal{U}$ possesses a $T_2$-completion $(S, \mathcal{U})$.

1.7 Corollary. Let $(X, \mathcal{V})$ be a non-compact uniform $T_2$-space. Then there exists a $T_2$-completion which is not a uniformity.

Proof. It is a well-known fact that $\mathcal{V}$ contains a totally bounded uniform $\mathcal{U}_0$. Then the completion $(S, \mathcal{U})$ of $(X, \mathcal{U}_0)$ is a $T_2$-compactification. Theorem 1.3 shows that $\tau(\mathcal{U})$ is a $T_2$-completion of $(X, \mathcal{V})$. The next proposition shows that $\tau(\mathcal{U})$ is not uniform on $S$.

For the second statement of the next proposition note that a locally compact Hausdorff space $(X, \mathcal{V})$ is an open subset in any (topological) Hausdorff extension $S$ of $(X, \mathcal{V})$.

1.8 Proposition. Let $X$ be dense in the space $(S, \mathcal{U})$ and $X \neq S$. Then $\tau(\mathcal{U})$ is not uniform and $\tau(\mathcal{U}) \neq \mathcal{U}$. If $(S, \mathcal{U})$ is a pointsymmetric Hausdorff space and if $(X, \tau(\mathcal{V}))$ is open in $(S, \tau(\mathcal{U}))$ then $(S, \tau(\mathcal{U}))$ is point-symmetric.

Proof. Let $y \in S$ with $y \notin X$. Then we have $\hat{V}_U^{-1}[y] = \{z \in S : y \in \hat{V}_U[z]\} = \{y\}$. Hence the induced topology of $\hat{V}_U^{-1}$ is discrete at $y \in S$. On the other side
\[\widetilde{V}_U[y] = \{y\} \cup (U[y] \cap X) \text{ is different from } \{y\} \text{ since } y \text{ is in the } \tau(W)\text{-closure of } X.\]

It follows that \(\widetilde{V}_U\) is not uniform.

Recall that a quasi-uniformity \(W\) is point-symmetric iff \(\tau(W) \subset \tau(W^{-1})\). Since \(\tau(W^{-1})\) is discrete at \(y \in S \setminus X\) we only need to consider the case \(y \in X\). Let \(\widetilde{V}_U[y] = V[y]\) be a neighborhood. Since \(W\) (and therefore \(\mathcal{U}\)) is point-symmetric we can find symmetric \(V_1 \in \mathcal{U}\), \(U_1 \in \mathcal{W}\) with \(V_1[y] \subset V[y]\) and \(U_1[y] \subset U[y]\).

Since \(X\) is an open subset we can assume that \(U_1[y] \subset X\). It suffices to show that \(\widetilde{V}_{U_1^{-1}}[y] \subset V[y]\). Let \(x \in \widetilde{V}_{U_1^{-1}}[y]\). Then \((x, y) \in \widetilde{V}_{U_1}\). If \(x\) is in \(X\) then \(y \in V_1[x]\) and, by symmetry of \(V_1\), \(x \in V_1[y] \subset V[y]\). If \(x \in S \setminus X\) then \(y \in (U_1[x] \cap X) \cup \{x\}\). Since \(y \in X\) we have \(y \neq x\), in particular \(y \in U_1[x]\). The symmetry yields \(x \in U_1[y] \subset X\), a contradiction.

\[\text{1.9 Corollary. Let } X \text{ be a completely regular Hausdorff space. Then the finest compatible uniformity and the finest compatible quasi-uniformity have a } T_2\text{-completion of the cardinality of } \beta(X).\]

\[\text{Proof. Let } \mathcal{U} \text{ be the filter considered in Corollary 1.9. Let } \mathcal{W} \text{ be the weak uniformity induced by the set } C^b(X, \mathbb{R}) \text{ of all bounded continuous real-valued functions. Then } \mathcal{W} \text{ and } \mathcal{U} \text{ are compatible and trivially } \mathcal{W} \subset \mathcal{U}. \text{ Moreover } \mathcal{W} \text{ is totally bounded and it is well known that the completion } \overline{\mathcal{W}} \text{ of } \mathcal{W} \text{ is the Stone-Čech compactification } \beta(X). \text{ Now apply Theorem 1.3.} \]

\[\text{1.10 Theorem. Let } (X, \mathcal{U}) \text{ be a completely regular quasi-uniform space. If } \mathcal{U} \text{ contains the Pervin quasi-uniformity } \mathcal{P} \text{ (with respect to } \tau) \text{ then } \mathcal{U} \text{ possesses a } T_2\text{-completion.}\]

\[\text{Proof. } \mathcal{P} \subset \mathcal{U} \text{ contains a compatible uniformity, see the proof of Theorem 3.11 in [14].} \]

\[\text{The next two results show that the quasi-uniformity } \widetilde{V}_U \text{ is almost never a compactification.}\]

\[\text{1.11 Proposition. Assume that } \mathcal{U} \text{ is precompact. Then } S \text{ is precompact with respect to } \widetilde{V}_U \text{ iff } S \setminus X \text{ is finite.}\]

\[\text{Proof. Choose } V \in \mathcal{U} \text{ and } U \in \mathcal{W}. \text{ If } \widetilde{V}_U \text{ is precompact there exists } y_1, \ldots, y_n \in S \text{ with } S \subset \bigcup_{i=1}^n \widetilde{V}_U[y_i]. \text{ Since } \widetilde{V}_U \subset X \cup \{y\} \text{ we obtain } S \subset X \cup \{y_1, \ldots, y_n\}. \text{ For the converse assume that } (X, \mathcal{U}) \text{ is precompact. Hence there exists } x_1, \ldots, x_m \in X \text{ with } X \subset V[x_1] \cup \ldots \cup V[x_m]. \text{ Let } S = X \cup \{y_1, \ldots, y_n\}. \text{ Then } S \subset \bigcup_{i=1}^m \widetilde{V}_U[x_i] \cup \bigcup_{j=1}^n \widetilde{V}_U[y_j]. \]

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1.12 Theorem. Let \((X, \mathcal{V})\) be a precompact quasi-uniform space and \((S, \mathcal{U})\) be a complete Hausdorff space such that \(\mathcal{U}\mid X \subset \mathcal{V}\) are compatible. Then the following statements are equivalent for \(\mathcal{U}\):

a) \(S\) is precompact.

b) \(S \setminus X\) is finite.

c) \(S\) is a Hausdorff compactification

d) \(S\) is regular.

Proof. Obviously c) implies d). For the converse note at first that \((X, \mathcal{V})\) is precompact and dense in \(S\). By Theorem 1.3 \(S\) is a complete space containing a dense precompact subspace \(X\). Since \(S\) is regular a well-known Corollary in [6, p. 53] shows that \(S\) is compact. Proposition 1.11 yields the equivalence of a) and b) and c) \(\Rightarrow\) a) is clear. For a) \(\Rightarrow\) c) note that \(S\) is complete (Theorem 1.3) and precompact and therefore compact. \(\Box\)

2. HAUSDORFF COMPACTIFICATIONS

2.1 Definition. Let \(\mathcal{U}\) and \(\mathcal{V}\) as in Definition 1.1. Define

\[\widehat{V}_U(S) := \bigcup_{x \in X} (\{x\} \times V[x]) \cup \bigcup_{y \in S \setminus X} \{y\} \times U[y].\]

Let \(\widehat{\mathcal{U}}(S)\) be the filter generated by the sets \(\widehat{V}_U(S)\) with \(U \in \mathcal{U}, V \in \mathcal{V}\).

As before, \(\widehat{\mathcal{U}}(S)\) is a quasi-uniformity and we have \(\mathcal{U} \subset \widehat{\mathcal{U}}(S) \subset \widehat{\mathcal{U}}\).

2.2 Proposition. The quasi-uniformity \((X, \mathcal{V})\) is an open subspace of \((S, \widehat{\mathcal{U}})\) and \(\widehat{\mathcal{U}}(S)\). In particular, if \(S\) is a compact regular space then \(X\) is locally compact.

Proof. Let \(x \in X\). Then \(x \in \widehat{V}_U[x] = V[x] \subset X\). Hence \(X\) is open in \(S\). The case \(\widehat{\mathcal{U}}(S)\) is similar. \(\Box\)

2.3 Proposition. If \((X, \mathcal{V})\) is precompact and \((S, \mathcal{U})\) is hereditarily precompact then \(S\) is precompact with respect to \(\widehat{\mathcal{U}}(S)\).

Proof. Let \(\widehat{V}_U(S)\) be given with \(V \in \mathcal{V}\) and \(U \in \mathcal{U}\). Since \(S\) is precompact with respect to \(\mathcal{V}\) and \(S \setminus X\) is precompact with respect to \(\mathcal{U}\mid (S \setminus X)\) we obtain \(X \subset V[x_1] \cup \ldots \cup V[x_m]\) and \((S \setminus X) \subset U[y_1] \cup \ldots \cup U[y_n]\) for some \(x_1, \ldots, x_m \in X\) and \(y_1, \ldots, y_n \in S \setminus X\). Now observe that \(\widehat{V}_U(S)[x_i] = V[x_i]\) and \(\widehat{V}_U[y_i] = U[y_i]\) for \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\). The proof is complete. \(\Box\)
The last proposition has an interesting consequence: Let \((X, \mathcal{V})\) be a precompact \(T_2\)-uniformity and let \((S, \mathcal{W})\) be the (unique) uniform Hausdorff completion of \(\mathcal{V}\). Then \((S, \mathcal{W}(S))\) is precompact and Hausdorff, cf. Proposition 1.2 and 2.3. If \(S\) is complete then \(S\) is a compact Hausdorff space and therefore \(X\) is locally compact by Proposition 2.2. Hence an analogue of Theorem 1.3 for \(\mathcal{W}(S)\) can only be expected for locally compact spaces. More precisely, we prove

2.4 Theorem. If \((X, \tau(\mathcal{V}))\) is open in \((S, \tau(\mathcal{W}))\) and \((S, \mathcal{W})\) is a complete quasi-uniformity such that \(\mathcal{W}|X \subset \mathcal{V}\) are compatible then \(S\) is complete with respect to \(\mathcal{W}(S)\).

Proof. Let \(\mathcal{F}\) be a \(\mathcal{W}(S)\)-Cauchy filter. Case 1 in the proof of 1.3 can be treated as in 1.3. Hence we can assume that there exists \(F_0 \in \mathcal{F}\) such that \(F_0 \cap X = \emptyset\). It is clear that \(\mathcal{F}\) is as well a \(\mathcal{W}\)-Cauchy filter. Hence there exists an adherent point \(y_0 \in S\) by \(\mathcal{W}\)-completeness. It suffices to show that \(y_0\) is an adherent point of \(\mathcal{F}\). At first we consider the case \(y_0 \in S \setminus X\). Let \(\mathcal{V}_U(S) = U[y_0]\) be a neighborhood of \(y_0\) and let \(F \in \mathcal{F}\). Then \(F \cap \mathcal{V}_U(S)[y_0] = F \cap U[y_0] \neq \emptyset\).

In the other case we have \(y_0 \in X\). Since \((X, \tau(\mathcal{V}))\) is open in \((S, \tau(\mathcal{W}))\) we can find \(U \in \mathcal{W}\) such that \(U[y_0] \subset X\). Hence \(F_0 \cap U[y_0] \subset F \cap X = \emptyset\), a contradiction. Hence \(y_0 \in X\) is impossible. \(\square\)

2.5 Theorem. If \((X, \mathcal{V})\) is locally compact and \((S, \mathcal{W})\) is compact Hausdorff such that \(\mathcal{W}|X \subset \mathcal{V}\) are compatible then \(S\) is compact with respect to \(\mathcal{W}(S)\).

Proof. Let \((T_x)_{x \in S}\) be an \(\mathcal{W}(S)\)-open covering of \(S\) with \(x \in T_x\). For \(x \in S \setminus X\) there exists \(U_x \in \mathcal{W}\) such that \(U_x[x] \subset T_x\). For \(x \in X\) there exists \(V_x \in \mathcal{V}\) such that \(x \in V_x[x] \subset (X \cap T_x)\) by local compactness. Since \(\mathcal{W}|X\) and \(\mathcal{V}\) are compatible there exists \(U_x \in \mathcal{W}\) such that \(x \in U_x[x] \subset V_x[x]\). Since \((U_x[x])_{x \in X}\) is a covering of \(S\) the \(\mathcal{W}\)-compactness implies that there exists a finite subcovering, say \(\{U_{x_1}[x_1], \ldots, U_{x_n}[x_n]\}\). Then \(\{T_{x_1}, \ldots, T_{x_n}\}\) is the desired finite subcovering. The proof is complete. \(\square\)

Recall that a topological \(T_2\)-compactification \(K\) of the topological space \(X\) consists of compact \(T_2\)-space \(K\) and a topological embedding \(i: X \rightarrow K\) such that \(i(X)\) is dense in \(K\). A quasi-uniform \(T_2\)-compactification of the quasi-uniform space \((X, \mathcal{V})\) is a compact quasi-uniform \(T_2\)-space \((K, \mathcal{V}_K)\) and a quasi-uniform embedding \(i: X \rightarrow K\) such that \(i(X)\) is dense in \(K\). Clearly every quasi-uniform compactification of \((X, \mathcal{V})\) induces a topological compactification; but observe that this correspondence is in general not injective, cf. Proposition 3.48 in [6].

It is a natural question whether for every topological \(T_2\)-compactification \(K\) of the quasi-uniform space \((X, \mathcal{V})\) (seen as a topological space) there exists a quasi-
uniformity \( \mathcal{U}_K \) on \( K \) such that \((K, \mathcal{U}_K)\) is a quasi-uniform \( T_2 \)-compactification of \((X, \mathcal{V})\). Since every compact \( T_2 \)-space has a (unique) compatible uniformity \( \mathcal{U}(K) \) which is the smallest compatible quasi-uniformity we obtain the following necessary condition for our problem:

(*) The restriction of the associated uniformity \( \mathcal{U}(K) \) to the subspace \( X \) is smaller than or equal to \( \mathcal{V} \).

It is shown in [6, p. 69] that (*) is also sufficient provided that \( \mathcal{V} \) is totally bounded. We show that (*) is sufficient provided that \( X \) is locally compact:

2.6 Theorem. Let \((X, \mathcal{V})\) be a locally compact quasi-uniform Hausdorff space and \( K \) a topological compactification of \((X, \tau(\mathcal{V}))\). Then \( K \) is a quasi-uniform \( T_2 \)-compactification of \((X, \mathcal{V})\) for a quasi-uniformity \( \overline{\mathcal{V}} \) on \( K \) iff (*) holds.

Proof. Suppose that (*) holds. Define \( \mathcal{U} := \mathcal{U}(K) \) and \( S := K \). Now Theorem 2.5 shows that \((K, \mathcal{U}_\mathcal{U}(S))\) is a compact space which contains \((X, \mathcal{V})\) as a quasi-uniform dense subspace. \( \square \)

References


Author's address: Fachbereich 11 Mathematik, Gerhard-Mercator-Universität Duisburg, Lotharstr. 65, D-47057 Duisburg, Federal Republic of Germany.