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On minimum locally $n$-(arc)-strong digraphs


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ON MINIMUM LOCALLY $n$-(ARC)-STRONG DIGRAPHS

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(Received September 9, 1994)

1. INTRODUCTION

Extensive studies have been devoted to the (global) connectedness in graphs and digraphs, one of the most important properties that a graph or digraph can possess (see, for instance, the surveys [2] and [8]). In 1974, G. Chartrand and R.E. Pippert [4] first defined locally connected and locally $n$-connected graphs and obtained some interesting results. Following [4], a variety of research [9–14] has been devoted to locally connected graphs. Recently, we first extended the study of local connectedness to digraphs (see [5] and [6]). In [5], we defined the locally $n$-strong digraphs and the locally $n$-arc-strong digraphs (See section 2 for definitions.), generalized some results of Chartrand and Pippert, and established relationships between local connectedness and global connectedness in digraphs, among which are the following theorems:

**Theorem A.** Any weakly connected and locally $n$-arc-strong digraph is $(n + 1)$-arc-strong.

**Theorem B.** Any weakly connected and locally $n$-strong digraph is $(n + 1)$-strong.

The aim of this paper is to further the study of locally $n$-(arc)-strong digraphs. We shall determine the minimum locally $n$-(arc)-strong digraphs and the minimum locally $n$-(arc)-strong oriented graphs. [Note: A minimum digraph with some property $\mathcal{P}$ is a digraph with minimum number of arcs in the digraphs with the property $\mathcal{P}$ which have minimum number of vertices.] Moreover, some results concerning tournaments are obtained, and the converses of the above Theorems A and B are shown to be not true.

¹ This research is supported in part by the RDG grant of the Pennsylvania State University.
2. Definitions

We follow the standard terminology and notation. A digraph $D = (V(D), A(D))$ is a finite nonempty set $V(D)$ of vertices together with a (possibly empty) set $A(D)$ of ordered pairs of distinct vertices of $D$ called arcs. An ordered pair $(u, v) \in A(D)$ is also called an arc from $u$ to $v$. A digraph $D$ is said to be weakly connected if its underlying undirected graph is connected. If there is a dipath from $u$ to $v$ for any pair $u$ and $v$ of vertices in $D$, then the digraph $D$ is said to be strongly connected, or simply said to be strong. The subdigraph induced by a nonempty subset $W \subset V(D)$ is denoted $(W)_D$. Let $u, v \in V(D)$. We say $u$ is a neighbor of $v$ if $(u, v) \in A(D)$ or $(v, u) \in A(D)$. The set of neighbors of $v$ in $D$ is denoted $N_D(v)$. The induced subdigraph $(N_D(v))_D$ is said to be the neighborhood of $v$. The outdegree of $v$ is denoted as $\text{od } v$ and the indegree of $v$ is denoted as $\text{id } v$. Let $\delta(D) = \min_{v \in V(D)} \{\text{id } v, \text{od } v\}$. If $\text{id } v = \text{od } v = \delta(D)$ for all $v \in V(D)$, $D$ is said to be diregular. Let $S$ and $T$ be two disjoint proper subsets of $V(D)$. We use $(S,T)_D$ to denote the set of arcs $(s,t)$ in $D$ with $s \in S$ and $t \in T$. When there is no confusion, we may simply use $(W)$, $(N(v))$ and $(S,T)$ to denote the corresponding $(W)_D$, $(N_D(v))_D$ and $(S,T)_D$, respectively.

Let $n \geq 1$. A digraph $D$ is said to be $n$-strong [$n$-arc-strong, resp.] if the removal of fewer than $n$ vertices [arcs, resp.] always results in a nontrivial strong digraph. Clearly, every $n$-strong digraph is $n$-arc-strong. Every $n$-strong [$n$-arc-strong, resp.] digraph is also $m$-strong [$m$-arc-strong, resp.] for $1 \leq m < n$. It should also be noted that $D$ is $1$-strong iff $D$ is $1$-arc-strong iff $D$ is a nontrivial strong digraph. The trivial strong digraph consisting of a single vertex is the only digraph that is strong but not 1-strong (or not 1-arc-strong).

A digraph $D$ is said to be locally strong [locally $n$-strong, locally $n$-arc-strong, resp.] if the neighborhood of every vertex of $D$ is strong [$n$-strong, $n$-arc-strong, resp.].

The associated digraph of a graph $G$, denoted as $D(G)$, is the digraph obtained from $G$ when each edge $e$ of $G$ is replaced by a pair of oppositely oriented arcs with the same ends as $e$.

For other terminologies not defined here we refer the reader to the book [3].

3. Main results

Theorem 1. The associated digraph $D(K_{n+2})$ of the complete graph $K_{n+2}$ is both the unique minimum locally $n$-strong digraph and the unique minimum locally $n$-arc-strong digraph.
Before giving the proof of Theorem 1, we list some needed simple facts as the following propositions.

**Proposition 1.** Let $D$ be an $n$-(arc)-strong digraph. Then $\delta(D) \geq n$, $|V(D)| \geq n + 1$, and $|A(D)| \geq n(n + 1)$.

The proof is easy and is omitted here.
From Proposition 1, we immediately get

**Proposition 2.** The associated digraph $D(K_{n+1})$ is both the unique minimum $n$-strong digraph and the unique minimum $n$-arc-strong digraph.

**Proof.** Clearly, $D(K_{n+1})$ is $n$-strong and $n$-arc-strong. Both the vertex number and the arc number reach the lower bounds given in Proposition 1.

**Proposition 3.** Let $D$ be a locally $n$-(arc)-strong digraph. Then $\delta(D) \geq n + 1$, $|V(D)| \geq n + 2$, and $|A(D)| \geq (n + 1)(n + 2)$.

**Proof.** By Theorem A and Proposition 1.

Now the proof of Theorem 1 goes as follows.

**Proof of Theorem 1.** From Proposition 2, $D(K_{n+2})$ is locally $n$-strong and locally $n$-arc-strong. Since both the vertex number and the arc number of $D(K_{n+2})$ reach the lower bounds given in Proposition 3, $D(K_{n+2})$ is a minimum locally $n$-strong and minimum locally $n$-(arc)-strong digraph.

The uniqueness is easily seen from the following:

If $D$ is a minimum locally $n$-(arc)-strong digraph, then by Proposition 3, $\delta(D) \geq n + 1$. Note that $|V(D)|$ must be not greater than the vertex number of $D(K_{n+2})$. Then, $|V(D)| = n + 2$. Thus we must have $ov = id = v = n + 1$ for all vertices in $D$. Therefore, $D = D(K_{n+2})$.

Now we turn to determine the minimum locally $n$-(arc)-strong oriented graphs. Recall that a digraph is said to be an oriented graph if its underlying graph is a simple graph. Such digraphs are widely used in applications of graph theory.

**Theorem 2.** A digraph $D$ is a minimum locally $n$-arc-strong oriented graph if and only if $D$ is a diregular tournament of $2n + 3$ vertices.

In the proof, we need the following lemmas where Lemma 1 is a rewritten version of a known result in [1].

**Lemma 1.** Let $D$ be an oriented graph. If $\delta(D) \geq \left\lfloor \frac{|V(D)|+2}{4} \right\rfloor$, then $D$ is $\delta(D)$-arc-strong.
Lemma 2. Let $D$ be a locally $n$-arc-strong oriented graph. Then $\delta(D) \geq n + 1$, $|V(D)| \geq 2n + 3$, and $|A(D)| \geq (n + 1)(2n + 3)$.

Proof. By Propositions 3, $\delta(D) \geq n + 1$. Then the other two inequalities immediately follow since $D$ is an oriented graph.

Now the proof of Theorem 2 goes as follows.

Proof of Theorem 2. We first prove the sufficiency. Let $D$ be a diregular tournament of $2n + 3$ vertices. By Lemma 1, it is easy to see that every neighborhood of a vertex in $D$ is $n$-arc-strong. So, $D$ is locally $n$-arc-strong. Since $|V(D)| = 2n + 3$ and $A(D) = (n + 1)(2n + 3)$, $D$ is a minimum locally $n$-arc-strong oriented graph by Lemma 2.

Now we prove the necessity. Let $D$ be a minimum locally $n$-arc-strong oriented graph. Since we have proved that a diregular tournament of $2n + 3$ vertices is a minimum locally $n$-arc-strong oriented graph, we have $|V(D)| = 2n + 3$, $|A(D)| = (n + 1)(2n + 3)$. By Lemma 2, $\delta(D) \geq n + 1$. Then we must have $\text{id}_v = \text{od}_v = n + 1$ for any vertex $v$ in $D$. Therefore, $D$ is a diregular tournament of $2n + 3$ vertices.

For the minimum locally $n$-strong oriented graphs, we have the following result.

Theorem 3. Every minimum locally $n$-strong oriented graph is a diregular tournament of $2n + 3$ vertices.

Before giving the proof we need to give a lemma, which also has its own interest.

Lemma 3. Let $D$ be a tournament. Then $D$ is locally $n$-strong if and only if $D$ is $(n + 1)$-strong.

Proof. The necessity is immediately seen from Theorem B. We only need to show the sufficiency.

Assume there is a tournament $D$ which is $(n + 1)$-strong but not locally $n$-strong. Then, there is a vertex $v$ in $D$ such that $\langle N(v) \rangle$ is not $n$-strong. Thus, we can find a proper subset $S$ of $N(v)$ such that $|S| \leq n - 1$ and $\langle N(v) \rangle - S$ is not strong. Let $S' = S \cup \{v\}$. Then $|S'| \leq n$, and $D - S' = \langle N(v) \rangle - S$ since $D$ is a tournament. Thus, $D - S'$ is not strong, which contradicts the assumption that $D$ is $(n + 1)$-strong.

It completes the proof of Lemma 3.

Now we prove Theorem 3 as follows.

Proof of Theorem 3. First we claim that for any positive integer $n$, there exists a diregular tournament of $2n + 3$ vertices which is locally $n$-strong. For instance, we may consider the right Cayley digraph $L(Z_{2n+3}, \{1, 2, \ldots, n + 1\})$ which
is a diregular tournament of $2n + 3$ vertices. (Recall that for an additive group $G$ and $S \subseteq G \setminus \{0\}$, the right Cayley digraph $L(G, S)$ is a digraph $D$ with $V(D) = G$ and $A(D) = \{(x, x + y) : y \in S\}$.) By a result of Y. O. Hamidoune [7, Proposition 5.1], $L(Z_{2n+3}, \{1, 2, \ldots, n + 1\})$ is $(n + 1)$-strong. Then it is locally $n$-strong by Lemma 3. So, our claim is true.

Let $D$ be a minimum locally $n$-strong oriented graph. By the above claim, $|V(D)| \leq 2n + 3$ and $|A(D)| \leq (n + 1)(2n + 3)$. Then by Lemma 2, we must have $|V(D)| = 2n + 3$ and $|A(D)| = (n + 1)(2n + 3)$. Moreover, from Lemma 2, $\delta(D) \geq n + 1$. Then we must have $\text{id} v = \text{od} v = n + 1$ for every vertex $v$ in $D$. Therefore, $D$ is a diregular tournament of $2n + 3$ vertices.

**Remark 1.** From Lemma 3, it seems natural to pose the following conjecture:

Let $D$ be a tournament. Then $D$ is locally $n$-arc-strong if and only if $D$ is $(n + 1)$-arc-strong.

However, this conjecture is false, which can be seen from Proposition 4 given at the end of this paper.

Note that Theorem 3 only gives a result parallel to the necessity part of Theorem 2. In fact, the converse of Theorem 3 does not hold for $n \geq 3$. It can be seen from the following result.

**Theorem 4.** For any integer $n \geq 3$, there exists a diregular tournament of $2n + 3$ vertices which is not locally $n$-strong.

**Proof.** We proceed in two steps.

Step 1. By induction on $n$, show that there is a diregular tournament $D_{2n+3}$ of $2n + 3$ vertices satisfying the following conditions: $V(D_{2n+3}) = X_n \cup Y_n \cup C$ where $X_n$, $Y_n$ and $C$ are pairwise disjoint, $|X_n| = |Y_n| = n$ and $\langle C \rangle$ is a dicycle of length 3; and $A(D_{2n+3}) \supseteq (X_n, C) \cup (C, Y_n)$.

For $n = 3$, the desired $D_9$ can be constructed as follows. Take three pairwise disjoint dicycles of length 3 and denote their vertex sets as $X_3$, $Y_3$ and $C$. Then add all arcs in $(X_3, C) \cup (C, Y_3) \cup (Y_3, X_3)$. It can be easily verified that this digraph is the desired $D_9$.

Now, assume that $D_{2k+3}$ has been constructed for $k \geq 3$. We construct a new tournament of two more vertices as follows. First, we add two new vertices $x$ and $y$ and add arcs $(y, x) \cup \{(x, y), C\} \cup (C, \{y\})$ so that we have $\text{id} x = \text{od} y = 2$, $\text{id} y = \text{od} x = 2$ and $\text{id} v = \text{od} v$ for every $v \in C$. Then, we arbitrarily take a subset $S \subseteq X_k \cup Y_k$ with $|S| = k + 1$, and let $\overline{S} = (X_k \cup Y_k) \setminus S$. Clearly, $|S| - |\overline{S}| = 2$. Then we add arcs $(S, \{x\}) \cup (\{x\}, \overline{S}) \cup (\{y\}, S) \cup (\overline{S}, \{y\})$. Let $X_{k+1} = X_k \cup \{x\}$ and $Y_{k+1} = Y_k \cup \{y\}$. Then it is easily seen that the obtained digraph is the desired $D_{2n+3}$. This completes the induction.
Step 2. Show that $D_{2n+3}$ is not locally $n$-strong.

Let $D = D_{2n+3} - X_n$. Then $V(D)$ can be decomposed as two disjoint subsets $C$ and $Y_n$. Since $(Y_n,C) = \emptyset$, $D$ is not strong. Note that $|X_n| = n$. Then we see that $D_{2n+3}$ is not $(n+1)$-strong. Hence, it is not locally $n$-strong by Theorem B.

It completes the proof of Theorem 4. \hfill \Box

Remark 2. The condition $n \geq 3$ in Theorem 4 is necessary since any diregular tournament of 5 (7, resp.) vertices is easily seen to be locally 1-strong (locally 2-strong, resp.). Therefore, the converse of Theorem 3 only holds for $n = 1, 2$.

Remark 3. It should be noted that the conclusion in Lemma 3 is not true for general digraphs, i.e., the converses of Theorems A and B are not true, which can be seen from the associated digraphs $D(K_{n+1,n+k})$ of the complete bipartite graphs $K_{n+1,n+k}$ (with $k \geq 1$).

It is easy to see the following facts:
(a) $G$ is connected iff $D(G)$ is strong;
(b) $G$ is $n$-connected iff $D(G)$ is $n$-strong;
(c) $G$ is $n$-edge-connected iff $D(G)$ is $n$-arc-strong;
(d) $G$ is locally $n$-connected iff $D(G)$ is locally $n$-strong;
(e) $G$ is locally $n$-edge-connected iff $D(G)$ is locally $n$-arc-strong (Note: $G$ is said to be locally $n$-edge-connected if the neighborhood of every vertex of $G$ is $n$-edge-connected.)

From these relationships between $G$ and $D(G)$, we can easily see that $D(K_{n+1,n+k})$ is $(n+1)$-strong and $(n+1)$-arc-strong but not locally $n$-(arc)-strong, since $K_{n+1,n+k}$ ($k \geq 1$) is $(n+1)$-connected and $(n+1)$-edge-connected but not locally $n$-(edge)-connected.

Finally, let us go back to the conjecture mentioned earlier. It is disproved by the following result:

**Proposition 4.** For any integer $n \geq 1$, there is a tournament which is $(n+1)$-arc-strong but not locally $n$-arc-strong.

**Proof.** Let $D$ be a diregular tournament of $2n + 3$ vertices. Let $S$ be a subset of $V(D)$ with $|S| = n - 1$, and let $\overline{S} = V(D) - S$. Then $|\overline{S}| = n + 4$. Let $D_1$ be an isomorphic copy of $D$ under the isomorphism $\varphi: V(D) \rightarrow V(D_1)$. Let $S_1 = \varphi(S)$ and $\overline{S}_1 = \varphi(\overline{S})$. Then we extend the digraph $D \cup D_1$ to a tournament $H$ by adding arcs between $V(D)$ and $V(D_1)$ so that it satisfies the condition $((V(D), V(D_1)) = \{(x, \varphi(x))|x \in S\}$. Then by Lemma 1 of [5] (which says that a digraph $D$ is $n$-arc-strong if and only if $|(S, \overline{S})_D| \geq n$ for every nonempty proper subset $S$ of $V(D)$ (where $\overline{S} = V(D) - S$), we see that $H$ is not $n$-arc-strong since $|(V(D), V(D_1))| = 322$
\[ |S| = n - 1 < n. \] Now we construct the desired tournament \( T \) from \( H \) by adding a new vertex \( x \) and adding all the arcs in \((\{x\}, S \cup \bar{S} \cup (\bar{S} \cup S, \{x\})\). It is easily seen that \( \delta(T) = n + 2 \).

Note that \[ \left\lceil \frac{V(T)+2}{4} \right\rceil = \left\lceil \frac{2(2n+3)+1}{4} + 2 \right\rceil = n + 2. \] Then by Lemma 1, \( T \) is \( \delta(T) \)-arc-strong, implying that \( T \) is \((n+1)\)-arc-strong. However, since \( N_T(x) = H \), \( T \) is not locally \( n \)-arc-strong. \( \square \)

Acknowledgement.

I would like to thank Professor Gary Chartrand for his kindly sending me a copy of his joint paper [4] with Professor Raymond E. Pippert which inspired this work.

References


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