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_Czechoslovak Mathematical Journal_, Vol. 46 (1996), No. 2, 331–333

Persistent URL: [http://dml.cz/dmlcz/127295](http://dml.cz/dmlcz/127295)

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CONSTRUCTION OF ALL HOMOMORPHISMS OF MONO-\(n\)-ARY ALGEBRAS

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(Received December 16, 1994)

In [3] a construction of all homomorphisms of a groupoid into another one is described. In the present paper we present a generalization of this result, i.e., a construction of all homomorphisms of an algebra with one \(n\)-ary operation into another algebra of the same type. The proofs are omitted because they may be easily obtained from the proofs of [3]. Our generalized construction is needed, e.g., if constructing all strong homomorphisms of a structure with one \(n+1\)-ary relation into another structure of the same type as described in Corollary 2 of [2].

Let \(n\) be an integer such that \(n \geq 2\). If \(A\) is an arbitrary set, we denote by \(A^n\) the Cartesian product \(\times \{A_i; 1 \leq i \leq n\}\) where \(A_i = A\) for any \(i\) with \(1 \leq i \leq n\).

Suppose that \(A, A'\) are sets and \(n \geq 2\) an integer. A mapping \(f\) of \(A^n\) into \((A')^n\) is said to be \(n\)-decomposable if there exists a mapping \(h\) of \(A\) into \(A'\) such that \(f(x_1, \ldots, x_n) = (h(x_1), \ldots, h(x_n))\) for any \((x_1, \ldots, x_n) \in A^n\). Then we write \(f = h^n\).

Let \(n \geq 1\) be an integer. We denote by \((A, o)\) an algebraic structure where \(o\) is an \(n\)-ary operation on the set \(A\). This structure will be called a mono-\(n\)-ary algebra. Furthermore, we denote by \(\text{ALG}n\) the category whose objects are mono-\(n\)-ary algebras and whose morphisms are homomorphisms of these algebras. (The symbol \(\text{ALG}n\) in [2] has a different meaning!)

Let \(A\) be a set, \(n \geq 2\) an integer. A unary operation \(w\) on the set \(A^n\) is said to be binding if for any \((x_1, \ldots, x_n) \in A^n\) the condition \(w(x_1, \ldots, x_n) = (y_1, \ldots, y_n)\) implies that \(x_i = y_{i-1}\) for any \(i\) with \(2 \leq i \leq n\). An algebra \((A^n, w)\) with a binding unary operation \(w\) will be called a binding unary \(n\)-algebra.

We now define a category \(\text{MAP}n\). Its objects are binding unary \(n\)-algebras, its morphisms are \(n\)-decomposable homomorphisms of these \(n\)-algebras.

We now present a functor \(F\) of the category \(\text{ALG}n\) into \(\text{MAP}n\) by presenting the object mapping \(F_0\) and the morphism mapping \(Fm\).
If \((A, o)\) is an object in the category \(\text{ALG}n\), we define \(\text{un}[o](x_1, \ldots, x_n) = (x_2, \ldots, x_n, o(x_1, \ldots, x_n))\) for any \((x_1, \ldots, x_n) \in A^n\). Clearly, \((A^n, \text{un}[o])\) is an object in the category \(\text{MAP}n\). We put

\[
F_0(A, o) = (A^n, \text{un}[o]).
\]

Let \((A, o), (A', o')\) be objects in \(\text{ALG}n\), \(h\) a homomorphism of \((A, o)\) into \((A', o')\). It is easy to see that \(h^n\) is a morphism of \(F_0(A, o)\) into \(F_0(A', o')\) in the category \(\text{MAP}n\). We put

\[
F_\text{m}(h) = h^n.
\]

Similarly as Theorem 5 in [3] we obtain

**Theorem.** Let \(n \geq 2\) be an integer. The functor \(F\) is an isomorphism of the category \(\text{ALG}n\) onto the category \(\text{MAP}n\).

A generalization of Corollary 3 in [3] reads as follows.

**Corollary.** Let \(n \geq 2\) be an integer, \((A, o), (A', o')\) mono-\(n\)-ary algebras.

(i) For any homomorphism \(h\) of \((A, o)\) into \((A', o')\) there exists an \(n\)-decomposable homomorphism \(f\) of \((A^n, \text{un}[o])\) into \(((A')^n, \text{un}[o'])\) such that \(f = h^n\).

(ii) If \(f\) is an \(n\)-decomposable homomorphism of \((A^n, \text{un}[o])\) into \(((A')^n, \text{un}[o'])\), then \(f = h^n\) and \(h\) is a homomorphism of \((A, o)\) into \((A', o')\).

Construction from [3] may be generalized as follows.

**Construction.** Let \(n \geq 2\) be an integer, let mono-\(n\)-ary algebras \((A, o), (A', o')\) be given.

Construct the mono-unary algebras \((A^n, \text{un}[o])\) and \(((A')^n, \text{un}[o'])\).

Construct all homomorphisms of \((A^n, \text{un}[o])\) into \(((A')^n, \text{un}[o'])\) using the construction described in [1].

Test the constructed homomorphisms and reject all of them that are not \(n\)-decomposable.

For any \(n\)-decomposable homomorphism \(f\) of \((A^n, \text{un}[o])\) into \(((A')^n, \text{un}[o'])\) construct the mapping \(h\) such that \(f = h^n\).

By Corollary, we obtain that any constructed mapping \(h\) is a homomorphism of \((A, o)\) into \((A', o')\) and that any homomorphism of \((A, o)\) into \((A', o')\) can be constructed in this way.
Application. Let $n \geq 1$ be an integer, $A, A'$ sets, $t$ a relation of arity $n + 1$ on $A$, $t'$ a relation of the same arity on $A'$. In Corollary 2 of [2] a construction of all strong homomorphisms of the structure $(A, t)$ into $(A', t')$ is described: We construct mono-$n$-ary algebras $(P(A), R[t])$ and $(P(A'), R[t'])$ where $P(A) = \{X; X \subseteq A\}, R[t](X_1, \ldots, X_n) = \{x \in A; (x_1, \ldots, x_n, x) \in t, x_1 \in X_1, \ldots, x_n \in X_n\}$ for any $X_1, \ldots, X_n$ in $P(A); P(A'), R[t']$ are defined in a similar way. Furthermore, we construct all homomorphisms of the first algebra into the other using [1] or the presented Construction. Then we choose all of them that are totally additive and atom-preserving in the sense of [2]. Any of them defines a strong homomorphism of $(A, t)$ into $(A', t')$ and any strong homomorphism of $(A, t)$ into $(A', t')$ can be obtained in this way.

It is easy to see that the above constructed isomorphism $F$ of the category $\textbf{ALG}^n$ onto $\textbf{MAP}^n$ is not the only possible isomorphism of $\textbf{ALG}^n$ onto a category of mono-unary algebras. The other isomorphisms define a relationship between mono-$n$-ary algebras and mono-unary algebras that is different from the relationship that has been presented here.

References


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