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# CONSTRUCTION OF ALL HOMOMORPHISMS OF MONO-$n$-ARY ALGEBRAS

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In [3] a construction of all homomorphisms of a groupoid into another one is described. In the present paper we present a generalization of this result, i.e., a construction of all homomorphisms of an algebra with one $n$-ary operation into another algebra of the same type. The proofs are omitted because they may be easily obtained from the proofs of [3]. Our generalized construction is needed, e.g., if constructing all strong homomorphisms of a structure with one $n+1$-ary relation into another structure of the same type as described in Corollary 2 of [2].

Let $n$ be an integer such that $n \geqslant 2$. If $A$ is an arbitrary set, we denote by $A^n$ the Cartesian product $\times\{A_i\,;\,1 \leqslant i \leqslant n\}$ where $A_i = A$ for any $i$ with $1 \leqslant i \leqslant n$.

Suppose that $A$, $A'$ are sets and $n \geqslant 2$ an integer. A mapping $f$ of $A^n$ into $(A')^n$ is said to be *$n$-decomposable* if there exists a mapping $h$ of $A$ into $A'$ such that $f(x_1, \ldots, x_n) = (h(x_1), \ldots, h(x_n))$ for any $(x_1, \ldots, x_n) \in A^n$. Then we write $f = h^n$.

Let $n \geqslant 1$ be an integer. We denote by $(A, o)$ an algebraic structure where $o$ is an $n$-ary operation on the set $A$. This structure will be called a *mono-$n$-ary algebra*. Furthermore, we denote by $\mathbf{ALG}n$ the category whose objects are mono-$n$-ary algebras and whose morphisms are homomorphisms of these algebras. (The symbol $\mathbf{ALG}n$ in [2] has a different meaning!)

Let $A$ be a set, $n \geqslant 2$ an integer. A unary operation $w$ on the set $A^n$ is said to be *binding* if for any $(x_1, \ldots, x_n) \in A^n$ the condition $w(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ implies that $x_i = y_{i-1}$ for any $i$ with $2 \leqslant i \leqslant n$. An algebra $(A^n, w)$ with a binding unary operation $w$ will be called a *binding unary $n$-algebra*.

We now define a category $\mathbf{MAP}n$. Its objects are binding unary $n$-algebras, its morphisms are $n$-decomposable homomorphisms of these $n$-algebras.

We now present a functor $F$ of the category $\mathbf{ALG}n$ into $\mathbf{MAP}n$ by presenting the object mapping $Fo$ and the morphism mapping $Fm$.

If $(A, o)$ is an object in the category $\mathbf{ALG}n$, we define $\mathbf{un}[o](x_1, \ldots, x_n) = (x_2, \ldots, x_n, o(x_1, \ldots, x_n))$ for any $(x_1, \ldots, x_n) \in A^n$. Clearly, $(A^n, \mathbf{un}[o])$ is an object in the category $\mathbf{MAP}n$. We put

$$Fo(A, o) = (A^n, \mathbf{un}[o]).$$

Let $(A, o)$, $(A', o')$ be objects in $\mathbf{ALG}n$, $h$ a homomorphism of $(A, o)$ into $(A', o')$. It is easy to see that $h^n$ is a morphism of $Fo(A, o)$ into $Fo(A', o')$ in the category $\mathbf{MAP}n$. We put

$$Fm(h) = h^n.$$

Similarly as Theorem 5 in [3] we obtain

**Theorem.** *Let $n \geqslant 2$ be an integer. The functor $F$ is an isomorphism of the category $\mathbf{ALG}n$ onto the category $\mathbf{MAP}n$.*

A generalization of Corollary 3 in [3] reads as follows.

**Corollary.** *Let $n \geqslant 2$ be an integer, $(A, o)$, $(A', o')$ mono-$n$-ary algebras.*

(i) *For any homomorphism $h$ of $(A, o)$ into $(A', o')$ there exists an $n$-decomposable homomorphism $f$ of $(A^n, \mathbf{un}[o])$ into $((A')^n, \mathbf{un}[o'])$ such that $f = h^n$.*

(ii) *If $f$ is an $n$-decomposable homomorphism of $(A^n, \mathbf{un}[o])$ into $((A')^n, \mathbf{un}[o'])$, then $f = h^n$ and $h$ is a homomorphism of $(A, o)$ into $(A', o')$.*

Construction from [3] may be generalized as follows.

**Construction.** Let $n \geqslant 2$ be an integer, let mono-$n$-ary algebras $(A, o)$, $(A', o')$ be given.

Construct the mono-unary algebras $(A^n, \mathbf{un}[o])$ and $((A')^n, \mathbf{un}[o'])$.

Construct all homomorphisms of $(A^n, \mathbf{un}[o])$ into $((A')^n, \mathbf{un}[o'])$ using the construction described in [1].

Test the constructed homomorphisms and reject all of them that are not $n$-decomposable.

For any $n$-decomposable homomorphism $f$ of $(A^n, \mathbf{un}[o])$ into $((A')^n, \mathbf{un}[o'])$ construct the mapping $h$ such that $f = h^n$.

By Corollary, we obtain that any constructed mapping $h$ is a homomorphism of $(A, o)$ into $(A', o')$ and that any homomorphism of $(A, o)$ into $(A', o')$ can be constructed in this way.

**Application.** Let $n \geqslant 1$ be an integer, $A$, $A'$ sets, $t$ a relation of arity $n+1$ on $A$, $t'$ a relation of the same arity on $A'$. In Corollary 2 of [2] a construction of all strong homomorphisms of the structure $(A,t)$ into $(A',t')$ is described: We construct mono-$n$-ary algebras $(\boldsymbol{P}(A), \mathbf{R}[t])$ and $(\boldsymbol{P}(A'), \mathbf{R}[t'])$ where $\boldsymbol{P}(A) = \{X;\ X \subseteq A\}$, $\mathbf{R}[t](X_1, \ldots, X_n) = \{x \in A;\ (x_1, \ldots, x_n, x) \in t, x_1 \in X_1, \ldots, x_n \in X_n\}$ for any $X_1, \ldots, X_n$ in $\boldsymbol{P}(A)$; $\boldsymbol{P}(A')$, $\mathbf{R}[t']$ are defined in a similar way. Furthermore, we construct all homomorphisms of the first algebra into the other using [1] or the presented Construction. Then we choose all of them that are totally additive and atom-preserving in the sense of [2]. Any of them defines a strong homomorphism of $(A,t)$ into $(A',t')$ and any strong homomorphism of $(A,t)$ into $(A',t')$ can be obtained in this way.

It is easy to see that the above constructed isomorphism $F$ of the category $\mathbf{ALG}n$ onto $\mathbf{MAP}n$ is not the only possible isomorphism of $\mathbf{ALG}n$ onto a category of mono-unary algebras. The other isomorphisms define a relationship between mono-$n$-ary algebras and mono-unary algebras that is different from the relationship that has been presented here.

*References*

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