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WEAK CALIBERS AND THE SCOTT-WATSON THEOREM

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Let k be an infinite cardinal number. A collection $\mathscr{U}$ of subsets of a space X is said to be point-k if each point $x \in X$ is in fewer than k members of $\mathscr{U}$. A collection $\mathscr{U}$ is locally-k at a point x if there is an open neighbourhood of x meeting fewer than k members of $\mathscr{U}$. If every point-k open cover of a space X is locally-k at a dense set of points then we say that X has weak caliber k. A space X has very weak caliber k if every point-k open cover $\mathscr{U}$ of X such that $|\mathscr{U}| \leqslant k$ is locally-k at a dense set of points. Recall that a space X has caliber k if every point-k collection of open sets has cardinality less than k. Obviously caliber $k \Rightarrow$ weak caliber $k \Rightarrow$ very weak caliber k. If X is a ccc space (i.e. every collection of pairwise disjoint non-empty open subsets of X is countable) and k is a cardinal of uncountable cofinality then it follows easily by Prop. 3.4 in [10] that X has caliber k iff it has weak caliber k.

X is a k-Baire space if the intersection of fewer than k dense open sets is dense [10]. Thus the $\aleph_1$-Baire spaces are the usual Baire spaces. It is well-known that a space X is a Baire space iff it has weak caliber $\aleph_0$ iff it has very weak caliber $\aleph_0$ ([2], [3]). Moreover, it is known that if k is regular and X is k^+-Baire then X has very weak caliber k [1]. If X is almost k-discrete (i.e. every non-empty intersection of fewer than k open sets has non-empty interior) and k is regular then X is k^+-Baire iff it is k-Baire and has very weak caliber k [1]. It would be interesting, for a regular cardinal k, to know whether there exists a space which has very weak caliber k but has not weak caliber k .

In the sequel no separation axiom is assumed, unless explicitly stated. A space X is almost k-metacompact if for every open cover $\mathscr{U}$ of X there are an open refinement $\mathscr{V}$ of $\mathscr{U}$ and an open dense subset D of X such that $\mathscr{V}$ is point-k on D. Almost $\aleph_0$-metacompact (almost $\aleph_1$-metacompact) spaces are called almost metacompact (almost metaLindelöf) [7]. The following property is a stronger one: X is quasi k-

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metacompact if for every open cover $\mathscr{U}$ of X there are an open refinement $\mathscr{V}$ of $\mathscr{U}$ and an open dense subset D of X such that $\mathscr{V}$ is point-k on D and for every $\mathscr{W} \subset \mathscr{V}$ with $|\mathscr{W}| \geqslant k$, it follows that $|\{W \cap D \colon W \in \mathscr{W}\}| \geqslant k$. Quasi $\aleph_0$-metacompact (quasi $\aleph_1$-metacompact) spaces are called quasi metacompact (quasi metaLindelöf). If k is a regular cardinal then every almost k-metacompact space is quasi k-metacompact. Let us consider an open cover $\mathscr{U}$ of an almost k-metacompact space X, let $\mathscr{V}$ be an open refinement of $\mathscr{U}$ and D an open dense subset of X such that $\mathscr{V}$ is point-k on D. Let us show that if $\mathscr{W} \subset \mathscr{V}$ and $\mathscr{G} = \{W \cap D \colon W \in \mathscr{W}\}$ has cardinality $< k$ then $|\mathscr{W}| < k$. Let $\lambda = |\mathscr{G}|$ and let $\mathscr{G} = \{G_\alpha : \alpha \in \lambda\}$. For every $G_\alpha \in \mathscr{G}$ let $\mathscr{D}(G_\alpha) = \{W \in \mathscr{W} : W \cap D = G_\alpha\}$. Take a point x in G_α. Then obviously $\mathscr{D}(G_\alpha) \subset \mathscr{V}_x = \{V \in \mathscr{V} : x \in V\}$, and since $x \in D$ and $\mathscr{V}$ is point-k on D it follows that $|\mathscr{D}(G_\alpha)| \leqslant |\mathscr{V}_x| < k$. Hence $\mathscr{W} = \bigcup_{\alpha<\lambda} \mathscr{D}(G_\alpha)$, $\lambda < k$, and k is regular, therefore $|\mathscr{W}| < k$.

X is weakly k-compact if each open cover $\mathscr{U}$ of X has a subfamily $\mathscr{V}$, $|\mathscr{V}| < k$, with a dense union ([5], see also [4]). Weakly $\aleph_0$-compact (weakly $\aleph_1$-compact) spaces are called weakly compact (weakly Lindelöf). Obviously a regular weakly compact space is compact.

A space X is feebly k-compact if every discrete family of non-empty open subsets of X has cardinality $< k$ (if X is a regular space this is equivalent to saying that every locally finite family of non-empty open subsets of X has cardinality $< k$). Feebly $\aleph_0$-compact (feebly $\aleph_1$ compact) spaces are called feebly compact (feebly Lindelöf). Clearly a Tychonoff space is feebly compact iff it is pseudocompact.

Remark 1. A space X is quasi-regular [8] if for every non-empty open subset V of X there is a non-empty open subset U of X such that $\overline{U} \subset V$. If X is a quasi-regular weakly k-compact space then it is feebly k-compact. Let us suppose that there is a discrete family $\mathscr{U} = \{U_\alpha \colon \alpha < k\}$ of non-empty open subsets of X. For each $\alpha < k$ let V_α be a non-empty open set such that $\overline{V}_\alpha \subset U_\alpha$. Set $V = X - \bigcup\{\overline{V}_\alpha : \alpha < k\}$; $\{V_\alpha : \alpha < k\}$ is a discrete family so V is an open subset of X. Then $\mathscr{U} \cup \{V\}$ is an open cover of X such that for each $\mathscr{V} \subset \mathscr{U}$ with $|\mathscr{V}| < k$, $\bigcup \mathscr{V}$ is not dense in X.

Lemma 2. *Let k be a regular cardinal and let X be feebly k-compact. If $\mathscr{U}$ is an open cover of X which is locally-k on a dense subset of X, then $\mathscr{U}$ contains a subfamily $\mathscr{V}$ such that $|\mathscr{V}| < k$ and $\overline{\bigcup \mathscr{V}} = X$.*

P r o o f. Let $\mathscr{U}$ be an open cover of X which is locally-k on a dense set D. Let $\mathscr{C}$ be the collection of all families $\mathscr{G}$ of open subsets of X such that

(i) $|\{U \in \mathscr{U} : G \cap U \neq \emptyset\}| < k$ for each $G \in \mathscr{G}$,

(ii) $|\{G \in \mathscr{G} : U \cap G \neq \emptyset\}| \leqslant 1$ for each $U \in \mathscr{U}$.

$(\mathscr{C}, \subseteq)$ is a poset and every linearly ordered subset of $\mathscr{C}$ has an upper bound, hence by Zorn's lemma there is a maximal element $\mathscr{M}$ of $\mathscr{C}$. Clearly $\mathscr{M}$ is a discrete family, moreover X is feebly k-compact so $|\mathscr{M}| < k$. Let $\mathscr{V} = \{U \in \mathscr{U} : U \cap V \neq \emptyset$ for some $V \in \mathscr{M}\}$. Since k is regular so $|\mathscr{V}| < k$.

It remains to show that $\overline{\bigcup \mathscr{V}} = X$. Suppose there is an $x \in D \cap (X - \overline{\bigcup \mathscr{V}})$, let W be an open neighbourhood of x such that $W \subseteq X - \overline{\bigcup \mathscr{V}}$ and $|\{U \in \mathscr{U} : W \cap U \neq \emptyset\}| < k$. Then $\mathscr{M} \cup \{W\}$ satisfies (i) and (ii) and $\mathscr{M}$ is not maximal, a contradiction. □

Lemma 3. *If X has weak caliber k and G is an open subset of X then G has weak caliber k.*

Proof. Let $\mathscr{U}$ be a point-k open cover of G. Then $\mathscr{V} = \mathscr{U} \cup \{X\}$ is a point-k open cover of X. X has weak caliber k, so $D = \{x \in X : \mathscr{V}$ is locally-k at $x\}$ is dense in X, therefore $\mathscr{U}$ is locally-k on the dense subset $D \cap G$ of G. If $x \in D \cap G$ then there is an open neighbourhood U_x of x in X such that $|\{V \in \mathscr{V} : V \cap U_x \neq \emptyset\}| < k$, therefore $G_x = U_x \cap G$ is an open neighbourhood of x in G such that $|\{U \in \mathscr{U} : U \cap G_x \neq \emptyset\}| < k$. □

Proposition 4. *Let X be a quasi k-metacompact space with weak caliber k. If $\mathscr{U}$ is an open cover of X then there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ which is locally-k at an open dense subset of X.*

Proof. Let $\mathscr{U}$ be an open cover of X, by hypothesis there are an open refinement $\mathscr{V}$ of $\mathscr{U}$ and an open dense subset D of X such that $\mathscr{V}$ is point-k on D and for every $\mathscr{W} \subset \mathscr{V}$ with $|\mathscr{W}| \geqslant k$, it follows that $|\{W \cap D : W \in \mathscr{W}\}| \geqslant k$. $\mathscr{A} = \{V \cap D : V \in \mathscr{V}\}$ is a point-k open cover of D, D is open in X and X has weak caliber k, hence by Lemma 3 D has weak caliber k. Therefore $G = \{x \in D : \exists$ an open neighbourhood U_x of x in D meeting fewer than k members of $\mathscr{A}\}$ is dense in D, obviously G is open in D and hence in X. To complete the proof we show that $\mathscr{V}$ is locally-k at the open dense subset G of X. Let $x \in G$, then there is an open neighbourhood U_x of x in D such that $|\mathscr{A}_x| < k$, where $\mathscr{A}_x = \{A \in \mathscr{A} : A \cap U_x \neq \emptyset\}$; obviously U_x is an open neighbourhood of x in X. Let $\mathscr{W} = \{V \in \mathscr{V} : V \cap U_x \neq \emptyset\}$, if $|\mathscr{W}| \geqslant k$ then by the quasi k-metacompactness of X it follows that $\{V \cap D : V \in \mathscr{W}\}$ is a subset of $\mathscr{A}_x$ having cardinality $\geqslant k$, a contradiction. Hence $\mathscr{V}$ is locally-k at x. □

Theorem 5. *Let k be a regular cardinal and let X be a space which has weak caliber k. If X is feebly k-compact and almost k-metacompact then X is weakly k-compact.*

Proof. Let k be a regular cardinal and let X be a feebly k-compact almost k-metacompact space which has weak caliber k. Let $\mathscr{U}$ be an open cover of X, X

is quasi k-metacompact (k is regular), hence it follows by Prop. 4 that there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ which is locally-k at an open dense subset of X. Then by Lemma 2 there exists a $\mathscr{W} \subset \mathscr{V}$ such that $|\mathscr{W}| < k$ and $\overline{\bigcup \mathscr{W}} = X$. For each $W \in \mathscr{W}$ choose an element $U(W)$ of $\mathscr{U}$ such that $W \subset U(W)$. $\mathscr{G} = \{U(W): W \in \mathscr{W}\}$ is a subcollection of $\mathscr{U}$ such that $|\mathscr{G}| < k$ and $\overline{\bigcup \mathscr{G}} = X$. So X is weakly k-compact. □

For the special case $k = \aleph_0$ we obtain the following result: every feebly compact almost metacompact Baire space is weakly compact.

It is known that a regular feebly compact space is a Baire space [6], therefore a regular space is weakly compact (and hence compact) if and only if it is feebly compact and almost metacompact ([7], Thm. 1).

In particular, we have the following

Corollary 6 (Scott-Watson theorem). *Every Tychonoff pseudocompact metacompact space is compact.*

Remark 7. Theorem 5, for $k = \aleph_1$, says that an almost metaLindelöf feebly Lindelöf space which has weak caliber $\aleph_1$ is weakly Lindelöf. The example given in [12] shows (as pointed out in [7]) that a Tychonoff pseudocompact metaLindelöf space need not be weakly Lindelöf. In [7] it is also shown that a regular Baire space is weakly Lindelöf iff it is feebly Lindelöf and almost θ-refinable.

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