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ON f -DOMINATION NUMBER OF A GRAPH

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1. INTRODUCTION

The domination number $\gamma(G)$ of a graph G is the smallest cardinality of a set D of vertices such that every vertex outside D has at least one neighbor in D . Extensive studies on domination number and domination-related topics have been done in the past thirty years. Recently, some new domination models have been proposed. For example, [4, 5] studied the k -domination number. For a positive integer k , a subset D of $V(G)$ is a k -dominating set of G if each vertex of $V(G) \setminus D$ is adjacent to at least k distinct vertices of D . A k -independent set T is a subset of $V(G)$ such that the maximum degree of the induced subgraph $G[T]$ of G is less than k . The k -domination number of G , denoted by $\gamma_k(G)$, is the cardinality of the smallest k -dominating set of G ([4, 5]). The k -independence number of G ([4, 5]), $\beta_k(G)$, is the cardinality of the largest k -independent set of G . Evidently, $\gamma_1(G)$ and $\beta_1(G)$ are, respectively, the ordinary domination number $\gamma(G)$ and the ordinary independence number $\beta(G)$.

The following result was conjectured by J.F. Fink and M.S. Jacobson ([4, 5]) and proved in [3].

Theorem 1. *For any simple graph G and positive integer k , we have $\gamma_k(G) \leq \beta_k(G)$.*

This theorem generalizes the inequality $\gamma \leq \beta$. Another upper bound for γ_k is the following

Theorem 2 ([1]). *Let n and k be positive integers, and G a graph with minimum degree $\delta(G) \geq \frac{n+1}{n}k - 1$. Then $\gamma_k(G) \leq \frac{np}{n+1}$, where $p = |V(G)|$.*

More upper bounds for γ_k can be found in [9]. In the same paper a general domination concept was introduced. For any integer-valued function f defined on

$V(G)$, a subset D of $V(G)$ is called an f -dominating set of G if $|N_G(x) \cap D| \geq f(x)$ for each $x \in V(G) \setminus D$, where $N_G(x)$ is the set of neighbors of x in G . Then the f -domination number of G , denoted by $\gamma_f(G)$, is defined to be the smallest cardinality of an f -dominating set of G . Obviously, if f is such that $f(x) = k$ for all $x \in V(G)$, then $\gamma_f(G)$ is exactly the k -domination number. If $T \subseteq V(G)$ satisfies $d_{G[T]}(x) < f(x)$ for all $x \in T$, then we call T an f -independent set of G . The maximum cardinality of f -independent sets of G is then defined to be the f -independence number, denoted by $\beta_f(G)$.

In this paper we initiate the study on γ_f and β_f . Basic results for these two invariants are discussed in the next section. Some upper bounds for γ_f are given in Section 3. In particular, Theorems 1-2 are generalized. In the last section some open problems are proposed. Throughout the paper G is a finite, undirected graph with no loops and multiedges, and $f, V(G) \rightarrow Z$ is an integer-valued function. For $D \subseteq V(G)$ and $x \in V(G)$, let $N_D(x) = N_G(x) \cap D$ and $d_D(x) = |N_D(x)|$. Let p and $\varepsilon(G)$ represent the number of vertices and the number of edges of G , respectively.

2. BASIC RESULTS

A subset S of $V(G)$ is called an f -transversal of G if it intersects all non- f -independent sets of G . The minimum cardinality of f -transversals of G is then defined to be the f -transversal number of G , denoted by $\alpha_f(G)$. The following Gallai-type equality is in fact a consequence of a more general result of [6].

Theorem 3. $\alpha_f(G) + \beta_f(G) = p$.

Proof. It can be shown that $S \subseteq V(G)$ is an f -transversal iff $V(G) \setminus S$ is an f -independent set. Then the theorem follows. □

Proposition 1. (1) If H is a spanning subgraph of G , then $\gamma_f(G) \leq \gamma_f(H)$;

(2) If $f': V(G) \rightarrow Z$ is another function satisfying $f(x) \leq f'(x)$ for all $x \in V(G)$, then $\gamma_f(G) \leq \gamma_{f'}(G)$ and $\beta_f(G) \leq \beta_{f'}(G)$.

Proposition 2. (1) If $f(x) > d(x)$ for some $x \in V(G)$, then x must belong to any f -dominating set of G ;

(2) If $f(x) < 1$ for a vertex x , then x can not be in any minimal f -dominating set of G .

Proposition 3. Let $M = \max_{x \in V(G)} f(x)$, then

$$\gamma_f(G) \geq \frac{1}{M} \left(\sum_{x \in V(G)} f(x) - \varepsilon(G) \right).$$

Proof. Let D be an f -dominating set of G with the smallest cardinality. Then

$$\sum_{x \in V(G)} f(x) - \sum_{x \in D} f(x) = \sum_{x \in V(G) \setminus D} f(x) \leq \varepsilon(G).$$

So $M \cdot \gamma_f(G) = M \cdot |D| \geq \sum_{x \in D} f(x) \geq \sum_{x \in V(G)} f(x) - \varepsilon(G)$. This completes the proof. \square

For any function $f: V(G) \rightarrow Z$, let $f^*: V(G) \rightarrow Z$ be a companion function defined by $f^*(x) = d(x) - f(x) + 1$, $x \in V(G)$. Then we have

Proposition 4.

- (1) $\gamma_f(G) + \beta_{f^*}(G) \leq p$;
- (2) $\gamma_{f^*}(G) + \beta_f(G) \leq p$.

Proof. Let T be a maximum f^* -independent set of G . Then $d_{G[T]}(x) \leq f^*(x) - 1$ for each $x \in T$. So $d_{V(G) \setminus T}(x) \geq d(x) - f^*(x) + 1 = f(x)$ for each $x \in T$. Thus $V(G) \setminus T$ is an f -dominating set of G , and (1) is true. Since $(f^*)^* = f$, (2) follows from (1) immediately. \square

Corollary 1. *If $f(x) \leq \frac{d(x)+1}{2}$ for all $x \in V(G)$, then $\gamma_f(G) + \beta_f(G) \leq p$.*

Proof. The given condition implies that $f(x) \leq f^*(x)$ for each $x \in V(G)$. Hence $\beta_f(G) \leq \beta_{f^*}(G)$ by Proposition 1(2). The corollary then follows from Proposition 4(1).

Corollary 1 generalizes a known result ([8]) that $\gamma(G) + \beta(G) \leq p$ if $p \geq 2$ and G contains no isolated vertices. \square

3. SOME UPPER BOUNDS FOR γ_f

As shown in Theorem 1, $\gamma_k(G) \leq \beta_k(G)$ for any positive integer k . Then we may naturally ask if $\gamma_f(G) \leq \beta_f(G)$ for any function f . The answer is affirmative. In fact we have the following more general result.

Theorem 4. *For any function $f: V(G) \rightarrow Z$, every f -independent set D of G such that $\sum_{x \in D} f(x) - \varepsilon(D)$ is maximum is an f -dominating set of G , where $\varepsilon(D)$ is the number of edges of $G[D]$.*

Proof. The proof is similar to that used in [3].

Suppose otherwise; then there must exist $v \in V(G) \setminus D$ such that $d_D(v) < f(v)$. Let $B = N_D(v)$, then $0 \leq |B| < f(v)$. Let

$$A = \{x \in B : d_D(x) = f(x) - 1\}$$

and let S be a maximal independent set of $G[A]$. Then $\Phi \subseteq S \subseteq A \subseteq B \subseteq D$. Let $C = (D \setminus S) \cup \{v\}$. Then C must be an f -independent set of G .

In fact,

$$\begin{aligned} d_C(v) &\leq |B| < f(v), \\ d_C(x) &\leq d_D(x) < f(x), \quad \forall x \in D \setminus B, \\ d_C(x) &\leq d_D(x) + 1 \leq (f(x) - 2) + 1 < f(x), \quad \forall x \in B \setminus A. \end{aligned}$$

Noting that S is a maximal independent set of $G[A]$, each $x \in A \setminus S$ is adjacent to at least one vertex in S . Hence

$$d_C(x) \leq (d_D(x) - 1) + 1 < f(x), \quad \forall x \in A \setminus S.$$

Thus C is indeed an f -independent set of G . We have

$$\varepsilon(C) = \varepsilon(D) - \sum_{x \in S} (f(x) - 1) + |B| - |S| = \varepsilon(D) - \sum_{x \in S} f(x) + |B|.$$

Hence,

$$\begin{aligned} \sum_{x \in C} f(x) - \varepsilon(C) &= \left(\sum_{x \in D} f(x) - \sum_{x \in S} f(x) + f(v) \right) - \left(\varepsilon(D) - \sum_{x \in S} f(x) + |B| \right) \\ &= \sum_{x \in D} f(x) - \varepsilon(D) + f(v) - |B| > \sum_{x \in D} f(x) - \varepsilon(D), \end{aligned}$$

contradicting the choice of D . This completes the proof. \square

Corollary 2. For any graph G and any function $f: V(G) \rightarrow Z$, we have $\gamma_f(G) \leq \beta_f(G)$.

Proof. By Theorem 4 there exists an f -dominating set D which is also an f -independent set. So $\gamma_f(G) \leq |D| \leq \beta_f(G)$. \square

Let f^* be defined as in Section 2, then we have

Corollary 3.

- (1) $\gamma_f(G) + \gamma_{f^*}(G) \leq p$;
- (2) $\gamma_f(G) \cdot \gamma_{f^*}(G) \leq \left(\frac{p}{2}\right)^2$.

PROOF. By Theorem 4 we can choose an f -dominating set of G which is also an f -independent set. Thus for any $x \in D$,

$$|N_G(x) \cap (V \setminus D)| \geq d_G(x) - (f(x) - 1) = f^*(x).$$

So $V \setminus D$ is an f^* -dominating set of G . This implies (1). (2) is a direct consequence of (1). \square

Combining Corollaries 1-2, we have

Corollary 4. *Let $f: V(G) \rightarrow Z$ be such that $f(x) \leq \frac{1}{2}(d(x) + 1)$, $\forall x \in V(G)$, then $\gamma_f(G) \leq \frac{1}{2}p$.*

Corollary 4 generalizes an early result of Ore which states that $\gamma(G) \leq \frac{1}{2}p$ if G has no isolated vertices.

The idea used in [1] can be applied to prove the following result, which generalizes Theorem 2.

Theorem 5. *Let n be a positive integer and let $f: V(G) \rightarrow Z$ be such that $f(x) \leq \frac{n}{n+1}(d_G(x) + 1)$, $\forall x \in V(G)$. Then $\gamma_f(G) \leq \frac{np}{n+1}$.*

PROOF. Let V_1, V_2, \dots, V_{n+1} be a partition of $V(G)$ such that $E' = E(G) \setminus \bigcup_{i=1}^{n+1} E(G[V_i])$ contains as many edges as possible. Then by a theorem of Erdős ([2]) $d_H(x) \geq \left\lceil \frac{n}{n+1} d_G(x) \right\rceil$, $\forall x \in V(G)$, where $H = (V(G), E')$ and $\lceil a \rceil$ is the smallest integer not less than a . The condition $f(x) \leq \frac{n}{n+1}(d_G(x) + 1)$ implies $d_G(x) \geq \frac{n+1}{n} f(x) - 1$. This gives

$$d_H(x) \geq \left\lceil \frac{n}{n+1} \left(\frac{n+1}{n} f(x) - 1 \right) \right\rceil = \left\lceil f(x) - \frac{n}{n+1} \right\rceil = f(x), \quad \forall x \in V(G).$$

Without loss of generality we may suppose $|V_1| = \max_{1 \leq i \leq n+1} |V_i|$. By the above discussion, $\bigcup_{i=2}^{n+1} V_i$ is an f -dominating set of G . Thus

$$\gamma_f(G) \leq p - |V_1| \leq p - \frac{p}{n+1} = \frac{np}{n+1}.$$

\square

Corollary 5. *Let $n_0 = \max_{x \in V(G)} \left\lceil \frac{f(x)}{f^*(x)} \right\rceil$ ($f(x) \neq d(x) + 1$ for all $x \in V(G)$). Then $\gamma_f(G) \leq \frac{n_0 p}{n_0 + 1}$.*

Note that this corollary generalizes Corollary 4.

For any $f_i: V(G) \rightarrow Z$ with $1 \leq f(x) \leq d_G(x)$, $x \in V(G)$, define a function $f-1: V(G) \rightarrow Z$ such that

$$(f-1)(x) = \max\{1, f(x) - 1\}, \quad x \in V(G).$$

Inductively define the function $f - (i + 1) = (f - i) - 1$ for any positive integer i . Then it is not difficult to see that $\gamma_{f-m} = \gamma(G)$, where $m = \max_{x \in V(G)} f(x) - 1$. To investigate the relation between γ_f and γ , we prove the following

Theorem 6. For any function $f: V(G) \rightarrow Z$ satisfying $1 \leq f(x) \leq d(x)$, $x \in V(G)$, we have

$$\gamma_f(G) \leq \frac{1}{2}(p + \gamma_{f-1}(G)).$$

Proof. Let D_1 be an $(f-1)$ -dominating set of G with the cardinality $\gamma_{f-1}(G)$, and

$$S = \{x \in V(G) \setminus D_1 : f(x) = 1\}.$$

Then

$$(f-1)(x) = \begin{cases} 1, & x \in S. \\ f(x) - 1, & x \in V(G) \setminus (D_1 \cup S). \end{cases}$$

Let A, B be, respectively, the set of non-isolated vertices and the set of isolated vertices of $G[V(G) \setminus (D_1 \cup S)]$. Let T be a minimum dominating set of $G[A]$. Then by Ore's theorem (mentioned earlier), $|T| \leq \frac{1}{2}|A|$.

It is easy to see that $D_1 \cup B \cup T$ is an f -dominating set of G , so

$$(1) \quad \gamma_f(G) \leq \gamma_{f-1}(G) + |B| + |T| \leq \gamma_{f-1}(G) + |B| + \frac{|A|}{2}.$$

On the other hand $D_1 \cup S \cup T$ is also an f -dominating set of G . In fact for any $x \in B$, $d_{D_1}(x) \geq (f-1)(x) = f(x) - 1$. If $d_{D_1}(x) = f(x) - 1 \leq d_G(x) - 1$, then x must be adjacent to a vertex of S . Thus $d_{D_1 \cup S \cup T}(x) \geq f(x)$. If $d_{D_1}(x) = f(x)$, then $d_{D_1 \cup S \cup T}(x) \geq f(x)$ as well. It is obvious that $d_{D_1 \cup S \cup T}(x) \geq f(x)$ for any $x \in A \setminus T$. So $D_1 \cup S \cup T$ is indeed an f -dominating set. Thus

$$(2) \quad \gamma_f(G) \leq \gamma_{f-1}(G) + |S| + |T| \leq \gamma_{f-1}(G) + |S| + \frac{|A|}{2}.$$

Combining (1) and (2) we get

$$\gamma_f(G) \leq \gamma_{f-1}(G) + \frac{1}{2}(|A| + |B| + |S|) = \frac{1}{2}(p + \gamma_{f-1}(G)).$$

□

Corollary 6. For any i with $1 \leq i \leq m = \max_{x \in V(G)} f(x) - 1$,

$$\gamma_f(G) \leq p - \frac{1}{2^i}(p - \gamma_{f-i}(G)).$$

In particular,

$$\gamma_f(G) \leq p - \frac{1}{2^m}(p - \gamma(G)).$$

For each k , $1 \leq k \leq \Delta(G)$, let

$$\mathcal{X}_k = \{X \subseteq V(G) : |X| = k\}.$$

Denote $\Gamma_G(X) = \bigcap_{x \in X} N_G(x)$ for each $X \in \mathcal{X}_k$, and $\Delta_k(G) = \max_{X \in \mathcal{X}_k} |\Gamma_G(X)|$. Then $\Delta_k(G) \geq 1$. Let $X \in \mathcal{X}_k$ be such that $|\Gamma_G(X)| = \Delta_k(G)$ and $S = V(G) \setminus (X \cup \Gamma_G(X))$. Then $V(G) \setminus \Gamma_G(X)$ is a k -dominating set of G . Thus

$$\gamma_k(G) \leq p - \Delta_k(G)$$

or, equivalently,

$$(3) \quad |\Gamma_G(X)| = \Delta_k(G) \leq p - n,$$

where $n = \gamma_k(G)$. Suppose

$$(4) \quad |\Gamma_G(X)| = p - n - r, \quad 0 \leq r < p - n.$$

Then

$$(5) \quad |S| = p - |X| - |\Gamma_G(X)| = n + r - k.$$

Note that for any $x \in X$ and $y \in \Gamma_G(X)$, $(S \setminus N_G(y)) \cup \{x, y\}$ is a dominating set of G . Hence

$$|S| - |S \cap N_G(y)| + 2 \geq \gamma(G),$$

i.e.

$$|S \cap N_G(y)| \leq n + r - k - \gamma(G) + 2.$$

Similarly, we have

$$|S \cap N_G(x)| \leq n + r - k - \gamma(G) + 2.$$

Let h_k be the maximum number of edges in a subgraph of G with $|S| = p - \Delta_k(G) - k$ vertices. Then we have

$$\begin{aligned}
 (6) \quad 2\varepsilon(G) &\leq 2\varepsilon(G[S]) + |X|(|X| - 1) + |X| |\Gamma_G(X)| \\
 &\quad + \sum_{y \in N_G(X)} (|N_G(y) \cap S| + |N_G(y)|) + \sum_{x \in X} |N_G(x) \cap S| \\
 &\leq 2h_k + k(k - 1) + k\Delta_k + \Delta_k[(n + r - k - \gamma(G) + 2) + \Delta] \\
 &\quad + k(n + r - k - \gamma(G) + 2) \quad (\Delta_k = \Delta_k(G), \Delta = \Delta(G)) \\
 &= 2h_k + k(k - 1) + k\Delta_k + (\Delta_k + k)(p - \Delta_k - k - \gamma(G) + 2) + \Delta_k \Delta \\
 &= 2h_k + k(p - \gamma(G) + 1) - \Delta_k^2 + \Delta_k(p - k + \Delta(G) - \gamma(G) + 2) \\
 &\leq 2h_k + k(p - \gamma(G) + 1) - \Delta_k^2 + (p - n)(p - k + \Delta(G) - \gamma(G) + 2).
 \end{aligned}$$

This leads to the following

Theorem 7. For each k , $1 \leq k \leq \Delta(G)$, let Δ_k and h_k be as before. Then

$$\gamma_k(G) \leq p - \left\lceil \frac{2(\varepsilon(G) - h_k) + \Delta_k^2 - k(p - \gamma(G) + 1)}{p - k + \Delta(G) - \gamma(G) + 2} \right\rceil.$$

Since $\gamma(G) \geq 1$ and $h_k \leq \frac{1}{2}(p - \Delta_k - k)(p - \Delta_k - k - 1)$, we obtain the following two corollaries.

Corollary 7. For each k with $1 \leq k \leq \Delta(G)$,

$$\gamma_k(G) \leq p - \left\lceil \frac{2(\varepsilon(G) - h_k) + \Delta_k^2 - kp}{p - k + \Delta(G) + 1} \right\rceil.$$

Corollary 8. $\gamma_k(G) \leq p - \left\lceil \frac{2\varepsilon(G) + 2(p - k)\Delta_k + k(p + \gamma(G) - k - 2) - p(p - 1) - \Delta_k}{p - k + \Delta(G) - \gamma(G) + 2} \right\rceil$.

Taking $k = 1$ in (6) we get

$$2\varepsilon(G) \leq 2h_1 + (p - \gamma(G) + 1) - \Delta^2 + \Delta(p + \Delta + 1 - \gamma(G)).$$

This implies a new upper bound for $\gamma(G)$.

Corollary 9. Let h_1 be the maximum number of edges in a subgraph of G having $p - \Delta - 1$ vertices. Then

$$(7) \quad \gamma(G) \leq p - \left\lceil \frac{2(\varepsilon(G) - h_1)}{\Delta(G) + 1} \right\rceil + 1.$$

Example 1. Let G be the graph obtained from the cycle of five edges by adding a chord. Then $p = 5$, $\varepsilon(G) = 6$, $\Delta(G) = 3$, $\gamma(G) = 2$, $\Delta_2(G) = 2$ and $h_2 = 0$. Theorem 7 gives $\gamma_2(G) \leq 3$. But it is easy to see $\gamma_2(G) \geq 3$. So $\gamma_2(G) = 3$. This shows that the upper bound in Theorem 7 is attainable.

Example 2. If G is the cycle with four edges, then it is easy to see that both sides of (7) equal 2. So the upper bound in Corollary 9 is attainable.

For any subgraph H of G , the restriction of the function $f: V(G) \rightarrow Z$ to $V(H)$ is also denoted briefly by f . Thus $\gamma_f(H)$ is well-defined. The technique used in the proof of Theorem 7 can be applied to prove the next result, which shows the connection of $\gamma_f(G)$ and $\gamma_f(H)$.

Theorem 8. Let t_q be the maximum number of edges in a subgraph of G with $p - q$ vertices, $1 \leq q \leq p - 1$. Then for any subgraph H of G with $q (> \gamma_f(H))$ vertices and without isolated vertices,

$$\gamma_f(G) \leq p - \left\lceil \frac{2(\varepsilon(G) - t_q) + (q - a)^2 - (p - \gamma(G))\gamma_f(H)}{p + \gamma_f(H) + \Delta(G) - \gamma(G) + 1} \right\rceil.$$

Proof. Suppose that X is an f -dominating set of H with $\gamma_f(H) = a$ vertices, and that $Y = V(H) \setminus X$, $S = V(G) \setminus V(H)$. Since $X \cup S = V(G) \setminus Y$ is an f -dominating set of G , we have

$$\gamma_f(G) \leq p - |Y|,$$

or equivalently, $|Y| \leq p - n$, where $n = \gamma_f(G)$. Suppose

$$|Y| = p - n - r, \quad 0 \leq r < p - n.$$

Then $|S| = p - a - |Y| = n + r - a$.

For any $y \in Y$, $(S \setminus N_G(y)) \cup X \cup \{y\}$ is a dominating set of G , hence

$$|S| - |S \cap N_G(y)| + |X| + 1 \geq \gamma(G),$$

i.e. $|S \cap N_G(y)| \leq n + r - \gamma(G) + 1$. Similarly, $|S \cap N_G(x)| \leq p - a - \gamma(G) + 1$ for each $x \in X$. We have

$$\begin{aligned}
 2\varepsilon(G) &\leq 2\varepsilon(G[S]) + |X|(|X| - 1) + |X||Y| \\
 &\quad + \sum_{y \in Y} (|S \cap N_G(y)| + |N_G(y)|) + \sum_{x \in X} |S \cap N_G(x)| \\
 &\leq 2t_q + a(a - 1) + a|Y| + |Y|(n + r - \gamma(G) + 1 + \Delta(G)) \\
 &\quad + a(p - a - \gamma(G) + 1) \\
 &= 2t_q + a(p - \gamma(G)) + a|Y| + |Y|(p - |Y| - \gamma(G) + \Delta(G) + 1) \\
 &= 2t_q + a(p - \gamma(G)) - |Y|^2 + |Y|(p + a + \Delta(G) - \gamma(G) + 1) \\
 &\leq 2t_q + a(p - \gamma(G)) - (q - a)^2 + (p - n)(p + a + \Delta(G) - \gamma(G) + 1).
 \end{aligned}$$

This gives

$$n \leq p - \frac{2(\varepsilon(G) - t_q) + (q - a)^2 - (p - \gamma(G))a}{p + a + \Delta(G) - \gamma(G) + 1}.$$

This completes the proof. □

4. REMARKS

A lot of problems concerning the f -domination number and the f -independence number can be proposed. Perhaps the most attractive one is whether there exist the Nordhaus–Gaddum type inequalities for γ_f . Such inequalities for γ have been shown in [7]. Naturally we can define the upper f -domination number $\Gamma_f(G)$ of G to be the maximum cardinality of a minimal f -dominating set of G . Also we can define the f -domatic number, $d_f(G)$, to be the maximum order of a partition of $V(G)$ into f -dominating sets. Another interesting invariant is $i_f(G)$, which is defined to be the smallest non-negative integer i such that $\gamma_{f-i}(G) = \gamma(G)$. Studies on these invariants are necessary, as well as interesting. For example, relations among γ_f , Γ_f , β_f , d_f , i_f and other graphical invariants, e.g. the domination number, the independence number, are valuable research topics. The lower bounds and the upper bounds for γ_f and β_f deserve further study as well.

This work is on-going and results will be published later.

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