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IDEALS AND CONGRUENCE KERNELS OF ALGEBRAS

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§ 1. INTRODUCTION

The concepts of a normal subgroup, a ring ideal and a lattice ideal were extended by A. Ursini in 1972 to the notion of an ideal in universal algebras with 0 [12]. In their 1984 paper [5] H.-P. Gumm and A. Ursini studied and characterized universal algebras $\mathcal{A}$ such that every ideal $I$ of $\mathcal{A}$ is its kernel (i.e. $I = [0]_\theta$) for a unique congruence $\theta$ of $\mathcal{A}$. Such an algebra is called ideal determined. As it is well-known ideal determined algebras include groups and rings but not all lattices. In this paper we study algebras $\mathcal{A}$ with a weaker property: every ideal of $\mathcal{A}$ is the kernel of some congruence of $\mathcal{A}$. In Theorem 10 we list 8 equivalent conditions for this property. Here three conditions refer to the kernels of congruences generated by certain sets of the form $\{0\} \times S$, one condition to a certain congruence permutability around 0 and three conditions relate ideals and unary polynomials or translations of fundamental operations.

In Corollary 12 we characterize all varieties $\mathcal{V}$ (with a nullary term 0) such that for every $\mathcal{A} \in \mathcal{V}$ each ideal is a congruence kernel. This condition requires that to each at least ternary term $q(x_1, \ldots, x_n)$ of $\mathcal{V}$ in which $x_1$ appears exactly once there exists a term $p(x_1, \ldots, x_n)$ of $\mathcal{V}$ satisfying the identities

\begin{align*}
p(0, 0, 0, x_4, \ldots, x_n) &= 0 \\
q(x, x, y, x_4, \ldots, x_n) &= p(q(y, x, y, x_4, \ldots, x_n), x, y, x_4, \ldots, x_n).
\end{align*}

Finally, in Proposition 13 we give a Mal'tsev type condition for varieties with a nullary term 0 such that each $\mathcal{A} \in \mathcal{V}$ is permutatile at 0 (i.e. $[0](\theta \lor \varphi) = \{a \in A; (0, x) \in \theta \text{ and } (x, a) \in \varphi \text{ for some } x \in A\}$).

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733
**Definitions 1.** Let $\mathcal{A} = (A; F)$ be an algebra and let $0$ be a fixed element of $A$. Let $f$ be an $n$-ary term operation of $\mathcal{A}$ and let $N \subseteq \{1, \ldots, n\}$. Following [5] call $f$ an $N$-ideal term operation (or briefly an ideal term operation) of $\mathcal{A}$ if $f(a_1, \ldots, a_n) = 0$ holds whenever $a_1, \ldots, a_n \in A$ satisfy $a_i = 0$ for all $i \in N$.

For example, let $\mathcal{A} = (A; +, - , \cdot , 0)$ be a ring. Then both $x_1 + x_2$ and $x_1 - x_2$ are $\{1, 2\}$-ideal term operations of $\mathcal{A}$. Similarly $x_1 \cdot x_2$ is an $N$-ideal term operation of $\mathcal{A}$ for both $N = \{1\}$ and $N = \{2\}$. Next for a lattice $\mathcal{L} = (L; \lor, \land, 0)$ with the least element $0$ clearly $x_1 \lor x_2$ is an $\{1, 2\}$-ideal term operation of $\mathcal{L}$ and $x_1 \land x_2$ is an $N$-ideal term operation of $\mathcal{L}$ for both $N = \{1\}$ and $N = \{2\}$.

Denote by $J_{\mathcal{A}}$ the set of all ideal term operations of $\mathcal{A}$. The following fact was noted in [5]:

**Proposition 2.** The set $J_{\mathcal{A}}$ is a subclone of the clone of term operations of $\mathcal{A}$.

**Proof.** Let $1 \leq i \leq n$. Clearly the $i$-th $n$-ary projection is an $\{i\}$-ideal term operation. Let $f, g \in J_{\mathcal{A}}$ be $m$-ary and $n$-ary. Then $f$ is an $M$-ideal and $g$ is an $N$-ideal term operation of $A$ for some $M \subseteq \{1, \ldots, m\}$ and $N \subseteq \{1, \ldots, n\}$. It is easy to see that the operation $f' \ast g$ obtained from $f$ by exchanging its variables also belongs to $J_{\mathcal{A}}$. Similarly $J_{\mathcal{A}}$ is closed under any fusion of variables. Finally set $p := m + n - 1$ and define $h := f \ast g$ as the $p$-ary operation on $A$ satisfying $h(a_1, \ldots, a_p) = f(a_1, \ldots, a_m, a_{m+1}, \ldots, a_p)$ for all $a_1, \ldots, a_p \in A$. Let $M = \{i_1, \ldots, i_k\}$ and $N = \{j_1, \ldots, j_1\}$ where $1 \leq i_1 < \ldots < i_k \leq m$ and $1 \leq j_1 < \ldots < j_1 \leq n$. We have two cases:

1) If $i_1 = 1$ then $h$ is a $\{j_1, \ldots, j_1, i_2 + n - 1, \ldots, i_k + n - 1\}$-ideal term operation of $\mathcal{A}$.

2) If $i_1 > 1$ the $h$ is an $\{i_1 + n - 1, \ldots, i_k + n - 1\}$-ideal term operation.

From Mal’cev’s formalism it follows that $J_{\mathcal{A}}$ is a clone. □

**Example 3.** Let $\mathcal{A} = (A; +, - , \cdot , 0, \{a: a \in A\})$ be an associative and commutative ring (with all possible nullary operations). Let $\{F_1, \ldots, F_m\}$ be a family of not necessarily distinct subsets of $\{1, \ldots, n\}$, let $a_1, \ldots, a_m \in A$ and let $r_{ij}$ ($i \in \{1, \ldots, m\}, j \in F_i$) be positive integers. Further let $N \subseteq \{1, \ldots, r\}$. The polynomial

$$f(x_1, \ldots, x_n) := \sum_{i=1}^{m} a_i \prod_{j \in F_i} x_j^{r_{ij}}$$

is an $N$-ideal term operation of $\mathcal{A}$ if and only if $N$ meets each $F_i$ ($i = 1, \ldots, m$).

**Definition 4.** A nonempty subset $I$ of $A$ is an ideal of $\mathcal{A}$ if for every $n$-ary $N$-ideal term operation $f$ of $\mathcal{A}$

$$(2) \quad a_i \in I \quad \text{ for all } \quad i \in N \Rightarrow f(a_1, \ldots, a_n) \in I$$

holds for all $a_1, \ldots, a_n \in A$.
Notice that for rings and lattices this definition agrees with the standard one. Consider a group \( \mathcal{A} = (A; \cdot, -1, 0) \). The operations \( f(x_1, x_2) \approx x_1 x_2^{-1}, g(x_1, x_2) \approx x_1^{-1} x_2, h(x_1, x_2) \approx x_2^{-1} x_1 x_2 \) are \( \mathcal{N} \)-ideal term operations for \( \mathcal{N} \) equal \( \{1, 2\}, \{1, 2\} \) and \( \{1\} \) respectively. It follows that every ideal of \( \mathcal{A} \) is a normal subgroup of \( \mathcal{A} \). Conversely, it is not difficult to verify that every normal subgroup of \( \mathcal{A} \) is an ideal of \( \mathcal{A} \).

Denote by \( J(\mathcal{A}) \) the set of all ideals of \( \mathcal{A} \). The poset \( J(\mathcal{A}) = (J(\mathcal{A}), \subseteq) \) is a complete lattice in which

\[
\bigwedge \{J_i; i \in I\} = \bigcap\{J_i; i \in I\}
\]

for every subset \( \{J_i\}_{i \in I} \) of \( J(\mathcal{A}) \). Thus for every \( S \subseteq A \) the ideal generated by \( S \) is the least ideal \( I(S) \) of \( \mathcal{A} \) containing \( S \). We have the following description of \( I(S) \) [5]:

**Lemma 5.** Let \( S \subseteq A \). Then \( I(S) \) is the set of all \( f(a_1, \ldots, a_n) \) where \( f \) is an \( \mathcal{N} \)-ideal term operation of \( \mathcal{A} \) and \( a_1, \ldots, a_n \in A \) satisfy \( a_i \in S \) for all \( i \in \mathcal{N} \).

**Proof.** Denote by \( K \) the set defined in Lemma 5. Clearly \( K \subseteq I(S) \). Moreover, \( S \subseteq K \) because \( \text{id}_A \) is a \( \{1\} \)-ideal term operation. Thus it suffices to show that \( K \in J(\mathcal{A}) \). Let \( g \) be an \( m \)-ary \( \mathcal{M} \)-ideal term operation and let \( a_1, \ldots, a_m \in A \) satisfy \( a_k \in K \) for all \( k \in M \).

1) First consider the case \( M = \emptyset \). Then \( g \) is constant with value 0 and 0 = \( g(a_1, \ldots, a_m) \in K \).

2) Thus let \( M \neq \emptyset \). Without loss of generality we may assume that \( M = \{1, \ldots, p\} \) for some \( p \leq m \). By the definition of \( K \), for each \( 1 \leq i \leq p \) we have \( a_i = f_i(b_{i1}, \ldots, b_{i1}) \) for some \( L_i \)-ideal term operation \( f_i \) and \( b_{i1}, \ldots, b_{i1} \in A \) such that \( b_{ij} \in S \) for all \( j \in L_i \) (\( i = 1, \ldots, p \)). Set \( L' = l_1 + \ldots + l_p \) and

\[
L := \bigcup_{j=1}^{p} (L_i + l_1 + \ldots + l_{j-1}) \quad (i = 1, \ldots, p);
\]

where for every set \( X \) of positive integers and a nonnegative integer \( a \), the symbol \( X + a \) stands for \( \{x + a : x \in X\} \). Further define an \((1 + m - p)\)-ary operation \( h \) on \( A \) by setting

\[
h(c_1, \ldots, c_{1l_1}, \ldots, c_{pl_p}, c_{l+1}, \ldots, c_{l+m-p}) :=
\]

\[
:= g(f_1(c_1, \ldots, c_{1l_1}), \ldots, f_p(c_{1l_p}, \ldots, c_{l+m-p}))
\]

for all \( c_{1l_1}, \ldots, c_{pl_p}, c_{l+1}, \ldots, c_{l+m-p} \in A \). It is easy to check that \( h \) is an \( L \)-ideal term operation of \( A \). Finally

\[
g(a_1, \ldots, a_m) = h(b_{11}, \ldots, b_{pl_p}, a_{l+1}, \ldots, a_m) \in K.
\]

\[\square\]
Notice that for $J_i \in J(\mathcal{A}) \ (i \in I)$ clearly

$$\bigvee_{i \in I} J_i = I\left(\bigcup_{i \in I} J_i\right)$$

and that $\{0\}$ and $A$ are the least and greatest elements of $J(\mathcal{A})$. We abbreviate $I(\{s_1, \ldots, s_n\})$ by $I(s_1, \ldots, s_n)$.

**Definition 6.** For $S \subseteq A$ and $\varrho \subseteq A^2$ the set $[S]_\varrho := \{a \in A : (s, a) \in \varrho$ for some $s \in S\}$ is the hull of $S$ in $\varrho$. In particular, the set $[0]_\varrho := \{\{0\}\}_\varrho$ is the kernel of $\varrho$. A subset $B$ of $A$ is a congruence kernel if $B = [0]_\theta$ for some congruence $\theta$ of $\mathcal{A}$. The following lemma extends a result from [5].

**Lemma 7.** If $\varrho$ is a reflexive subuniverse of $\mathcal{A}^2$ then the kernel of $\varrho$ is an ideal of $\mathcal{A}$.

**Proof.** Let $f$ be an $n$-ary $N$-ideal term of $\mathcal{A}$ and $a_1, \ldots, a_n \in A$ satisfy $a_i \in I := [0]_\varrho$ for all $i \in N$. Set $b_i := 0$ for all $i \in N$ and $\hat{b}_i := a_i$ otherwise. Then $\beta := f(b_1, \ldots, b_n) = 0$ and $(b_i, a_i) \in \varrho$ due to $(0, a_i) \in \varrho$ for $i \in N$ and $(a_i, a_i) \in \varrho$ otherwise. Thus for $\alpha := f(a_1, \ldots, a_n)$ we have $(0, \alpha) = (\beta, \alpha) = (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \varrho$ proving $\alpha \in [0]_\varrho$. □

For the proof of the next theorem we need the following minute sharpening of a well-known result.

**Definition 8.** Let $f$ be an $n$-ary operation on $A$, let $1 \leq i \leq n$ and let $a_1, \ldots, a_n \in A$. The selfmap $r$ of $A$ defined by

$$r(x) \approx f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

is an $i$-translation (or shortly a translation) of $f$. For $\mathcal{A} = (A; F)$ denote by $P(\mathcal{A})$ and $T(\mathcal{A})$ the sets of all unary polynomials of $\mathcal{A}$ and of all translations of operations from $F$, respectively. Further, let $M(\mathcal{A})$ denote the monoid of selfmaps of $A$ generated by $T(\mathcal{A})$ and set

$$\mathcal{A}_P := (A; P(\mathcal{A})), \quad \mathcal{A}_M := (A; M(\mathcal{A})), \quad \mathcal{A}_T := (A, T(\mathcal{A})).$$

Clearly $\mathcal{A}_P, \mathcal{A}_M$ and $\mathcal{A}_T$ are unary algebras on $A$ and $P(\mathcal{A}) \supseteq M(\mathcal{A}) \supseteq T(\mathcal{A})$. The following simple example shows that $M(\mathcal{A})$ may be a proper submonoid of $P(\mathcal{A})$.

Let $\mathcal{N}_5 = (N_5; \lor, \land)$ denote the 5-element nonmodular lattice with $N_5 = \{0, a, b, c, 1\}$ and $0 < a < b < 1 > c > 0$. Set $p(x) \approx (x \lor b) \land (x \lor c)$. A direct check shows that

$$p(0) = 0, \quad p(a) = p(b) = b, \quad p(c) = c, \quad p(1) = 1.$$
Clearly $p \in P(\mathcal{A}_5')$. We show that $p \not\in M(\mathcal{A}_5')$. The translations of $\mathcal{A}_5'$ are the selfmaps $x \mapsto x \lor k$ and $x \mapsto x \land k$ with $k \in N_5$. Every map from $M(\mathcal{A}_5')$ can be expressed

\begin{equation}
(\ldots((x \lor k_1) \land h_1) \lor \ldots \lor k_n) \land h_n
\end{equation}

for suitable $n > 0$ and $k_1, \ldots, k_n, h_1, \ldots, h_n \in N_5$. Suppose $p \in M(\mathcal{A}_5')$. Choose a representation (3) of $p$ with the least possible $n$. From $p(1) = 1$ we obtain

\begin{equation}
(\ldots(h_1 \lor k_2) \land \ldots) \lor k_n = 1 = h_n
\end{equation}

while $p(0) = 0$ yields $\ldots(k_1 \land h_1) \lor \ldots \lor k_n) \land 1 = 0$ i.e.

\begin{equation}
(\ldots(k_1 \land h_1) \lor \ldots) \land h_{n-1} = 0 = k_n.
\end{equation}

By the minimality of $n$ we obtain $n = 1$ and $p(x) \approx (x \lor 0) \land 1 \approx x$. However, this contradicts $p(a) = b$. Thus $p \not\in M(N_5)$.

We have:

**Lemma 9.** Let $\mathcal{A} = (A; F)$ be an algebra. Then

(i) $\text{Con} \mathcal{A} = \text{Con} \mathcal{A}_P = \text{Con} \mathcal{A}_M = \text{Con} \mathcal{A}_T$.

(ii) The following are equivalent for $S \subseteq A$:

(a) $S$ is a block of a congruence of $\mathcal{A}$.

(b) $S \cap g(S) \neq \emptyset \Rightarrow g(S) \subseteq S$

(c) (5) holds for all $g \in P(\mathcal{A})$.

Proof. (i) From $P(\mathcal{A}) \supseteq M(\mathcal{A}) \supseteq T(\mathcal{A})$ and the fact that $P(\mathcal{A})$ is the set of unary polynomials of $\mathcal{A}$ we obtain $\text{Con} \mathcal{A} \subseteq \text{Con} \mathcal{A}_P \subseteq \text{Con} \mathcal{A}_M \subseteq \text{Con} \mathcal{A}_T$. To prove $\text{Con} \mathcal{A}_T \subseteq \text{Con} \mathcal{A}$ let $\theta \in \text{Con} \mathcal{A}_T$, let $f \in F$ be $n$-ary and let

\begin{equation}
(a_1, b_1), \ldots, (a_n, b_n) \in \theta.
\end{equation}

For $i = 0, \ldots, n$ set

\begin{equation}
c_i = f(b_1, \ldots, b_i, a_{i+1}, \ldots, a_n)
\end{equation}

and notice that $c_0 = f(a_1, \ldots, a_n)$ while $c_n = f(b_1, \ldots, b_n)$. For $i = 1, \ldots, n$ denote by $t_i$ the translation

\begin{equation}
t_i(x) \approx f(b_1, \ldots, b_{i-1}, x, a_{i+1}, \ldots, a_n).
\end{equation}
As $t_i \in T$ and $\theta \in \text{Con} \, \mathcal{A}/\mathcal{F}$, we have
\[(c_{i-1}, c_i) = (t_i(a_i), t_i(b_i)) \in \theta.\]

By transitivity,
\[(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) = (c_{m}, c_n) \in \theta.\]

(Notice that in this standard proof the symmetry of $\theta$ has not been used and so (i) holds if we replace $\text{Con}$ by Quao where Quao $\mathcal{A}$ denotes the set of all compatible quasiorders (= reflexive and transitive relations). The equality Quao $\mathcal{A} = \text{Quao} \, \mathcal{A}$ was observed in [7], p. 10). 

(ii) Let $S \subseteq A$.  (a)$\Rightarrow$(b): If $S$ is a block of some $\theta \in \text{Con} \, \mathcal{A}$, then clearly every polynomial $g$ of $\mathcal{A}$ satisfies (5). (b)$\Rightarrow$(c): Trivial. (c)$\Rightarrow$(a): Let (5) hold for every $g \in M(\mathcal{A})$. Denote by $\theta$ the reflexive and transitive hull of the binary relation $\bigcup \{g(S^2) : g \in M(\mathcal{A})\}$. It is easy to verify that $\theta \in \text{Con} \, \mathcal{A}_M$. As $\text{Con} \, \mathcal{A}_M = \text{Con} \, \mathcal{A}$ by (i), it remains to show that $S$ is a block of $\theta$. As $\text{id}_A \in M(\mathcal{A})$ clearly $S^2 \subseteq \text{id}_A(S^2) \subseteq \theta$. Suppose to the contrary that $S$ is not a block of $\theta$. By the definition of $\theta$ there exist $s, s' \in S$ and $g \in M(\mathcal{A})$ such that $g(s) \in S$ while $g(s') \notin S$ in contradiction to (5).

In this paper we study algebras $\mathcal{A}$ with 0 such that every ideal of $\mathcal{A}$ is a congruence kernel. The next theorem characterizes such algebras. As usual, for a binary relation $\rho$ on $A$ we denote by $Cg(\rho)$ the least congruence of $\mathcal{A}$ containing $\rho$. For $\rho = \{(a, b)\}$ we abbreviate $Cg(\{a, b\})$ by $Cg(a, b)$.

**Theorem 10.** The following are equivalent for an algebra $\mathcal{A} = (A, F)$ with 0:

(i) Every ideal of $\mathcal{A}$ is a congruence kernel.

(ii) $I(S) = [0]Cg(\{0\} \times S)$ for every subset $S$ of $A$.

(iii) $I(S) = [0]Cg(\{0\} \times S)$ for every finite subset $S$ of $A$.

(iv) $I(S) = [I(S \setminus \{s\})]Cg(0, s)$ for every finite nonempty subset $S$ of $A$ and each $s \in S$.

(v) For every finite subset $S = \{s_1, \ldots, s_n\}$ of $A$ and $\theta_i = Cg(0, s_i)$ ($i = 1, \ldots, n$)
\[I(S) = [0](\theta_1 \circ \ldots \circ \theta_n).\]

(vi) For every ideal $I$ of $\mathcal{A}$, all $a, b \in I$ and every $p \in P(\mathcal{A})$
\[p(a) \in I \Rightarrow p(b) \in I.\]

(vii) For every ideal $I$ of $\mathcal{A}$, all $a, b \in I$ and every $m \in M(\mathcal{A})$
\[m(a) \in I \Rightarrow m(b) \in I.\]
(viii) \( p(a) \in I(a, b, p(b)) \) for all \( a, b \in A \) and every \( p \in P(\mathcal{A}) \).
(ix) \( m(a) \in I(a, b, m(b)) \) for all \( a, b \in A \) and every \( m \in M(\mathcal{A}) \).

**Proof.** (i) \( \Rightarrow \) (ii): Let (i) hold and let \( S \subseteq A \). The \( \mathcal{I}(S) = [0] \tau \) for some \( \tau \in \text{Con.} \mathcal{A} \). Set \( \theta := Cg(\{0\} \times S) \). From \( S \subseteq \mathcal{I}(S) = [0] \tau \) we obtain \( \{0\} \times S \subseteq \tau \) and so \( [0] \theta \subseteq [0] \tau \). Clearly \( S \subseteq [0] \theta \). By Lemma 7 the set \( [0] \theta \) is an ideal of \( \mathcal{A} \) and therefore \( \mathcal{I}(S) \subseteq [0] \theta \). Together \( \mathcal{I}(S) \subseteq [0] \theta \subseteq [0] \tau = \mathcal{I}(S) \); hence \( \mathcal{I}(S) = [0] \theta \) proving (ii). Next (ii) \( \Rightarrow \) (iii) is trivial.

(iii) \( \Rightarrow \) (iv): Let (iii) hold and let \( S = \{s_1, \ldots, s_n\} \) be a finite subset of \( A \). Set \( S' := \{s_1, \ldots, s_{n-1}\} \); \( K := \mathcal{I}(S') \) and \( \theta := Cg(0, s_n) \).

1. Let \( n = 1 \). Then \( \mathcal{I}(\emptyset) = \{0\} \). Applying (iii) to \( S = \{s_1\} \) we obtain the required \( \mathcal{I}(S) = [0] \theta = [I(\emptyset)]\theta = [I(S')]\theta \).

2. Thus let \( n > 1 \). To prove \( \mathcal{I}(S) \subseteq [I(S')]\theta \) let \( w \in \mathcal{I}(S) \) be arbitrary. By Lemma 5 we have \( w = f(a_1, \ldots, a_m) \) for an \( m \)-ary \( M \)-ideal term operation \( f \) of \( \mathcal{A} \) and \( a_1, \ldots, a_m \in A \) such that \( a_i \in S \) for all \( i \in M \). If \( M = \emptyset \) then \( f \) is constant with value 0 and \( w = 0 \in [I(S')]\theta \). Thus let \( M \neq \emptyset \). For notational simplicity let \( M = \{1, \ldots, p\} \) for some \( 1 \leq p \leq m \). Without loss of generality we may assume that each \( s_i \) appears at most once among \( a_1, \ldots, a_p \). (Indeed, if some \( s_i \) appears more than once, it suffices to fuse the corresponding variables). We distinguish two cases. (1) Let \( s_n \notin \{a_1, \ldots, a_p\} \). Then \( w \in [I(S')] \subseteq [I(S')]\theta \) and we are done. (2) Thus let \( s_n \in \{a_1, \ldots, a_p\} \), e.g. let \( s_n = a_1 \). Set \( v := f(0, a_2, \ldots, a_m) \). Again from Lemma 5 and \( \mathcal{I}(S') = \mathcal{I}(S' \cup \{0\}) \) we obtain that \( v \in \mathcal{I}(S') \). Moreover, \( (v, w) \in \theta \) because \( f \) is a term operation of \( \mathcal{A} \). Together we have the required \( w \in [I(S')]\theta \) and \( \subseteq \). To prove \( \mathcal{I}(S) \supseteq [I(S')]\theta \) let \( w \in [I(S')]\theta \). Then \( (v, w) \in \theta \) for some \( v \in \mathcal{I}(S') \). By (iii) clearly \( \mathcal{I}(S') = [0]Cg(\{0\} \times S') \). Thus \( (0, w) \in Cg(\{0\} \times S') \vee \theta = Cg(\{0\} \times S') \vee Cg(0, s_n) = Cg(0 \times S) \). Thus \( w \in [0]Cg(\{0\} \times S) \) and so by (iii) we have \( w \in \mathcal{I}(S) \). Thus (iv) holds.

(iv) \( \Rightarrow \) (v) Let (iv) hold and let \( S = \{s_1, \ldots, s_n\} \subseteq A \). For \( i = 1, \ldots, n \) set \( \theta_i := Cg(0, s_i) \) and \( S_i := \{s_1, \ldots, s_i\} \). From (iv) we get \( \mathcal{I}(S_1) = [I(\emptyset)]\theta_1 = [0]\theta_1 \). By an easy induction we obtain

\[
\mathcal{I}(S) = \mathcal{I}(S_n) \subseteq (\ldots (([0]\theta_1)\theta_2) \ldots) = [0](\theta_1 \circ \theta_2 \circ \ldots \circ \theta_n).
\]

(v) \( \Rightarrow \) (iii): Let (v) hold and let \( S = \{s_1, \ldots, s_n\} \subseteq A \). For \( i = 1, \ldots, n \) set \( \theta_i := Cg(0, s_i) \). Further set \( \sigma := Cg(\{0\} \times S) \) and \( K := [0]\sigma \). Notice that \( \sigma = \theta_1 \vee \ldots \vee \theta_n \) (in the lattice of equivalences on \( A \)). By Lemma 7 the set \( K \) is an ideal of \( \mathcal{A} \). Clearly \( S \subseteq K \) and whence \( \mathcal{I}(S) \subseteq K \). To prove \( K \subseteq \mathcal{I}(S) \) let \( v \in K \), i.e. \( (0, v) \in \sigma = \theta_1 \vee \ldots \vee \theta_n \). There exist \( m \geq 1 \), \( 0 = b_0, b_1, \ldots, b_m = v \) in \( A \) and \( j_0, j_1, \ldots, j_{m-1} \in \{1, \ldots, n\} \) such that \( (b_i, b_{i+1}) \in \theta_j \) for \( i = 0, \ldots, m - 1 \). We need the following:
Claim. \([0](\theta_1 \circ \ldots \circ \theta_n) = [0](\theta_{\pi(1)} \circ \ldots \circ \theta_{\pi(n)})\) for every permutation \(\pi\) of \(\{1, \ldots, n\}\).

Proof of the claim. Apply (v) to \(S = \{s_{\pi(1)}, \ldots, s_{\pi(n)}\}\) to obtain \(I(S) = [0](\theta_{\pi(1)} \circ \ldots \circ \theta_{\pi(n)})\).

Using repeatedly the claim we obtain \((0, v) \in \theta_1 \circ \ldots \circ \theta_n\). Hence \(v \in [0](\theta_1 \circ \ldots \circ \theta_n) = I(S)\) by (v). Thus \(K \subseteq I(S)\) and (iii) holds.

(iii)\(\Rightarrow\)(ii): Let (iii) hold and let \(S \subseteq A\). Set \(\sigma := Cg(\{0\} \times S)\). Again by Lemma 7 and \(S \subseteq \{0\}\sigma\) we have \(I(S) \subseteq \{0\}\sigma\). For the converse let \(v \in \{0\}\sigma\). Then \((0, v) \in \sigma\). The congruence \(\sigma\) is compactly generated and so \((0, v) \in \sigma' := Cg(\{0\} \times S')\) for some finite subset \(S'\) of \(S\). From (iii) we obtain \(v \in \{0\}\sigma' = I(S) \subseteq I(S)\). Thus \([0]\sigma \subseteq I(S)\).

(ii)\(\Rightarrow\)(i): Trivial. (i)\(\Leftrightarrow\)(vi)\(\Leftrightarrow\)(vii): Lemma 9 (ii) (a)\(\Leftrightarrow\)(b)\(\Leftrightarrow\)(c).

(vi)\(\Rightarrow\)(viii): Let (vi) hold and let \(a, b \in A\) and \(p \in \Pi(\mathcal{A})\). Set \(I := I(a, b, p(b))\). As \(p(b) \in I\), the condition (vi) yields \(p(a) \in I\). (viii)\(\Rightarrow\)(ix): Trivial.

(xi)\(\Rightarrow\)(i): Let (ix) hold. Suppose to the contrary that (i) does not hold. Then there exists an ideal \(S\) of \(\mathcal{A}\) which is the kernel of no congruence of \(\mathcal{A}\). By Lemma 9 (ii) (c) \(\Rightarrow\) (a) there exist \(m \in M(\mathcal{A})\) and \(a, b \in S\) such that \(m(a) \not\in S\) while \(m(b) \in S\). Observe that by (ix) we have \(m(a) \in I(a, b, m(b)) \subseteq I(S) = S\) in contradiction to \(m(a) \not\in S\).

Corollary 11. Let \(\mathcal{A}\) be such that to every two-element subset \(T\) of \(A\) there exists a binary term operation \(p_T\) of \(\mathcal{A}\) satisfying \(p_T(0, 0) = 0\) and \(Cg(\{0\} \times T) \subseteq Cg(0, t)\) for some \(t = p_T(a, b)\) with \(a, b \in T\). Then every ideal of \(\mathcal{A}\) is a congruence kernel if and only if \(I(x)\) is a congruence kernel for every \(x \in A\).

Proof. \((\Rightarrow)\) Obvious. \((\Leftarrow)\) Let \(I(x)\) be a congruence kernel for all \(x \in A\). We need the following:

Claim. For every finite subset \(S\) of \(A\) we have \(Cg(\{0\} \times S) = Cg(0, s)\) for some \(s \in I(S)\).

Proof of the claim. By induction on \(n := |S|\). The claim is evident for \(n \leq 1\). Thus assume that the claim holds for some \(n \geq 1\) and let \(S = \{s_1, \ldots, s_{n+1}\}\). Set \(S' := \{s_1, \ldots, s_n\}\). By the induction hypothesis \(Cg(\{0\} \times S') = Cg(0, s')\) for some \(s' \in I(S')\). Set \(T := \{s', s_{n+1}\}\) and \(\theta := Cg(\{0\} \times T)\). By the hypothesis \(\theta \subseteq Cg(0, t)\) for some \(t := p_T(a, b)\) with \(a, b \in T\). Clearly \((0, t) = (p_T(0, 0), p_T(a, b)) \in \theta\); whence \(Cg(0, t) \subseteq \theta\) and \(\theta = Cg(0, t)\). As \(p_T(0, 0) = 0\), clearly \(p_T\) is an \(\{1, 2\}\)-ideal term operation and so \(t \in I(T) \subseteq I(S)\). This concludes the induction step.

For the remaining part, we verify (iii) from Theorem 10. Let \(S\) be a finite subset of \(A\). By the claim and the hypothesis \(I(S) \subseteq [0]Cg(\{0\} \times S) = [0]Cg(0, s) = I(s) \subseteq I(S)\).
For varieties we obtain:

**Corollary 12.** The following conditions are equivalent for a variety \( \mathcal{V} \) of algebras of the same type with a nullary term \( 0 \):

(i) Every ideal of each \( \mathcal{A} \in \mathcal{V} \) is a congruence kernel.

(ii) To every \( n \geq 3 \) and each term \( q(x_1, \ldots, x_n) \) of \( \mathcal{V} \) in which \( x_1 \) occurs exactly once, there exists an \( n \)-ary term \( p \) of \( \mathcal{V} \) satisfying the following identities:

\[ p(0, 0, x_1, \ldots, x_n) = 0, \]

\[ q(x_1, x_2, \ldots, x_{n-1}) = p(q(x_2, x_1, x_3, \ldots, x_{n-1}), x_1, \ldots, x_{n-1}). \]

**Proof.** (i) \( \Rightarrow \) (ii): Let (i) hold, let \( n > 1 \) and let \( q(x_1, \ldots, x_n) \) be an \( n \)-ary term of \( \mathcal{V} \) in which \( x_1 \) occurs exactly once (e.g. \( (x_2 \land x_3) \lor (x_4 \land (x_3 \lor (x_1 \land x_2))) \) is such a term in the variety of lattices). Denote by \( \mathcal{L} \) the free algebra of \( \mathcal{V} \) on \( n - 1 \) generators \( x_1, \ldots, x_{n-1} \). For every \( z \in Z \) set

\[ m(z) := q(z, x_1, \ldots, x_{n-1}). \]

It is easy to see that \( m \in M(\mathcal{L}) \) (in the above example \( m = t_1 \circ t_2 \circ t_3 \circ t_4 \) where \( t_1(z) \approx (x_1 \land x_2) \lor z \), \( t_2(z) \approx x_3 \land z \), \( t_3(z) \approx x_2 \lor z \), \( t_4(z) \approx z \land x_1 \)). By assumption \( \mathcal{L} \in \mathcal{V} \) satisfies (i) and therefore by Theorem 10 (i) \( \Rightarrow \) (iii) the algebra \( \mathcal{L} \) also satisfies (ix). For \( a = x_1 \) and \( b = x_2 \) we obtain \( m(x_1) \in I(x_1, x_2, m(x_2)) \) where by (8)

\[ m(x_1) = q(x_1, x_1, \ldots, x_{n-1}), m(x_2) = q(x_2, x_1, \ldots, x_{n-1}). \]

Set \( S := \{ x_1, x_2, q(x_2, x_1, \ldots, x_{n-1}) \} \). From \( m(x_1) \in I(S) \) and Lemma 5 we obtain

\[ q(x_1, x_1, \ldots, x_{n-1}) = m(x_1) = g(a_1, \ldots, a_k) \]

where \( g \) is an \( N \)-ideal term operation of \( \mathcal{L} \) and \( a_1, \ldots, a_k \in Z \) satisfy \( a_i \in S \) for all \( i \in N \).

Notice that each \( a_i \in Z \setminus S \) is of the form \( h_i(x_1, \ldots, x_{n-1}) \) for some term \( h_i \) of \( \mathcal{V} \). It follows that

\[ g(a_1, \ldots, a_k) = p(q(x_2, x_1, \ldots, x_{n-1}), x_1, \ldots, x_{n-1}). \]

for some \( \{1, 2, 3\} \)-ideal term operation \( p \) of \( \mathcal{V} \). Thus (ii) holds. (ii) \( \Rightarrow \) (i): Let (ii) hold, let \( \mathcal{A} \in \mathcal{V} \), let \( a_1, a_2 \in A \) and let \( m \in M(\mathcal{A}) \). Then there exists \( k \geq 1 \), a \( k \)-ary term \( r(x_1, \ldots, x_k) \) of \( \mathcal{V} \) and \( a_3, \ldots, a_{k+1} \in A \) such that (1) \( x_1 \) appears at most once in \( r \) and (2) \( m(x) = r^{\mathcal{A}}(x, a_3, \ldots, a_{k+1}) \) for all \( x \in A \) (where, as usually \( r^{\mathcal{A}} \) denotes
the k-ary term operation of \( A \) which to arbitrary \( b_1, \ldots, b_k \in A \) assigns the value calculated in \( s/ \) according to \( r \). Set \( n := k + 2 \) and define the \( n \)-ary term \( q \) of \( r \) by

\[
q(x_1, \ldots, x_n) = r(x_1, x_1, \ldots, x_n)
\]

(i.e. \( q \) differs from \( r \) only in two dummy variables). By (ii) to \( q \) there exists an \( n \)-ary term \( p \) of \( r \) satisfying (6) and (7). Now

\[
(*) \quad m(a_1) = q^{df}(a_1, a_1, a_2, \ldots, a_{n-1}) = p^{df}(q^{df}(a_2, a_1, a_2, a_3, \ldots, a_{n-1}, a_1, \ldots, a_{n-1}) = p^{df}(r^{df}(a_2, \ldots, a_{n-1}), a_1, \ldots, a_{n-1}).
\]

According to (6) the operation \( p^{df} \) is an \( \{1, 2, 3\} \)-ideal term of \( s/ \). Now (*) and Lemma 5 show that \( m(a_1) \in I\{a_1, a_2, m(a_2)\} \). Thus (ix) of Theorem 10 is satisfied and so (i) holds.

Example 13.

1) Consider the variety of all groups (with the neutral element 0). For \( n \geq 3 \) each term \( q(x_1, \ldots, x_n) \) in which \( x_1 \) occurs exactly once is of the form \( ax_1'b \) where \( a \) and \( b \) are terms in \( x_2, \ldots, x_n \) and \( j \in \{-1, 1\} \). Put

\[
p(x_0, \ldots, x_{n-1}) := x_0b^{-1}(x_1, \ldots, x_{n-1})x_2^{-j}x_1^jb(x_1, \ldots, x_{n-1}).
\]

Clearly \( p \) satisfies (6). We check (7). Abbreviate \((x_1, \ldots, x_{n-1}) \) by \( u \) and set \( \alpha := a(u) \) and \( \beta := b(u) \). Then \( q(x_1, u) = \alpha x_1^j\beta, q(x_2, u) = \alpha x_2^{-j}\beta \) and

\[
p(q(x_2, u), u) = q(x_2, u)\beta^{-1}x_2^{-j}x_1^j\beta = \alpha x_2^{-j}\beta\beta^{-1}x_2^{-j}x_1^j\beta = \alpha x_1^{-j}\beta = q(x_1, u)
\]

proving (7). From Corollary 12 we obtain that every group ideal is a congruence kernel. As group ideals are exactly the normal subgroups this is just the elementary fact relating normal subgroups and group congruences.

2) Consider the variety \( r \) of distributive lattices with 0. Let \( n \geq 3 \) and let \( q(x_1, \ldots, x_n) \) be a term of \( r \). Then \( q \) can be written as (1) \( (x_1 \wedge a) \vee b \) or (2) \( x_1 \vee b \) where \( a \) and \( b \) are terms of \( r \) in variables \( x_2, \ldots, x_n \). Consider the case (1). Set

\[
p(x_1, \ldots, x_n) := (x_1 \wedge b) \vee (x_2 \wedge a).
\]

Clearly \( p \) satisfies (6). We check (7). Again abbreviate \((x_1, \ldots, x_{n-1}) \) by \( u \) and \( a(u) \) and \( b(u) \) by \( \alpha \) and \( \beta \). Now

\[
p(q(x_2, u), u) = (q(x_2, u) \wedge \beta) \vee (x_2 \wedge \alpha) = ((x_2 \wedge \alpha) \vee \beta) \wedge (x_1 \wedge \alpha) = \beta \vee (x_1 \wedge \alpha) = q(x_1, u).
\]
The case (2) is similar but simpler.

From Corollary 12 we obtain that every ideal of a distributive lattice is a congruence kernel. This is a known result [4]; in fact, in [4] it is also shown that among lattices only distributive lattices have this property.

Following [2, 3, 5] we say that $\mathcal{A}$ is permutable at 0 if $[0](\theta \circ \psi) = [0](\psi \circ \theta)$ for all $\theta, \psi \in \text{Con} \mathcal{A}$. We have

**Proposition 14.** Let $\mathcal{V}$ be a variety of algebras of the same type such that 0 is a nullary term of $\mathcal{V}$. Then

1) The following are equivalent:

(i) Every $\mathcal{A} \in \mathcal{V}$ is permutable at 0.

(ii) Every $\mathcal{A} \in \mathcal{V}$ is permutable at 0.

(iii) $b(x, x) \approx 0, \ b(x, 0) \approx x$

for a binary term $b$ of $\mathcal{V}$, and

(iv) $t(x, x, y) \approx y, \ t(0, x, x) \approx 0$

for a ternary term $t$ of $\mathcal{V}$.

2) If $\mathcal{V}$ satisfies one of (i)–(iii), then for every $\mathcal{A} \in \mathcal{V}$ each ideal of $\mathcal{A}$ is a congruence kernel.

**Proof.** 1) The equivalence of (i)–(iii) is shown in [5] pp. 48–49. 2) Let (iii) hold for $\mathcal{V}$ and let $t$ be a term of $\mathcal{V}$ satisfying (10). Let $\mathcal{A} \in \mathcal{V}$ and let $I$ be an ideal of $\mathcal{A}$. We verify the condition (vi) of Theorem 10. Let $p \in P(\mathcal{A})$ satisfy $p(i) \in I$ for some $i \in I$ and let $i' \in I$. There exists an $m$-ary term operation $q$ of $\mathcal{A}$ and $a_2, \ldots, a_m \in A$ such that $p(x) \approx q(x, a_2, \ldots, a_m)$. Set

$$s(x_1, \ldots, x_{m+2}) \approx t(x_1, q(x_2, x_4, \ldots, x_{m+2}), q(x_3, x_4, \ldots, x_{m+2})).$$

By the second half of (10)

$$s(0, 0, 0, x_4, \ldots, x_{m+2}) \approx t(0, q(0, x_4, \ldots, x_{m+2}), q(0, x_4, \ldots, x_{m+2})) \approx 0$$

and so $s$ is an $\{1, 2, 3\}$-ideal term operation of $\mathcal{A}$. By the first half of (10) and the definition of $s$

$$p(i') = t(p(i), p(i), p(i')) = s(p(i), i, i', a_2, \ldots, a_m).$$

Here $p(i), i, i' \in I$ and so $p(i') \in I$ as well. □

743
Example 15. Consider the variety $\mathcal{V}$ of all pseudocomplemented meet-semilattices $\mathcal{S} = (A; \wedge, *, 0)$ with 0 (i.e. for every $a \in A$ the element $a^*$ is the greatest element $y$ such that $a \wedge y = 0$). The term $b(x, y) \approx x \wedge y^*$ satisfies (9) and therefore every ideal of a pseudocomplemented meet-semilattice with 0 is a congruence kernel.

References


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