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Sandi Klavžar; Uroš Milutinović

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GRAPHS $S(n,k)$ AND A VARIANT OF THE TOWER OF HANOI PROBLEM

SANDI KLAVŽAR and UROŠ MILUTINOVIĆ, Maribor

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Summary. For any $n \geqslant 1$ and any $k \geqslant 1$, a graph $S(n,k)$ is introduced. Vertices of $S(n,k)$ are n-tuples over $\{1,2,\ldots,k\}$ and two n-tuples are adjacent if they are in a certain relation. These graphs are graphs of a particular variant of the Tower of Hanoi problem. Namely, the graphs $S(n,3)$ are isomorphic to the graphs of the Tower of Hanoi problem. It is proved that there are at most two shortest paths between any two vertices of $S(n,k)$. Together with a formula for the distance, this result is used to compute the distance between two vertices in $O(n)$ time. It is also shown that for $k \geqslant 3$, the graphs $S(n,k)$ are Hamiltonian.

1. INTRODUCTION

In Lipscomb [10, 11] a relation $\sim$ is introduced on the set of infinite sequences with values from an arbitrary set. This relation is defined in order to obtain some universal topological spaces. A natural question arises whether the relation $\sim$ restricted to the finite case yields any interesting structure. This is indeed used in Milutinović [13, 14] to obtain some more topological results on universal spaces. Direct connections with the Sierpiński gasket (triangular Sierpiński curve) are established in [12, 13, 14].

We use a slightly modified relation $\sim$ to define a class of graphs $S(n,k)$. The set of vertices of $S(n,k)$ is the Cartesian product of n sets $\{1,2,\ldots,k\}$, while the edges are defined according to the relation $\sim$. There are several classes of graphs defined on the Cartesian product of sets and/or using certain relations to define edges. Vertices of the most important graph products are Cartesian products of vertices of the factor graphs, see Feigenbaum and Schäffer [4], or Imrich and Izbicki [9] for the definitions. Among Cartesian products of graphs, Hamming graphs play

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a very special role, cf. Bandelt, Mulder and Wilkeit [1], or Wilkeit [15]. Note that hypercubes are binary Hamming graphs. The edges of Hamming graphs are defined with a particular relation, Hamming distance, among the corresponding tuples. We may henceforth consider the graphs $S(n,k)$ as being of "Hamming type".

In our investigation of these graphs we came across the well-known Tower of Hanoi problem. Although the problem is more than 100 years old [2], only recently a correct treatment of regular states was given by Hinz [5]. In fact there were several approaches before based on the wrong assumption that the largest disk moves at most once. We will not going into details here. We only refer to the papers [5] and [6] of Hinz for the large bibliography on the topic, historical overview, correct treatment and an algorithmic aspect of the problem.

The present paper is organized as follows. In the next section we define graphs $S(n,k)$. It is shown that the graphs $S(n,k)$ are graphs of a variant of the Tower of Hanoi problem and that the graphs $S(n,3)$ are isomorphic to the graphs of the Tower of Hanoi problem. We also demonstrate that graphs $S(n,k)$ are Hamiltonian for $k \geqslant 3$. In Section 3 the shortest path problem is studied. We first prove a formula for the distance between any pair of vertices. It is also proved that there are at most two shortest paths between any pair of vertices. These two results enable us to compute the distance between any two vertices of $S(n,k)$ in $O(n)$ time. Finally we explicitly construct all the shortest paths.

2. Graphs $S(n,k)$ and the Tower of Hanoi

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. For a graph G let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. As usual, the distance between vertices u and v of a graph G is the shortest path distance and will be denoted by $d(u,v)$.

For any $k \geqslant 1$ and any $n \geqslant 1$ we define a graph $S(n,k)$ as follows. Its vertex set is

$$V(S(n,k)) = \{1,2,\ldots,k\}^n$$

and two different vertices $I = (i_1, i_2, \ldots, i_n)$ and $J = (j_1, j_2, \ldots, j_n)$ are adjacent if $I \sim J$, where

$$I \sim J \iff \exists h \in \{1,2,\ldots,n\}$$

such that

$$\begin{aligned} &\text{i)} && \forall t,\ t < h \implies i_t = j_t, \\ &\text{ii)} && i_h \neq j_h, \\ &\text{iii)} && \forall t,\ t > h \implies i_t = j_h \ \&\ j_t = i_h. \end{aligned}$$

We point out that h may equal n, in which case the condition iii) is formally true being empty. In the rest of the paper we will write $i_1 i_2 \ldots i_n$ instead of $(i_1, i_2, \ldots, i_n)$ for brevity.

We use the notation $S(n,k)$ because our original motivation is related to Sierpiński.

For any $n \geqslant 1$, $S(n,1)$ is isomorphic to the one vertex graph K_1 and for any $n \geqslant 1$, $S(n,2)$ is isomorphic to the path on 2^n vertices P_{2^n}. Hence these paths play an analogous role among graphs $S(n,k)$ as hypercubes among the Hamming graphs. Furthermore, for any $k \geqslant 1$, $S(1,k)$ is the complete graph on k vertices. More interesting graphs appear when $k \geqslant 3$ and $n \geqslant 2$. For instance, the graph $S(3,4)$ is shown on Fig. 1.

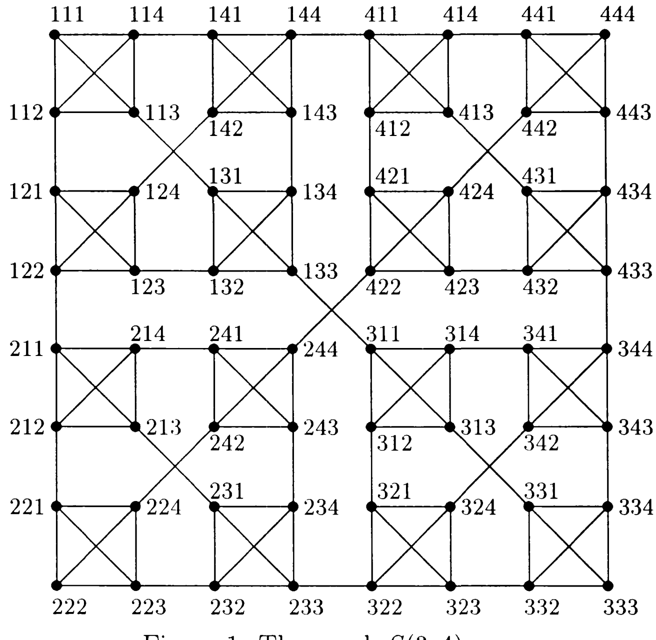

Figure 1. The graph $S(3,4)$

The problem of the Tower of Hanoi (the problem of TH for short) is well-known, thus we will not repeat the definition here, see for instance Hinz [5, 7]. The problem with three pegs is well understood. However, if we have more that three pegs it is still an open problem to determine the minimum number of moves needed to transfer n disks from one peg to another, cf. Hinz [7].

Consider the following variant of the TH with n disks and k pegs. Regular and perfect states are the same as in the classical problem: a state is *regular* if no larger disk lies on a smaller one, and a regular state with all disks on a single peg is called *perfect*. Legal moves are defined as follows. Suppose we have a regular state in which the t topmost disks on a peg i are the t smallest disks. Then if the $(t+1)$-st smallest disk is on a peg $j \neq i$ we are allowed to switch the t disks from the peg i with the disk on the peg j (see Fig. 2). Besides such switches the only other legal moves are arbitrary moves of the smallest disk. Let us henceforth call this variant of the TH the *switching Tower of Hanoi* or *STH* for short.

Note that a switch preserves regular states and that the switching operation is reversible. Therefore we can define the (undirected) graph of STH as usual: its vertices are regular states and two vertices are adjacent if we can move from one state to the other by a legal move. Then we have

Theorem 1. *Let $n \geqslant 1$ and $k \geqslant 1$. Then the graph of STH with n disks and k pegs is isomorphic to the graph $S(n,k)$.*

Proof. It is obvious that regular states of STH bijectively correspond to the sequences

$$i_1 i_2 \dots i_n \in \{1, 2, \dots, k\}^n,$$

according to the interpretation that $i_j = h$ means that the j-th largest disk is on the peg h. Recalling the definition of $\sim$ we then easily see that two vertices of the graph of STH are adjacent if and only if the corresponding sequences are in the relation $\sim$. $\square$

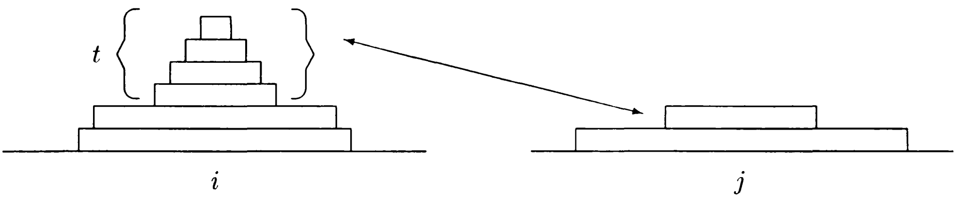

Figure 2. A legal move

Theorem 1 in particular implies that STH is also defined for two pegs, which is not the case with the classical problem. In addition, as $S(n,2)$ is a path, there is exactly one (shortest) path between any two regular states of STH with two pegs.

The interpretation of vertices as sequences used in the proof of Theorem 1 is just opposite to the one used in [3, 5, 7] for the interpretation of the TH (see the proof of Theorem 2). Since legal moves are quite different, the corresponding graphs of TH and STH would be expected to differ (even with the reinterpretation of the vertices of the TH graph by switching the order among the disks). This is in general indeed the case. However, to our surprise we get the same graphs in the case $k = 3$.

Theorem 2. *For any $n \geqslant 1$, the graph $S(n,3)$ is isomorphic to the graph of the TH with n disks.*

Proof. Let TH_n be the graph of the TH with n disks and three pegs. Its vertices are sequences $i_1 i_2 \dots i_n \in \{1,2,3\}^n$, according to the interpretation that $i_j = h$ means that the j-th smallest disk is on the peg h, cf. [5, 7].

By induction on n we construct isomorphisms $f_n \colon S(n,3) \longrightarrow TH_n$. For $n = 1$ both graphs are complete graphs on three vertices.

Let $n \geqslant 2$ and consider a partition of $V(S(n,3))$ into sets V_1, V_2 and V_3, where V_i consists of all vertices beginning with i, $i = 1,2,3$. Then for any i and j, $i \neq j$ there is exactly one edge between V_i and V_j, i.e. the edge between the vertices $ijj \dots j$ and $jii \dots i$. We will call such an edge a *bridging* edge.

In a similar way consider a partition of $V(TH_n)$ into sets W_1, W_2 and W_3, where W_i consists of all vertices ending with i, $i = 1,2,3$. Then for any i and j, $i \neq j$ there is exactly one bridging edge between W_i and W_j, i.e. the edge between the vertices $k \dots kkj$ and $k \dots kki$, $k \neq i$, $k \neq j$.

Then we may isomorphically map V_i onto W_i, using f_{n-1} and an appropriate automorphism (induced by a permutation of the set $\{1,2,3\}$) of TH_{n-1} for adjustment, in such a way that the ends of the bridging edges are mapped onto the corresponding ends of the bridging edges. Considering the three maps as one map from $V(S(n,k))$ onto $V(TH_n)$ yields the map f_n. □

It is interesting to observe that, given any regular state of STH, we can return to it in such a way that we visit every regular state exactly once. In other words:

Proposition 3. *For any $n \geqslant 1$ and any $k \geqslant 3$ the graph $S(n,k)$ is Hamiltonian.*

Proof. For $n = 1$ the proposition is trivial since $S(1,k)$ is a complete graph. Let $n \geqslant 2$ and consider the sequence of paths $P_1, P_2, \dots, P_k$, where P_1 is a path between the vertices $1kk \dots k$ and $122 \dots 2$, P_k between the vertices $k(k-1)(k-1) \dots (k-1)$ and $k11 \dots 1$, and for $i = 2,3,\dots,k-1$, P_i is a path between the vertices $i(i-1)(i-1) \dots (i-1)$ and $i(i+1)(i+1) \dots (i+1)$. We claim that the paths P_i can be constructed in such a way that they include all the vertices beginning with i, $i = 1,2,\dots,k$.

To prove the claim it is enough to see that for any i, j and g, $j \neq g$, there is a path between $ijj \dots j$ and $igg \dots g$ which goes through all vertices beginning with i. Obviously that reduces the induction argument to the statement that $jj \dots j$ and $gg \dots g$, $j \neq g$, may be connected in $S(n,k)$ by a path going through all vertices (for all n). Without loss of generality assume $j = 1$ and $g = k$. By the induction hypothesis we may find a path from $11 \dots 1$ to $12 \dots 2$ through all vertices beginning with 1. Add the edge between $12 \dots 2$ and $21 \dots 1$ to the path. By the same argument we may find a path from $21 \dots 1$ to $23 \dots 3$ through all vertices beginning with 2. Continue this procedure until $(k-1)k \dots k$ is joined to $k(k-1) \dots (k-1)$ and a path from $k(k-1)\dots(k-1)$ to $kk\dots k$ through all vertices beginning with k is added at the end.

It follows that the paths $P_1, P_2, \dots, P_k$ form a Hamiltonian cycle. $\square$

3. Shortest paths in $S(n,k)$-graphs

Define

$$\varrho_{i,j} = \begin{cases} 1; & i \neq j, \\ 0; & i = j. \end{cases}$$

(The symbol has been chosen in this way, since rho graphically resembles the Kronecker's delta symbol put upside down.) In addition, let

$$\mathscr{P}^{i}_{j_1,\dots,j_m} = \varrho_{i,j_1}\varrho_{i,j_2}\cdots\varrho_{i,j_m},$$

where the right-hand side term is a binary number, rhos representing its digits. Also, let V_i be the set of vertices of $S(n,k)$ consisting of all vertices beginning with i.

Lemma 4. *Let $I = ii\dots i$ and $J = j_1j_2\dots j_n$ be vertices of $S(n,k)$. Then $d(I,J) = \mathscr{P}^{i}_{j_1,\dots,j_n}$ and there is exactly one shortest path between I and J. In particular, for $i \neq j$, $d(ii\dots i, jj\dots j) = 2^n - 1$.*

Proof. By induction on n. The statement is trivial for $n = 1$.

Let $n \geqslant 2$.

If $i = j_1$ then by the induction hypothesis, the shortest path inside V_i has the length $\mathscr{P}^{i}_{j_2,\dots,j_n} = \mathscr{P}^{i}_{j_1,\dots,j_n}$. Consider now a path Q between I and J which is not completely in V_i. Let g, $g \neq i$, be such that the vertex $gi\dots i$ is the last vertex of Q not belonging to V_i. Then Q contains a subpath from $ig\dots g$ to I in V_i which has by induction length at least $2^{n-1} - 1 \geqslant \mathscr{P}^{i}_{j_2,\dots,j_n}$. Therefore $|Q| > \mathscr{P}^{i}_{j_2,\dots,j_n}$.

Let $i \neq j_1$. Then by the induction hypothesis, among all paths between I and J containing the edge between $ij_1\dots j_1$ and $j_1i\dots i$, there is a unique shortest one. Its length is $(2^{n-1} - 1) + 1 + \mathscr{P}^{i}_{j_2,\dots,j_n} = \mathscr{P}^{i}_{j_1,\dots,j_n} \leqslant 2^n - 1$. Consider a path Q between I and J containing an edge between the vertices $ig\dots g$ and $gi\dots i$, where g is chosen as above. Then $|Q| \geqslant (2^{n-1} - 1) + 1 + (2^{n-1} - 1) + 1 = 2^n$. Thus Q is not a shortest path.

Note finally that there is only one shortest path in both cases. $\square$

Theorem 5. *Let $I = i_1i_2\dots i_n$ and $J = j_1j_2\dots j_n$ be vertices of $S(n,k)$ such that $i_1 = j_1, \dots, i_{\ell-1} = j_{\ell-1}$ and $i_\ell \neq j_\ell$, $\ell \geqslant 1$. Then $d(I,J) = 1$ for $\ell = n$, and otherwise, $d(I,J)$ is equal to*

$$\min\{\mathscr{P}^{j_\ell}_{i_{\ell+1},\dots,i_n} + 1 + \mathscr{P}^{i_\ell}_{j_{\ell+1},\dots,j_n},\ \mathscr{P}^{h}_{i_{\ell+1},\dots,i_n} + 1 + 2^{n-\ell} + \mathscr{P}^{h}_{j_{\ell+1},\dots,j_n} \mid h \neq i_\ell, j_\ell\}.$$

Proof. By induction on n. If $n = 1$ then $\ell = 1$ and $d(I, J) = 1$ as claimed.

By a similar argument as in Lemma 4 we first note that it suffices to consider paths in the subgraph of $S(n, k)$ induced by the vertices beginning with $i_1 \dots i_{\ell-1}$. Omitting $i_1 \dots i_{\ell-1}$ from the vertices gives a natural isomorphism between the subgraph and $S(n - \ell + 1, k)$. If $\ell > 1$ then the theorem holds by the induction hypothesis. Hence it remains to prove the statement for $\ell = 1$.

Let $n \geqslant 2$. For brevity let $i_1 = i$ and $j_1 = j$. Consider a shortest path Q among those paths between I and J which have vertices only from $V_i \cup V_j$. Then, by Lemma 4, $|Q| = \mathscr{P}^{j}_{i_2,\dots,i_n} + 1 + \mathscr{P}^{i}_{j_2,\dots,j_n}$, because Q must contain the edge between the vertices $ij \dots j$ and $ji \dots i$. Also by the lemma, Q is unique. We will call such a path the *direct* path between I and J.

Consider now a shortest path Q' among the paths between I and J with vertices only from $V_i \cup V_j \cup V_h$, $h \neq i, j$ where $Q \cap V_h \neq \emptyset$. Since Q' must contain the edges between $ih \dots h$ and $hi \dots i$, and between $hj \dots j$ and $jh \dots h$, Lemma 4 implies $|Q'| = \mathscr{P}^{h}_{i_2,\dots,i_n} + 1 + (2^{n-1} - 1) + 1 + \mathscr{P}^{h}_{j_2,\dots,j_n}$. Furthermore, Lemma 4 also implies uniqueness of Q' (for a fixed h). We call such a path the *V_h-path* between I and J.

Clearly, for the direct path Q we have $|Q| < 2^n$ and thus the distance between I and J is strictly less than 2^n. But since any path containing also vertices from V_g and V_h, where i, j, h and g are pairwise different, has length at least $2^{n-1} + 2^{n-1} + 1 = 2^n + 1$, the theorem follows. □

From the computational point of view, Theorem 5 can be used to compute $d(I, J)$ in $O(nk)$ time. The next theorem will enable us to improve this complexity.

Theorem 6. *There are at most two shortest paths between any two vertices of $S(n, k)$.*

Proof. Let $I = i_1 i_2 \dots i_n$ and $J = j_1 j_2 \dots j_n$ be vertices of $S(n, k)$ and assume without loss of generality that $i_1 \neq j_1$. For brevity let $i_1 = i$ and $j_1 = j$. Note that the proof of Theorem 5 implies that the length of the direct path between I and J is $\mathscr{P}^{j}_{i_2,\dots,i_n} + 1 + \mathscr{P}^{i}_{j_2,\dots,j_n}$, while the length of the V_h-path is $\mathscr{P}^{h}_{i_2,\dots,i_n} + 1 + 2^{n-1} + \mathscr{P}^{h}_{j_2,\dots,j_n}$ for any $h \neq i, j$.

We distinguish several cases.

Case 1: $i_2 = i, j_2 = j$.

Any V_h-path is of length at least $2^{n-2} + 1 + 2^{n-1} + 2^{n-2}$, because $\varrho_{h,i_2} = \varrho_{h,j_2} = 1$. Since the direct path is of length at most $2^n - 1$, it is the unique shortest path in this case.

Case 2: $i_2 = j$. Any V_h-path is of length at least $2^{n-2} + 2^{n-1} + 1$, because $\varrho_{h,j_2} = 1$. Since $\varrho_{j,i_2} = 0$, the direct path is of length at most $(2^{n-2} - 1) + 1 + (2^{n-1} - 1)$. Hence it is again the unique shortest path between I and J.

The case $j_2 = i$ is treated analogously.

Case 3: $i_2 = i$, $j_2 = h$, $h \neq i, j$. Let $g \neq i, j, h$. Then $\varrho_{g,i_2} = \varrho_{g,j_2} = 1$. As in Case 1 it follows that the length of the V_g-path is at least $2^n + 1$. Thus a shortest path can only be the direct path or the V_h-path. (Consider for example vertices 113 and 233 of $S(3,4)$ on Figure 1 to see that both paths may be shortest.)

The case $j_2 = j$ and $i_2 = h$ is treated analogously.

Case 4: $i_2 = j_2 = h$, $h \neq i, j$. This case can be treated exactly as the previous one. (Consider for example vertices 133 and 231 of $S(3,4)$ in Figure 1 to see that the direct path and the V_h-path may have equal length. Also consider vertices 122 and 322 to see that the V_h-path may be shorter than the direct one.)

Case 5: $i_2 = g$, $j_2 = h$, i, j, g and h are pairwise different. Let $f \neq i, j, g, h$. Then as in the previous two cases we get that the V_f-path cannot be a shortest path.

The length of the direct path is equal to

$$\mathscr{P}^{\,j}_{g,i_3,\ldots,i_n} + 1 + \mathscr{P}^{\,i}_{h,j_3,\ldots,j_n} = 2^{n-1} + \mathscr{P}^{\,j}_{i_3,\ldots,i_n} + 1 + \mathscr{P}^{\,i}_{j_3,\ldots,j_n},$$

which is in turn equal to 2^{n-1} plus the length of the direct path between $ii_3 \ldots i_n$ and $jj_3 \ldots j_n$ in $S(n-1,k)$.

The length of the V_h-path is equal to

$$\mathscr{P}^{\,h}_{g,i_3,\ldots,i_n} + 1 + 2^{n-1} + \mathscr{P}^{\,h}_{h,j_3,\ldots,j_n} = 2^{n-1} + \mathscr{P}^{\,h}_{i_3,\ldots,i_n} + 1 + 2^{n-2} + \mathscr{P}^{\,h}_{j_3,\ldots,j_n},$$

which is equal to 2^{n-1} plus the length of the V_h-path between $ii_3 \ldots i_n$ and $jj_3 \ldots j_n$ in $S(n-1,k)$. An analogous statement holds for the V_g-path.

This proves that shortest paths between $igi_3 \ldots i_n$ and $jhj_3 \ldots j_n$ in $S(n,k)$ correspond to shortest paths between $ii_3 \ldots i_n$ and $jj_3 \ldots j_n$ in the graph $S(n-1,k)$. These paths can only be the direct path, the V_{i_3}-path, or the V_{j_3}-path. Hence if $\{i_3, j_3\} \neq \{g, h\}$, at most two paths among the direct path, the V_g-path and the V_h-path, may be the required shortest paths in $S(n,k)$. If $\{i_3, j_3\} = \{g, h\}$ we may use the initial argument once again. Finally, if $\{i_t, j_t\} = \{g, h\}$ holds for $t = 3, 4, \ldots, n-1$, shortest paths between $igi_3 \ldots i_n$ and $jhj_3 \ldots j_n$ in $S(n,k)$ correspond to shortest paths between ii_n and jj_n in $S(2,k)$. In $S(2,k)$ the direct path between ii_n and jj_n is of length $\varrho_{i,j_n} + \varrho_{j,i_n} + 1 \leqslant 3$. The length of the V_h-path is $\varrho_{h,j_n} + \varrho_{h,i_n} + 2^1 + 1 \geqslant 3$. Clearly, if the V_h-path is a shortest path then its length must be equal to 3, which is possible only if $i_n = j_n = h$. Analogously we see that the V_g-path may be a shortest path only if $i_n = j_n = g$. We conclude that at most two of these three paths may be shortest paths. $\square$

The proof of Theorem 6 in particular shows that $d(I,J)$, where again without loss of generality $i_1 \neq j_1$, from Theorem 5 is obtained as minimum of

$$\mathscr{P}^{j_1}_{i_2,\ldots,i_n} + 1 + \mathscr{P}^{i_1}_{j_2,\ldots,j_n}$$

and

$$\min\{\mathscr{P}^{h}_{i_2,\ldots,i_n} + 1 + 2^{n-1} + \mathscr{P}^{h}_{j_2,\ldots,j_n} \mid h \in \{i_2, j_2\} \setminus \{i_1, j_1\}\}.$$

This yields

Corollary 7. *The distance between any two vertices of $S(n,k)$ can be computed in $O(n)$ time.*

When we know the distance between two vertices we can also easily find all shortest paths, i.e. one or two of them. To see this it is enough to construct a shortest path between $I = i_1 i_2 \ldots i_n$ and $J = i_1 j_1 \ldots j_1$ of length $\mathscr{P}^{j_1}_{i_2,\ldots,i_n}$. Indeed, using such paths together with the corresponding bridging edges gives shortest paths between the original vertices.

Consider a sequence σ of vertices starting with I and ending with J where the next term is obtained from the previous one analogously to the way one represents addition of 1 in binary notation. The beginning of σ thus is

$$\begin{gathered}
i_1 \ldots i_{n-3} i_{n-2} i_{n-1} i_n, \\
i_1 \ldots i_{n-3} i_{n-2} i_{n-1} j_1, \\
i_1 \ldots i_{n-3} i_{n-2} j_1 i_{n-1}, \\
i_1 \ldots i_{n-3} i_{n-2} j_1 j_1, \\
i_1 \ldots i_{n-3} j_1 i_{n-2} i_{n-2}, \ \ldots
\end{gathered}$$

In the case when two consecutive terms of the above list are equal we of course omit the redundant one. That means that if $i_\ell = j_\ell$ then there are $2^{n-\ell}$ such redundant terms. Thus σ is indeed the shortest path between I and J of the desired length.

Note that the shortest path between $ii\ldots i$ and $jj\ldots j$, $i \neq j$, is obtained exactly by adding 1 in binary notation if we replace i by 0 and j by 1. Moreover, this describes the path $S(n,2)$.

To conclude the paper we remark that in view of Theorem 2, Theorems 5 and 6 (in the case $k = 3$) offer an alternative approach to the classical TH problem.

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Authors' address: University of Maribor, PF, Koroška cesta 160, 2000 Maribor, Slovenia.