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STRONG COMPACT ELEMENTS IN MULTIPLICATIVE LATTICES

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Throughout we assume that L is a C-lattice. It is well known that a Noether lattice is a principal element lattice if and only if every maximal element is weak meet principal (see Theorem 5 of [6]). Also it is known that if L is principally generated, then L is a principal element lattice if and only if L is an M-lattice satisfying the ascending chain condition (see Theorem 6 of [4]). In this paper, we introduce strong compact elements, P-weak meet principal elements and P-principal elements and using them, principal element lattices and almost principal elements lattices are characterized.

For any $a \in L$, we define a^w by $a^w = \bigwedge\limits_{n=1}^{\infty} a^n$. The reader is referred to [1] and [3] for general background and terminology.

We begin with the following definitions.

Definition 1. An element $a \in L$ is said to be a strong compact element if both a and a^w are compact elements.

Definition 2. A prime element $m \in L$ is said to be P-weak meet principal (P-principal) if every prime element $q \leqslant m$ is weak meet principal (principal).

Obviously, idempotent compact elements, compact nilpotent elements and complemented elements are examples of strong compact elements. Also L satisfies the ascending chain condition if and only if every element is strong compact. Observe that L is a principal element lattice if and only if every prime element is P-principal. If L is principally generated, then L is an M-lattice if and only if every maximal element is P-weak meet principal (see Theorem 1.4 of [7]).

Lemma 1. *Let m be a maximal element of L. If m is weak meet principal, then m^k is weak meet principal for all $k \in \mathbb{Z}^+$.*

Proof. We show that m^{r+1} is weak meet principal if m^r is. Let $a \leqslant m^{r+1}$ for some $a \in L$. If $m^{r+1} = m^r$, then we are through. Suppose $m^{r+1} < m^r$. Then

$a \leqslant m^{r+1} < m^r$, so $a = m^r b$ for some $b \in L$. Since $m^r b \leqslant m^{r+1}$ and m^{r+1} is m-primary, it follows that $m^r \leqslant m^{r+1}$ or $b \leqslant m$. In the first case, we are done. In the second case, $b = mc$ for some $c \in L$. Then $a = m^r b = m^r(mc) = m^{r+1}c$ and hence m^{r+1} is weak meet principal. □

Lemma 2. *Let m be a maximal element of L with $m^k \neq m^{k+1}$ for all $k \in \mathbb{Z}^+$. If m is weak meet principal, then*

(i) *m^w is a prime element.*

(ii) *$mm^w = m^w$.*

(iii) *If p is a prime element such that $p < m$, then $p \leqslant m^w$.*

Proof. (i) Suppose x and y are two compact elements such that $xy \leqslant m^w$. Since $xy \leqslant m$, it follows that either $x \leqslant m$ or $y \leqslant m$. Without loss of generality, assume that $y \leqslant m$. If $x \nleqslant m$, then $y \leqslant m^k$ for all $k \in \mathbb{Z}^+$ as $xy \leqslant m^w$ and each m^k is m-primary. So assume that $x \leqslant m$. If $x \nleqslant m^w$ and $y \nleqslant m^w$, then $x \leqslant m^r$, $x \nleqslant m^{r+1}$ and $y \leqslant m^s$, $y \nleqslant m^{s+1}$ for some $r, s \in \mathbb{Z}^+$. By Lemma 1, m^r and m^s are weak meet principal, so $x = m^r a$ and $y = m^s b$ for some $a, b \in L$. Note that $a \nleqslant m$ and $b \nleqslant m$. Then $xy = m^{r+s}ab \leqslant m^{r+s+1}$. As m^{r+s+1} is m-primary and $ab \nleqslant m$, it follows that $m^{r+s} \leqslant m^{r+s+1}$, a contradiction. Therefore m^w is a prime element.

(ii) Since $m^w \leqslant m$ and m is weak meet principal, we have $m^w = ma$ for some $a \in L$. Again since $ma \leqslant m^w$, $m^w < m$ and by (i), m^w is a prime element, it follows that $a \leqslant m^2$, so $m^w = ma \leqslant mm^w$ and hence $m^w = mm^w$.

(iii) Suppose p is a prime element such that $p < m$. If $p \nleqslant m^w$, then $p \leqslant m^k$ and $p \nleqslant m^{k+1}$ for some $k \in \mathbb{Z}^+$. By Lemma 1, $p = m^k a$ for some $a \in L$. Note that $a \nleqslant m$, so $m^k \leqslant p$ and $p = m^k$. This shows that $p = m$, a contradiction. Therefore $p \leqslant m^w$. □

Lemma 3. *Suppose L is a join principally generated and let m be a maximal element which is weak meet principal and $m^k \neq m^{k+1}$ for all $k \in \mathbb{Z}^+$. If m^w is compact, then*

(i) $\operatorname{rank} m = 1$,

(ii) *$m^w = 0_m$ and*

(iii) *$q = m^w$ or $q = m^k$ ($k \in \mathbb{Z}^+$) for every primary element $q \leqslant m$.*

Proof. (i) By Lemma 2(i), m^w is a prime element and $m^w < m$. Suppose $p < m$ is a prime element. By Lemma 2(iii), $p \leqslant m^w$. As m^w is compact and $mm^w = m^w$ (by Lemma 2(ii)), by Lemma 1.1 of [2], $m \vee (0 : m^w) = 1$ and so $m^w \leqslant p$. Therefore $p = m^w$ and hence $\operatorname{rank} m = 1$.

(ii) Since $m \vee (0 : m^w) = 1$ and 1 is compact, it follows that $m \vee x = 1$ for some compact element $x \in L$ such that $xm^w = 0$; $m^w \leqslant 0_m$. Obviously $0_m \leqslant m^w$ as m^w is a prime element. Hence $m^w = 0_m$.

(iii) By (i), $\operatorname{rank} m = 1$. Suppose q is m-primary. Then, by imitating the proof of Lemma 2(iii), it can be easily shown that $q = m^k$ for some $k \in \mathbb{Z}^+$. The remaining part is obvious. □

Definition 3. A maximal element m of L is said to be a Δ-prime if p^n is p-primary for all prime elements $p < m$ and for all $n \in \mathbb{Z}^+$.

Every maximal element m with $\operatorname{rank} m = 0$ is a Δ-prime element. Complemented maximal elements are Δ-prime elements. Note that, if L is a principally generated M-lattice, then every maximal is a Δ-prime element. In fact, if L is generated by compact join principal elements and if every semiprimary element is primary, then every maximal element is a Δ-prime element (see Theorem 4.2, Corollary 3.2 and Corollary 3.5 of [2]).

Lemma 4. *Let L be a quasilocal with maximal element m. Suppose m is weak meet principal and* $\bigwedge\limits_{k=1}^{\infty} m^k = 0$. *Then every nonzero element is a power of m. Further, every element is principal.*

Proof. Let a $(a < 1)$ be a nonzero element of L. Then $a \leqslant m^k$ and $a \nleqslant m^{k+1}$ for some $k \in \mathbb{Z}^+$. By Lemma 1, $a = m^k c$ for some $c \in L$. Note that $c \nleqslant m$ and so $c = 1$ as L is quasilocal. Therefore $a = m^k$. This shows that every nonzero element is a power of m. Note that m is weak join principal and so principal as L is a chain. Consequently, every element is principal. □

Lemma 5. *Let L be a join principally generated quasilocal lattice with maximal element m. Assume that m is weak meet principal and m^w is compact. Then, every element is principal.*

Proof. If $m^k = m^{k+1}$ for some $k \in \mathbb{Z}^+$, then $m^w = m^k$ and $mm^w = m^w$. If $m^k \neq m^{k+1}$ for all $k \in \mathbb{Z}^+$, then by Lemma 2, $mm^w = m^w$. As m^w is compact, by Lemma 1.1 of [2], $m^w = 0$ and hence by Lemma 4, every element is principal. □

An element $a \in L$ is simple if there is no element $x \in L$ such that $a^2 < x < a$.

Lemma 6. *Let L be a join principally generated quasilocal lattice with maximal element m. Assume that m is the join of weak meet principal elements. If m is strong compact and simple, then every element is principal.*

Proof. If $m = m^2$, then we are through. Suppose $m^2 < m$. Choose any weak meet principal element $a \leqslant m$ such that $a \nleqslant m^2$. Then $m = m^2 \vee a$. As m is compact, by Lemma 1.1 of [2], $m = a$ which is weak meet principal. Now the result follows from Lemma 5. □

Theorem 1. *Suppose L is principally generated and let m be a maximal Δ-prime element of L. Then the following statements are equivalent:*

(i) *m is P-principal.*

(ii) *m is P-weak meet principal and strong compact element of L.*

(iii) *m is strong compact and weak meet principal.*

(iv) *m is strong compact and every m-primary element is a power of m.*

(v) *m is strong, compact and simple.*

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 2 and (ii) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (iv). Suppose (iii) holds. Then $mm^w = m^w$ (see the proof of Lemma 5) and since m is strong compact $(m^w)_m = 0_m$. But $(m^w)_m = \bigwedge\limits_k (m_m)^k$ and so by Lemma 4, L_m is a principal element lattice. Consequently, every m-primary element is a power of m. Thus (iv) holds. (iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (i). Suppose (v) holds. By Lemma 6, L_m is a principal element lattice. As m is locally principal and compact, it follows that m is principal. Note that $\operatorname{rank} m \leqslant 1$. If $\operatorname{rank} m = 0$, then we are through. Suppose $\operatorname{rank} m = 1$. Then $p = 0_m$ is the only prime element properly contained in m. As m is a Δ-prime, p^2 is p-primary and therefore $p^2 = p = 0_m$ (by Lemma 3). Since m^w is compact, by Lemma 3(ii), 0_m is compact and hence p is an idempotent compact element and so by Lemma 1.1 of [2], p is complemented element. Again by Lemma 2.2 of [2], p is principal. Thus (i) holds and this completes the proof of the theorem. □

Theorem 2. *Suppose L is principally generated. Then the following statements are equivalent:*

(i) *L is a principal element lattice.*

(ii) *Every maximal element is P-principal.*

(iii) *Every maximal element is P-weak meet principal and strong compact.*

(iv) *Each maximal element is strong compact and weak meet principal.*

(v) *For every maximal element $m \in L$, m is strong compact and every m-primary element is a power of m.*

(vi) *For each maximal element $m \in L$, m is strong compact and simple.*

Proof. (i) $\Leftrightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi), see the proof of Theorem 1. We show that (vi) $\Rightarrow$ (i). Suppose (vi) holds. By Lemma 6, L is an almost principal element lattice. Note that $\dim L \leqslant 1$. Let p be a prime element of L. Then p is locally principal. Also p is either maximal or $p = m^w$ for all maximal elements m such that $p < m$. Therefore, by hypothesis, p is compact and hence p is principal. Thus every prime element is principal. Consequently every element is principal. This completes the proof of the theorem. □

Theorem 3. *Let L be generated by weak join principal elements and let p be a P-weak meet principal element. Then* $\operatorname{rank} p \leqslant 1$. *If $q < p$ is a prime element, then* $q = 0_p$.

Proof. Suppose q and r are prime elements such that $r < q < p$. As p is P-weak meet principal, we have $q = pq$. Choose any weak join principal element $x \leqslant q$ such that $x \nleqslant r$. Then $x = qa$ for some $a \in L$.

So $x = qa = (pq)a = px$ and therefore $p \vee (0 : x) = 1$. Since $x \nleqslant r$, $(0 : x) \leqslant r \leqslant p$, a contradiction. Therefore $\operatorname{rank} p \leqslant 1$.

Now assume that q is a prime element such that $q < p$. Obviously $0_p \leqslant q$. If $x \leqslant q$ is any weak join principal element, then $x = xp$ (by the above argument), so $p \vee (0 : x) = 1$ and hence $p \vee (0 : a) = 1$ for any compact element $a \leqslant q$. Consequently $q \leqslant 0_p$. This shows that $q = 0_p$. □

Corollary 1. *Let L be generated by weak join principal elements. Suppose L is quasilocal with maximal element m. If m is P-weak meet principal, then every element is principal.*

Proof. By Theorem 3, $\operatorname{rank} m \leqslant 1$. If $\operatorname{rank} m = 0$, then by Lemma 2, $m^k = m^{k+1}$ for some $k \in \mathbb{Z}^+$. By Lemma 1, m^k is weak meet principal and hence weak join principal. Consequently $m^k = 0$ and so by Lemma 5, every element is principal. If $\operatorname{rank} m = 1$, then by Lemma 2, and by Theorem 3, $m^w = 0$ and hence by Lemma 5, every element is principal. □

Definition 4. A maximal element m of L is said to be P^*-weak meet principal, if m_m is a P-weak meet principal element of L_m.

Lemma 7. *Let L be generated by compact weak join principal elements and let m be a maximal element of L. If m is P^*-weak meet principal, then L_m is a principal element lattice.*

Proof. The lemma follows from Corollary 1. □

Theorem 4. *Let L be generated by compact weak join principal elements. If every maximal element m is P^*-weak meet principal, then L is an almost principal element lattice.*

Proof. The theorem follows from Lemma 7. □

Theorem 5. *Let L be generated by compact weak join principal elements. If every maximal element is strong compact and P^*-weak meet principal, then every element is principal.*

Proof. By Theorem 4, L is an almost principal element lattice. So every maximal element is locally principal. Since every maximal element is compact, it follows that every maximal element is principal. Again by hypothesis and Theorem 2(iv), every element is principal. □

Theorem 6. *Let L be generated by compact weak join principal elements. Then the following statements are equivalent:*

(i) *L contains only a finite number of minimal prime elements and every maximal element is P^*-weak meet principal.*

(ii) *L is a finite direct sum of almost principal element domains and special principal element lattices.*

Proof. Suppose (i) holds. By Theorem 4, L is an almost principal element lattice and so it is an r-lattice. Let $p_1, p_2, \dots, p_n$ be the minimal prime elements of L. Suppose p_i $(1 \leqslant i \leqslant k)$ are nonmaximal prime elements and p_j $(k+1 \leqslant j \leqslant n)$ are maximal elements. By hypothesis, $\operatorname{rank} m \leqslant 1$ for every maximal element $m \in L$. As L_{p_j} $(k+1 \leqslant j \leqslant n)$ is a special principal element lattice, $0_{p_j} = p_{j_{p_j}}^{\ell_j}$ for $k+1 \leqslant j \leqslant n$ and if $\operatorname{rank} m = 1$, for some maximal element $m \in L$, then $0_m = p_{i_m}$ for some i $(1 \leqslant i \leqslant k)$. Therefore $0_m = (p_1 \wedge \dots \wedge p_k \wedge p_{k+1}^{\ell_{k+1}} \wedge \dots \wedge p_n^{\ell_n})_m$ for every maximal element $m \in L$ and hence $0 = p_1 \wedge \dots \wedge p_k \wedge p_{k+1}^{\ell_{k+1}} \wedge \dots \wedge p_n^{\ell_n}$. As the p_i's $(1 \leqslant i \leqslant n)$ are pairwise comaximal, we have $L \cong L/p_1 \times L/p_2 \times \dots \times L/p_k \times L/p_{k+1}^{\ell_{k+1}} \times \dots \times L/p_n^{\ell_n}$. Note that, for $1 \leqslant i \leqslant k$, L/p_i is a domain and an almost principal element lattice. So each L/p_i is an almost principal element domain. For $k+1 \leqslant j \leqslant n$, $L/p_j^{\ell_j}$ is a quasi-local, almost principal element lattice and hence it is a special principal element lattice.

Now assume that (ii) holds. Let $L = L_1 \times L_2 \times \dots \times L_k \times L_{k+1} \times \dots \times L_n$, where each L_i $(1 \leqslant i \leqslant k)$ is an almost principal element domain and each L_j $(k+1 \leqslant j \leqslant n)$ is a special principal element lattice. Note that each L_i is an r-lattice in which every compact element is principal and hence L is an r-lattice in which every compact element is principal. Let m be a maximal element of L. Then $m = (1, 1, \dots, m_i, \dots, 1)$, where m_i is a maximal element of L_i. If L_i is a two element chain, then m is a complemented element and so it is P^*-weak meet principal. So assume that each L_i is not a two element chain. Note that $\operatorname{rank} m \leqslant 1$. If $m_i \in L_i$ $(1 \leqslant i \leqslant k)$, then $\operatorname{rank} m = 1$ and m is nonidempotent. As L_m is totally ordered, m_m is principal (by Lemma 7 of [5]) and $p = (1, 1, \dots, 0_i, \dots, 1)$ is the only prime element contained in m which is also a complemented element and so p_m is a principal element of L_m. Therefore, if $m_i \in L_i$ $(1 \leqslant i \leqslant k)$, then m is p^*-weak meet principal.

So assume that $m_i \in L_i$ $(k+1 \leqslant i \leqslant n)$. Then $\operatorname{rank} m = 0$ and $m_i^k = 0_i$ for some $k \in \mathbb{Z}^+$. Note that m^k is a complemented element. As L_i is not a two element chain, $m \neq m^2$, and so by Lemma 7 of [5], m_m is principal in L_m. Thus every maximal element is P^*-weak meet principal. Obviously L contains only a finite number of minimal prime elements. This completes the proof of the theorem. □

Theorem 7. *Let L be generated by compact weak join principal elements. Then L is a finite direct sum of almost principal element domains if and only if L satisfies the following conditions:*

(i) *L contains only finitely many minimal prime elements.*
(ii) *Every maximal element is P^*-weak meet principal.*
(iii) *For every maximal element $m \in L$, L_m is a principal element domain.*

Proof. Suppose L satisfies the conditions (i), (ii) and (iii). By (iii), L is an almost principal element lattice and so it is an r-lattice. By (i) and (ii), there exist pairwise comaximal prime elements $p_1, p_2, \ldots, p_k, p_{k+1}, \ldots, p_n$ such that for $k+1 \leqslant j \leqslant n$, p_j's are maximal elements and $0 = p_1 \wedge \cdots \wedge p_k \wedge p_{k+1}^{\ell_{k+1}} \wedge \cdots \wedge p_n^{\ell_n}$ (see the proof of Theorem 6). Let $k+1 \leqslant j \leqslant n$. Since L_{p_j} is a domain, $0_{p_j} = p_j^{\ell_j}$, it follows that $0_{p_j} = p_j$ and hence $p_j = p_j^2 = p_j^{\ell_j}$. Therefore $0 = p_1 \wedge \cdots \wedge p_k \wedge p_{k+1} \wedge \cdots \wedge p_n$. As p_i's are comaximal, $L \cong L/p_1 \times \cdots \times L/p_n$ and each L/p_i is an almost principal element domain.

The converse follows from the proof of Theorem 6. □

Corollary 2. *Let L be generated by compact weak join principal elements. Then L is a finite direct product of principal element domains if and only if L satisfies the following conditions:*

(i) *L contains only finitely many minimal prime elements.*
(ii) *Every maximal element is P^*-weak meet principal.*
(iii) *For every maximal element $m \in L$, L_m is a principal element domain.*
(iv) *Every maximal element is strong compact.*

Proof. Suppose L is a finite direct product of principal element domains. By Theorem 7, L satisfies the conditions (i), (ii) and (iii). Since each factor is a principal element domain, it follows that L is a principal element lattice and so L is a Noether lattice. Consequently every element is strong compact.

The converse follows from Theorem 4, Theorem 5 and Theorem 7. □

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