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DIAGONAL BLOCKS OF TWO MUTUALLY INVERSE POSITIVE DEFINITE BLOCK MATRICES

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1. INTRODUCTION AND PRELIMINARIES

In 1964, the first named author gave [2] a complete characterization of ordered $2n$-tuples of positive numbers

$$a_1, \ldots, a_n, b_1, \ldots, b_n$$

with the following property: There exists a positive definite matrix A of order n such that its diagonal entries are $a_1, \ldots, a_n$ and such that the diagonal entries of A^{-1} are $b_1, \ldots, b_n$.

The result consists of two conditions:

C1° $a_i b_i \geqslant 1,\ i = 1, \ldots, n;$

C2° $2 \max\limits_{i=1,\ldots,n} (\sqrt{a_i b_i} - 1) \leqslant \sum\limits_{i=1}^{n} (\sqrt{a_i b_i} - 1).$

In the present paper we intend to treat the same problem in the case of block matrices.

The condition C1° which is a restatement of a generalized Hadamard inequality has a complete analogy in the block case. In the absence of commutativity, it turns out that the notion of the spectral geometric mean recently introduced by the authors [3] has an interesting application: the spectral geometric means of the corresponding diagonal blocks of A and A^{-1} have to be greater than or equal to identity matrices of appropriate sizes in the Loewner ordering.

The generalization of condition C2° is more complicated. We have succeded to present a complete characterization for the case that the number of block rows n is two. For the case $n > 2$ we give a necessary condition and a sufficient condition.

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For the purposes of this paper, it will be convenient to introduce a terminological convention.

Consider two n-tuples of square matrices

$$(A_1, \dots, A_n), \quad (B_1, \dots, B_n)$$

such that, for each j, the matrix B_j is of the same order as A_j.

We shall say the *completion problem for* $A_1, \dots, A_n$ *and* $B_1, \dots, B_n$ *has a solution* if the following condition is satisfied:

There exists a positive definite matrix A whose diagonal blocks are $A_1, \dots, A_n$ and such that the diagonal blocks of A^{-1} are $B_1, \dots, B_n$.

We shall need the following two basic lemmata:

Lemma 1.1. *Given nonsingular square matrices $M_1, \dots, M_n$ of appropriate orders the completion problem for*

$$M_1 A_1 M_1^*, \dots, M_n A_n M_n^*, M_1^{*-1} B_1 M_1^{-1}, \dots, M_n^{*-1} B_n M_n^{-1}$$

has a solution if and only if the completion problem for

$$A_1, \dots, A_n, B_1, \dots, B_n$$

has a solution.

Proof. If M is the block-diagonal matrix $\operatorname{diag}(M_1, \dots, M_n)$ and if A solves the second problem then clearly MAM^* solves the first and conversely. □

Lemma 1.2. *Let A and B be positive definite matrices of the same order. Then there exists a nonsingular matrix M and a diagonal matrix D such that*

$$MAM^* = M^{*-1} B M^{-1} = D.$$

The diagonal entries of the matrix D are the positive square roots of the eigenvalues of AB.

Proof. Define the matrix Q by

$$Q = (A^{-\frac{1}{2}} (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}} A^{-\frac{1}{2}})^{\frac{1}{2}}$$

where, for a positive definite matrix P, $P^{\frac{1}{2}}$ means the (unique) positive definite square root of P. It is easily checked that

$$QAQ = Q^{-1} B Q^{-1}. \tag{1}$$

Let

$$(2) \qquad QAQ = UDU^*$$

be the spectral decomposition of QAQ, D being the diagonal matrix of the eigenvalues and U a unitary matrix of the eigenvectors. Since by (1)

$$UD^2U^* = QABQ^{-1},$$

the diagonal entries of D are as asserted. If we define the matrix M by

$$M = U^*Q,$$

it is immediate by (2) and (1) that the relations in the theorem are satisfied. □

2. Results

We shall consider n by n Hermitian matrices partitioned into m block rows and m block columns. We shall assume that the partitioning is symmetric, corresponding to the decomposition $N = N_1 \cup \ldots \cup N_m$ of the set $N = \{1, \ldots, n\}$ where the cardinalities $|N_j| = n_j$ need not be equal. In fact, we shall assume that N_1 is the set of the first n_1 indices, N_2 the set of the next n_2 indices etc. so that we can write, say, $A = (A_{ik})$ where A_{ik} is an $n_i \times n_k$ submatrix of A. We shall now define the *block quasi-Hadamard product* $A \circ B$ of two such Hermitian block matrices $A = (A_{ik})$ and $B = (B_{ik})$ as the (again Hermitian) m by m matrix with the entries

$$(3) \qquad (A \circ B)_{ik} = \mathrm{tr} A_{ik} B_{ki}, \quad i, k = 1, \ldots, m.$$

(Observe that this quasi-Hadamard product for the case of 1 by 1 blocks is the usual Hadamard product of A and B^*.)

We shall express now the quadratic form corresponding to $A \circ B$ in terms of the matrices A and B as follows. Given a vector $x = (x_1, \ldots, x_m)^T$ we denote by X the diagonal matrix

$$X = \mathrm{diag}(x_1 I_1, \ldots, x_m I_m)$$

where I_j stands for the identity matrix of order n_j.

Lemma 2.1. *In the notation from above,*

$$(4) \qquad ((A \circ B)x, x) = \mathrm{tr} AXBX^*,$$

$$(5) \qquad (A \circ B)e = (\mathrm{tr}(AB)_{11}, \ldots, \mathrm{tr}(AB)_{mm})^T$$

where $e = (1, \ldots, 1)^T$.

Proof. We have

$$\begin{aligned}
\mathrm{tr}AXBX^* &= \mathrm{tr}\sum_{i,k}(AX)_{ik}(BX^*)_{ki} \\
&= \mathrm{tr}\sum_{i,k}(A_{ik}x_k)(B_{ki}x_i^*) \\
&= \sum_{i,k} x_i^* x_k \mathrm{tr}A_{ik}B_{ki} \\
&= ((A \circ B)x, x).
\end{aligned}$$

The proof of (5) is immediate. □

In the same notation as above, denote by M the diagonal matrix

$$M = \mathrm{diag}(n_1, \ldots, n_m).$$

Theorem 2.2. *Let* A *be a positive definite block matrix. Then the matrix*

$$(6) \qquad P = A \circ A^{-1} - M$$

is positive semidefinite and satisfies

$$(7) \qquad Pe = 0.$$

Proof. Since A is positive definite there exists a nonsingular upper triangular matrix T such that $A = T^*T$. We have thus by (4)

$$\begin{aligned}
(A \circ A^{-1})x, x) &= \mathrm{tr}AXA^{-1}X^* \\
&= \mathrm{tr}T^*TXT^{-1}T^{-1*}X^* \\
&= \mathrm{tr}(TXT^{-1})(TXT^{-1})^* \\
&= \sum_{i,k=1}^{n} |(TXT^{-1})_{ik}|^2 \\
&\geqslant \sum_{i=1}^{n} |(TXT^{-1})_{ii}|^2 \\
&= \sum_{j=1}^{m} n_j |x_j|^2 \\
&= (Mx, x).
\end{aligned}$$

Also,

$$(A \circ A^{-1})e = Me$$

by (5) which implies (7). □

Let us return now to the completion problem for block positive definite matrices.We shall first prove:

Theorem 2.3. *Let $A_1, \ldots, A_m, B_1, \ldots, B_m$ be positive definite matrices such that for each $j = 1, \ldots, m$, both A_j and B_j are of the same order n_j. For each $j = 1, \ldots, m$, denote by D_j the diagonal matrix whose diagonal entries are the positive square roots of the eigenvalues of $A_j B_j$. Then the completion problem $(A_1, \ldots, A_m, B_1, \ldots, B_m)$ has a solution if and only if the completion problem $(D_1, \ldots, D_m, D_1, \ldots, D_m)$ has a solution.*

Proof. Follows from Lemmata 1.1 and 1.2. □

Theorem 2.4. *Let $A = (A_{ik})$ be a positive definite block matrix with m block rows, let $A^{-1} = (B_{ik})$. Then:*

1° *for each $j = 1, \ldots, m$, all eigenvalues of $A_{jj}B_{jj}$ are greater than or equal to one;*

2° $2 \max\limits_{j=1,\ldots,m} \sqrt{\mathrm{tr} A_{jj} B_{jj} - n_j} \leqslant \sum\limits_{j=1,\ldots,m} \sqrt{\mathrm{tr} A_{jj} B_{jj} - n_j}.$

Proof. 1° follows from Theorem 2.3 and C1°. To prove 2°, we shall apply the idea of [1] to the matrix P from Theorem 2.2. Since P is positive semidefinite with row sums zero, it is the Gram matrix of a system of m vectors, say $u_1, \ldots, u_m$ whose sum is zero and thus form a closed polygon. Therefore the length of the largest vector, $\max\limits_k \sqrt{(u_k, u_k)}$, does not exceed the sum of the remaining lengths. Since $(u_j, u_j) = \mathrm{tr} A_{jj} B_{jj} - n_j$ in our notation, 2° follows. □

Remark 2.5. Let us mention that the condition 1° can be equivalently formulated in other ways, e.g. that the matrix $A^{\frac{1}{2}} B A^{\frac{1}{2}}$ is greater than or equal to the identity matrix in the Loewner ordering, or, that the *spectral geometric mean* of the matrices A and B introduced in [3], which is the common value in (1) has this property.

Remark 2.6. Using Theorem 2.3, the above completion problem simplifies in the sense that just conditions about the diagonal entries of the matrices D_i, i.e. about the square roots of the eigenvalues of $A_i B_i$ (or, equivalently, $B_i A_i$) are to be found.

Let us denote the diagonal entries of D_j by $d_1^{(j)}, \ldots, d_{n_j}^{(j)}, j = 1, \ldots, m$. Theorem 2.4 then rephrases as follows:

Theorem 2.7. *A necessary condition for the numbers $d_k^{(j)}$ that the completion problem $D_1, \dots, D_m, D_1, \dots, D_m$ be solvable is:*

1° *all the numbers $d_k^{(j)}$ satisfy*

$$d_k^{(j)} \geqslant 1, \quad k = 1, \dots, n_j, \quad j = 1, \dots, m;$$

2° $$2 \max_{j=1,\dots,m} \sqrt{\sum_{k=1}^{n_j} (d_k^{(j)} - 1)} \leqslant \sum_{j=1}^{m} \sqrt{\sum_{k=1}^{n_j} (d_k^{(j)} - 1)}.$$

To find a sufficient condition for the numbers $d_k^{(j)}$, observe that by C1° and C2° which are (necessary and) sufficient for the case $n_1 = \ldots = n_m = 1$, the condition C2° reads:

C 2′° $$2 \max_{k=1,\dots,m} (d_1^{(k)} - 1) \leqslant \sum_{k=1}^{m} (d_1^{(k)} - 1).$$

Theorem 2.8. *Let D_j, $j = 1, \dots, m$, be diagonal matrices with corresponding diagonal entries $d_1^{(j)}, \dots, d_{n_j}^{(j)}$, $j = 1, \dots, m$. The completion problem for $D_1, \dots, D_m, D_1, \dots, D_m$ has a solution if all the numbers $d_k^{(j)}$ are greater than or equal to one and there exists for each j an ordering of the $d_k^{(j)}$'s and a system of eventual gaps such that in the array*

$$\begin{array}{ccccccccccc}
\dots & d_{i_1}^{(1)}, & \dots & \dots & d_{i_2}^{(1)}, & \dots & \dots & d_{i_{n_1}}^{(1)}, & \dots \\
\dots & \dots & d_{k_1}^{(2)}, & \dots & \dots & d_{k_2}^{(2)}, & \dots, & d_{k_{n_2}}^{(2)}, & \dots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\dots & d_{l_1}^{(m)}, & \dots & d_{l_2}^{(m)}, & \dots, & d_{l_{n_m}}^{(m)}, & \dots & \dots & \dots
\end{array}$$

each column (the gaps are not considered) satisfies the condition C2′°.

P r o o f. By the result [2], there exist solutions for the completion problem corresponding to each column of the presented array. We assume that its dimension is in each column equal to the number of the $d_k^{(j)}$'s in that column. If we then reorder simultaneously the rows and columns of the direct sum of these solutions starting first with the diagonal entry in the first row of the array in the original ordering (if there is not a gap), continuing with the second row of the array in the original ordering etc. and ending with the last row in the original ordering, we obtain a solution of the problem $D_1, \dots, D_m, D_1, \dots, D_m$. □

Example 2.9. For $n_1 = 4$, $n_2 = 4$, $n_3 = 3$, let

$$\begin{array}{cccc}
7, & 5, & 4, & 3, \\
5, & 3, & 2, & 2, \\
8, & 6, & 2, &
\end{array}$$

be the array of Theorem 2.8. The third column and the fourth column do not satisfy the condition C2′°. However, if we perform appropriate permutations in each row to obtain (with just one gap in the last row) the array

$$\begin{matrix} 7, & 4, & 3, & 5, \\ 2, & 3, & 2, & 5, \\ 8, & 6, & 2, & \end{matrix}$$

then all columns satisfy this condition (even with equality).

Let, say, $A = (a_{ik})$ be a 3-by-3 matrix which completes the first column, $B = (b_{ik})$ the second column, $C = (c_{ik})$ the third and $D = (d_{ik})$ a 2-by-2 matrix completing the fourth column. Then the following 11-by-11 matrix (zeros are not marked) solves the problem:

$$\begin{pmatrix}
7 & & & & & & a_{12} & & a_{13} & & \\
 & 5 & & & d_{12} & & & & & & \\
 & & 4 & & & b_{12} & & & & b_{13} & \\
 & & & 3 & & & & c_{12} & & & c_{13} \\
 & d_{12} & & & 5 & & & & & & \\
 & & b_{12} & & & 3 & & & & & \\
a_{12} & & & & & & 2 & & & & \\
 & & & c_{12} & & & & 2 & & & c_{13} \\
a_{13} & & & & & & & & 8 & & \\
 & & b_{13} & & & & & & & 6 & \\
 & & & c_{13} & & & & c_{23} & & & 2
\end{pmatrix}.$$

Before we state the last theorem let us make the following observation:

Remark 2.10. For two numbers p_1, p_2, the inequality

$$2 \max_{i=1,2} p_i \leqslant \sum_{i=1}^{2} p_i$$

holds if and only if $p_1 = p_2$. Thus in this case, the solution of the 2×2 problem for p_1, p_2, p_1, p_2 exists if and only if $p_1 = p_2$ and $p_1 \geqslant 1$; indeed, the solution is $P = \begin{pmatrix} p_1 & \sqrt{p_1^2 - 1} \\ \sqrt{p_1^2 - 1} & p_1 \end{pmatrix}$ since $P^{-1} = \begin{pmatrix} p_1 & -\sqrt{p_1^2 - 1} \\ -\sqrt{p_1^2 - 1} & p_1 \end{pmatrix}$.

This enables us to prove that the sufficient condition in Theorem 2.8 is also necessary if $m = 2$.

Theorem 2.11. *Let A_1, B_1 be positive definite matrices of the same order and let A_2, B_2 be positive definite matrices of the same order. Then the completion* *problem A_1, A_2, B_1, B_2 has a solution if and only if all the eigenvalues of both matrices A_1B_1 and A_2B_2 are greater than or equal to one and those greater than one in A_1B_1 coincide with those of A_2B_2 (including multiplicities).*

Proof. Let first the completion problem for A_1, A_2, B_1, B_2 have a solution, let

$$A = \begin{pmatrix} A_1 & A_{12} \\ A_{12}^* & A_2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} B_1 & B_{12} \\ B_{12}^* & B_2 \end{pmatrix}$$

be this solution. Then

$$\begin{aligned} A_1B_1 + A_{12}B_{12}^* &= I_1, \\ B_2A_2 + B_{12}^*A_{12} &= I_2 \end{aligned}$$

which implies that the matrices $A_1B_1 - I_1$ and $B_2A_2 - I_2$ have the same non-zero eigenvalues including multiplicities. In the notation of Theorem 2.7 this means that those numbers $d_k^{(1)}$ which are greater than one coincide with such numbers $d_j^{(2)}$ including multiplicities.

Conversely, let this last condition be satisfied. Observe that in such case, the sufficient condition of Theorem 2.8 is fulfilled. □

Remark 2.12. The problem of necessary and sufficient conditions for the block case and $m > 2$ remains open.

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