Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 3, 385–394

Persistent URL: `http://dml.cz/dmlcz/127364`

Czechoslovak Mathematical Journal, 47 (122) (1997), Praha

VECTOR-VALUED PSEUDO ALMOST PERIODIC FUNCTIONS

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(Received August 20, 1992)

Abstract. Vector-valued pseudo almost periodic functions are defined and their properties are investigated. The vector-valued functions contain many known functions as special cases. A unique decomposition theorem is given to show that a vector-valued pseudo almost periodic function is a sum of an almost periodic function and an ergodic perturbation.

Keywords: almost periodic functions, pseudo almost periodic functions

MSC 1991: 42A75, 43A60

In [13, 14], we defined and investigated the space of numerical pseudo almost periodic functions, which is a new generalization of the almost peridic functions; as for the space of almost periodic functions and some of its generalizations, pseudo almost periodic functions have many applications in the theory of differential equations. In this paper, we deal with vector-valued pseudo almost periodic functions.

Throughout this paper, X denotes a Banach space and $\mathbb{J}_a$ stands for $[a,\infty)$ when $a \in \mathbb{R}$ and for $\mathbb{R}$ when $a = -\infty$; $C(\mathbb{J}_a, X)$ denotes the space of all bounded continuous functions from $\mathbb{J}_a$ to X. Also, m denotes Lebesgue measure on $\mathbb{R}$.

Definition 1. A subset P of $\mathbb{J}_a$ is said *to be relatively dense in* $\mathbb{J}_a$ if there exists a number $l > 0$ such that

$$[t, t+l] \cap P \neq \emptyset \qquad (t \in \mathbb{J}_a).$$

Definition 2. A closed subset C of $\mathbb{J}_a$ is said *to be an ergodic zero set in* $\mathbb{J}_a$ if $m(C \cap [a,t])/(t-a) \to 0$ as $t \to \infty$ ($m(C \cap [-t,t])/2t \to 0$ as $t \to \infty$, when $a = -\infty$).

Since $\lim\limits_{t\to\infty} m(C \cap [a,t])/(t-a) = \lim\limits_{t\to\infty} m(C \cap [a,t])/t$ for $a \in \mathbb{R}$, we will use the latter limit.

Proposition 3. *Let C be an ergodic zero set in $\mathbb{J}_a$. Then for any $\delta > 0$ and $L > 0$, there exists an interval $(u,v) \subset \mathbb{J}_a$ with the properties that $v - u > L$ and*

$$m(C \cap (u,v)) < \delta.$$

Proof. If such a (u,v) does not exist, one sees readily that $\liminf_{t\to\infty} m(C \cap [a,t])/t \geqslant \delta/2L$ ($\liminf_{t\to\infty} m(C \cap [-t,t])/2t \geqslant \delta/2L$ when $a = -\infty$). □

Proposition 4. *Let P be relatively dense in $\mathbb{J}_a$ and let C be an ergodic zero set in $\mathbb{J}_a$. Then for any given interval $[c,d] \subset \mathbb{R}$ and $\delta > 0$, there exist $(u,v) \subset \mathbb{J}_a$ and $\tau \in P$ such that*

$$[c,d] + \tau \subset (u,v),$$

and

$$m(C \cap (u,v)) < \delta.$$

Proof. Let $l > 0$ be the number for P as in Definition 1 and let $L = l + (d-c)$. By Proposition 3, there exists an interval $(u,v) \subset \mathbb{J}_a$ such that $m(C \cap (u,v)) < \delta$ and $L < v - u$. Since we can assume that $u - c \in \mathbb{J}_a$, we can choose $\tau \in [u-c, u-c+l] \cap P$. If $t \in [c,d]$,

$$u < c + \tau \leqslant t + \tau \leqslant d + \tau \leqslant d + u - c + l < v,$$

that is, $[c,d] + \tau \subset (u,v)$. □

Definition 5. A function $f \in \mathcal{C}(\mathbb{J}_a, X)$ is said *to be pseudo almost periodic* if for each $\varepsilon > 0$, there are a number $\delta > 0$, a relatively dense subset $P(\varepsilon)$ of $\mathbb{J}_a$, and an ergodic zero subset C_ε of $\mathbb{J}_a$ such that

$$\|f(t) - f(t+\tau)\| < \varepsilon \qquad (\tau \in P(\varepsilon),\ t, t+\tau \in \mathbb{J}_a \setminus C_\varepsilon), \tag{1}$$

and

$$\|f(t') - f(t'')\| < \varepsilon \qquad (t', t'' \in \mathbb{J}_a \setminus C_\varepsilon,\ |t' - t''| < \delta). \tag{2}$$

$\mathcal{PAP}(\mathbb{J}_a, X)$ will denote the set of all pseudo almost periodic functions from $\mathbb{J}_a$ to X and $\mathcal{PAP}_0(\mathbb{J}_a, X)$ is defined to be the set of all the functions $f \in \mathcal{C}(\mathbb{J}_a, X)$ with the property that $1/t \int_a^t \|f(x)\|\,\mathrm{d}x \to 0$ as $t \to \infty$ ($1/2t \int_{-t}^t \|f(x)\|\,\mathrm{d}x \to 0$ as $t \to \infty$, when $a = -\infty$).

Remarks 6. Under some restrictions on a and C_ε in Definition 5, the functions defined there reduce to familar ones which have been extensively investigated. For example,

(1) when $a = -\infty$, so $\mathbb{J}_a = \mathbb{R}$, and $C_\varepsilon = \emptyset$, $\mathcal{PAP}(\mathbb{R}, X) = \mathcal{AP}(\mathbb{R}, X)$, the space of almost periodic functions [1, 3, 4, 5, 8].
(2) when $a = 0$ and $C_\varepsilon = \emptyset$, $\mathcal{PAP}(\mathbb{J}_0, \mathbb{C}) = \mathcal{SAP}(\mathbb{J}_0)$, the space of strongly almost periodic functions [2, 6].
(3) when $a = 0$ and C_ε is bounded, $\mathcal{PAP}(\mathbb{J}_0, \mathbb{C}) = \mathcal{AP}(\mathbb{J}_0)$, the space of almost periodic functions [2, 6].
(4) when $a \in \mathbb{R}$ and C_ε is bounded, $\mathcal{PAP}(\mathbb{J}_a, X) = \mathcal{AAP}(\mathbb{J}_a, X)$, the space of asymptotically almost periodic functions [10, 11, 12].

In all the cases mentioned in Remarks 6, (2) in Definition 5 is a consequence of (1). However, we will show in Example 14 that (2) is independent of (1).

The proofs of the following two propositions are straightforward, we omit them.

Proposition 7. *A function $\varphi \in \mathcal{C}(\mathbb{J}_a, X)$ is in $\mathcal{PAP}_0(\mathbb{J}_a, X)$ if and only if, for $\varepsilon > 0$, the set $C_\varepsilon = \{t \in \mathbb{J}_a \colon \|\varphi(t)\| \geqslant \varepsilon\}$ is an ergodic zero set in $\mathbb{J}_a$.*

Proposition 8. *Let C_i, $i = 1, 2, \ldots, n$, be ergodic zero sets. Then $C = \bigcup_{i=1}^{n} C_i$ is also an ergodic zero set in $\mathbb{J}_a$.*

Let $g \in \mathcal{C}(\mathbb{R}, X)$ and let $\varepsilon > 0$. Set

$$P(\varepsilon) = \{\tau \in \mathbb{R} \colon \|g(t) - g(t+\tau)\| < \varepsilon \ \text{ for all } t \in \mathbb{R}\}.$$

Then, from Remark 6 (1), $g \in \mathcal{AP}(\mathbb{R}, X)$ if and only if $P(\varepsilon)$ is relatively dense in $\mathbb{R}$.

If $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{PAP}_0(\mathbb{J}_a, X)$, set $f = g\big|_{\mathbb{J}_a} + \varphi$. Then $f \in \mathcal{PAP}(\mathbb{J}_a, X)$. For, the almost periodicity of g implies that there is a relatively dense subset $P(\varepsilon/3) \subset \mathbb{R}$ such that

$$\|g(t) - g(t+\tau)\| < \frac{\varepsilon}{3} \qquad (t \in \mathbb{R},\ \tau \in P(\varepsilon/3)).$$

The uniform continuity of g [5, Theorem 6.2] implies that there is a number $\delta > 0$ such that

$$\|g(t') - g(t'')\| < \frac{\varepsilon}{3} \qquad (t', t'' \in \mathbb{R},\ |t' - t''| < \delta).$$

Set

$$C_\varepsilon = \left\{t \in \mathbb{J}_a \colon \|\varphi(t)\| \geqslant \frac{\varepsilon}{3}\right\};$$

by Proposition 7, C_ε is an ergodic zero set of $\mathbb{J}_a$. Now it is easy to show that f satisfies Definition 5.

The next theorem shows the converse: any function $f \in \mathcal{PAP}(\mathbb{J}_a, X)$ has a unique decomposition like this. As in [13], we will call g the almost periodic component and φ the ergodic perturbation respectively of f. Before stating the theorem, we need the following lemmas.

Lemma 9. *Let P be relatively dense in $\mathbb{J}_a$ and let C be an ergodic zero set in $\mathbb{J}_a$. For each $\tau \in P$, set $B_\tau = \{t \in \mathbb{R} \colon t + \tau \in C \cup (\mathbb{R} \setminus \mathbb{J}_a)\}$ ($B_\tau = \{t \in \mathbb{R} \colon t + \tau \in C\}$ when $a = -\infty$) and*

$$B = \bigcap_{\tau \in P} B_\tau. \tag{3}$$

Then $m(B) = 0$.

Proof. To show that $m(B) = 0$, it suffices to show that for any interval $[c, d] \subset \mathbb{R}$ and $\delta > 0$, $m([c, d] \cap B) < \delta$. Note that $t \in \mathbb{R} \setminus B$ if and only if there is a $\tau \in P$ such that $t + \tau \in \mathbb{J}_a \setminus C$. By Proposition 4, there exist $(u, v) \subset \mathbb{J}_a$ and $\tau \in P$ such that

$$[c, d] + \tau \subset (u, v),$$

and

$$m(C \cap (u, v)) < \delta.$$

This means that $m([c, d] \cap B) < \delta$. □

Lemma 10. *Let P be relatively dense in $\mathbb{J}_a$, let C be an ergodic zero set in $\mathbb{J}_a$, and let $t_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. Then for any $\delta > 0$, there exist a $\tau \in P$ and a $\Delta t \in [0, \delta)$ such that $t_i + \Delta t + \tau \in \mathbb{J}_a \setminus C$, $i = 1, 2, \ldots, n$.*

Proof. Suppose that $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$. Consider the interval $[t_1, t_n + \delta]$. Proposition 4 shows that there exist an interval $(u, v) \subset \mathbb{J}_a$ and a $\tau \in P$ such that $[t_1, t_n + \delta] + \tau \subset (u, v)$ and $m((u, v) \cap C) < \delta/n$. Set

$$F_i = \{0 \leqslant t < \delta \colon t_i + t + \tau \in C\},$$

and

$$F = \bigcup_{i=1}^{n} F_i.$$

Since $m(F_i) \leqslant m((u, v) \cap C)$, $i = 1, 2, \ldots, n$, $m(F) < \delta$. Therefore $[0, \delta) \setminus F \neq \emptyset$. We can choose $\Delta t \in [0, \delta) \setminus F$ as required. □

We are now going to prove the main result of the paper. Since the result for $\mathbb{R} = \mathbb{J}_{-\infty}$ is a simple corollary of that for $\mathbb{J}_a$, $a \in \mathbb{R}$ (see Remark 12 (3)), we will discuss only the latter.

Theorem 11. *A function $f \in \mathcal{C}(\mathbb{J}_a, X)$ is pseudo almost periodic if and only if there is a unique function $g \in \mathcal{AP}(\mathbb{R}, X)$ such that $f - g\big|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$.*

Proof. We only need to show the only if part.

Choose a sequence of positive numbers $\{\varepsilon_n\}$ decreasing to zero. Let δ_n, $P(\varepsilon_n/7)$ and C_n be for ε_n as in Definition 5. For $P(\varepsilon_n/7)$ and C_n, we have $B_n \subset \mathbb{R}$ from (3) of Lemma 9 with $m(B_n) = 0$. Without loss of generality, we may assume that $C_n \subset C_{n+1}$ for all $n \in \mathbb{N}$ since we can replace C_{n+1} by $C_n \cup C_{n+1}$, which still satisfies Definitions 2 and 5. Set $Q(\varepsilon_n) = P(\varepsilon_n/7) \cup P'(\varepsilon_n/7)$, where $P'(\varepsilon_n/7) = \{\tau\colon -\tau \in P(\varepsilon_n/7)\}$. $Q(\varepsilon_n)$ is relatively dense in $\mathbb{R}$.

In the proof of Lemma 9, we pointed out that for each $t \in \mathbb{R} \setminus B_n$, we can choose a $\tau_{n,t} \in P(\varepsilon_n/7)$ such that $t + \tau_{n,t} \in \mathbb{J}_a \setminus C_n$. Define a function f_n on $\mathbb{R} \setminus B_n$ by

$$f_n(t) = f(t + \tau_{n,t}). \tag{4}$$

f_n is well-defined on $\mathbb{R} \setminus B_n$.

Set

$$B = \bigcup_{n=1}^{\infty} B_n.$$

Since $m(B_n) = 0$, $n = 1, 2, \ldots$, $m(B) = 0$.

We will show that the sequence $\{f_n\}$ converges uniformly to a function $g \in \mathcal{AP}(\mathbb{R}, X)$ on $\mathbb{R} \setminus B$. First of all, we show that each f_n satisfies

$$\|f_n(t) - f_n(t + \tau)\| < \varepsilon_n, \qquad (\tau \in Q(\varepsilon_n),\ t, t + \tau \in \mathbb{R} \setminus B_n) \tag{5}$$

and

$$\|f_n(t') - f_n(t'')\| < \varepsilon_n, \qquad (t', t'' \in \mathbb{R} \setminus B_n,\ |t' - t''| < \delta_n). \tag{6}$$

We show (6) first. According to (4),

$$\|f_n(t') - f_n(t'')\| = \|f(t' + \tau_{n,t'}) - f(t'' + \tau_{n,t''})\|, \tag{7}$$

where $t' + \tau_{n,t'}$, $t'' + \tau_{n,t''} \in \mathbb{J}_a \setminus C_n$. Lemma 10, along with the facts that C_n is closed and f is continuous at $t' + \tau_{n,t'}$, $t'' + \tau_{n,t''} \in \mathbb{J}_a \setminus C_n$, implies that there are a

$\tau \in P(\varepsilon_n/7)$ and $\Delta t \in [0, \delta_n)$ such that

$$\begin{gathered} t' + \tau_{n,t'} + \Delta t + \tau,\ t'' + \tau_{n,t'} + \Delta t + \tau, \\ t'' + \Delta t + \tau,\ t'' + \tau_{n,t''} + \Delta t + \tau, \\ t' + \tau_{n,t'} + \Delta t,\ t'' + \tau_{n,t''} + \Delta t \in \mathbb{J}_a \setminus C_\varepsilon \end{gathered}$$

and

$$\begin{aligned} &\|f(t' + \tau_{n,t'}) - f(t' + \tau_{n,t'} + \Delta t)\| < \varepsilon_n/7, \\ &\|f(t'' + \tau_{n,t''}) - f(t'' + \tau_{n,t''} + \Delta t)\| < \varepsilon_n/7. \end{aligned} \tag{8}$$

It follows from (1), (2) and (8) that

$$\begin{aligned} &\|f(t' + \tau_{n,t'}) - f(t'' + \tau_{n,t''})\| \\ &\quad \leqslant \|f(t' + \tau_{n,t'}) - f(t' + \tau_{n,t'} + \Delta t)\| \\ &\qquad + \|f(t' + \tau_{n,t'} + \Delta t) - f(t' + \tau_{n,t'} + \Delta t + \tau)\| \\ &\qquad + \|f(t' + \tau_{n,t'} + \Delta t + \tau) - f(t'' + \tau_{n,t'} + \Delta t + \tau)\| \\ &\qquad + \|f(t'' + \tau_{n,t'} + \Delta t + \tau) - f(t'' + \Delta t + \tau)\| \\ &\qquad + \|f(t'' + \Delta t + \tau) - f(t'' + \tau_{n,t''} + \Delta t + \tau)\| \\ &\qquad + \|f(t'' + \tau_{n,t''} + \Delta t + \tau) - f(t'' + \tau_{n,t''} + \Delta t)\| \\ &\qquad + \|f(t'' + \tau_{n,t''} + \Delta t) - f(t'' + \tau_{n,t''})\| \\ &\quad < \varepsilon_n. \end{aligned} \tag{9}$$

Similarly, we can show (5) in the case that $\tau \in P(\varepsilon_n/7)$ and $t,\ t + \tau \in \mathbb{R} \setminus B_n$.

If $\tau \in P'(\varepsilon_n/7)$ and $t, t+\tau \in \mathbb{R}\setminus B_n$, set $T = t+\tau$ and $\tau' = -\tau$. Then $\tau' \in P(\varepsilon_n/7)$ and $t = T + \tau'$. Therefore

$$\|f_n(t) - f_n(t + \tau)\| = \|f_n(T) - f_n(T + \tau')\| < \varepsilon_n.$$

Now we show that the sequence $\{f_n\}$ converges uniformly on $\mathbb{R} \setminus B$. In fact, for $t \in \mathbb{R}\setminus B$, by (4) $f_m(t) = f(t+\tau_{m,t})$ and $f_n(t) = f(t+\tau_{n,t})$, where $t+\tau_{m,t} \in \mathbb{J}_a \setminus C_m$ and $t + \tau_{n,t} \in \mathbb{J}_a \setminus C_n$. Say, $m > n$, so $C_m \supset C_n$ and $\mathbb{J}_a \setminus C_m \subset \mathbb{J}_a \setminus C_n$. Note that $\varepsilon_n > \varepsilon_m$. In (9), replace t', t'' by t, $\tau_{n,t'}$ and $\tau_{n,t''}$ by $\tau_{m,t}$ and $\tau_{n,t}$ respectively, and ε_n by ε_m, and choose $\tau \in P(\varepsilon_m/7)$; we get

$$\begin{aligned} \|f_m(t) - f_n(t)\| &= \|f(t + \tau_{m,t}) - f(t + \tau_{n,t})\| \\ &< \frac{4\varepsilon_m}{7} + \frac{3\varepsilon_n}{7} < \varepsilon_n. \end{aligned} \tag{10}$$

Thus there is a function g on $\mathbb{R} \setminus B$ such that $f_n \to g$ uniformly on $\mathbb{R} \setminus B$ as $n \to \infty$. For $\varepsilon > 0$, we choose j_0 such that $\varepsilon_{j_0} < \varepsilon/5$ and

$$\|g(t) - f_{j_0}(t)\| < \frac{\varepsilon}{5} \qquad (t \in \mathbb{R} \setminus B). \tag{11}$$

Now we show three assertions.

(i) If a sequence $\{t_n\} \subset \mathbb{R} \setminus B$ is Cauchy, so is $\{g(t_n)\}$. For, by (6) and (11)

$$\begin{aligned} \|g(t_n) - g(t_m)\| \leqslant \|g(t_n) - f_{j_0}(t_n)\| + \|f_{j_0}(t_n) - f_{j_0}(t_m)\| \\ + \|f_{j_0}(t_m) - g(t_m)\| < \varepsilon. \end{aligned}$$

This implies that g is continuous on $\mathbb{R} \setminus B$ and extends uniquely to $\mathbb{R}$ by continuity.

(ii) $g \in \mathcal{AP}(\mathbb{R}, X)$. By (5) and (11) one can similarly show that, for all $t \in \mathbb{R}$ and $\tau \in Q(\varepsilon_{j_0})$,

$$\begin{aligned} &\|g(t) - g(t+\tau)\| \\ &\quad \leqslant \|g(t) - g(t+\Delta t)\| + \|g(t+\Delta t) - f_{j_0}(t+\Delta t)\| \\ &\qquad + \|f_{j_0}(t+\Delta t) - f_{j_0}(t+\Delta t+\tau)\| + \|f_{j_0}(t+\Delta t+\tau) - g(t+\Delta t+\tau)\| \\ &\qquad + \|g(t+\Delta t+\tau) - g(t+\tau)\| < \varepsilon, \end{aligned}$$

where, as before, a small number $\Delta t > 0$ is chosen such that $t+\Delta t, t+\Delta t+\tau \in \mathbb{R}\setminus B$, $\|g(t)-g(t+\Delta t)\| < \varepsilon/5$, and $\|g(t+\tau)-g(t+\Delta t+\tau)\| < \varepsilon/5$. Since $Q(\varepsilon_{j_0})$ is relatively dense in $\mathbb{R}$, $g \in \mathcal{AP}(\mathbb{R}, X)$.

(iii) $f - g\big|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$. In fact, if $x \in \mathbb{J}_a \setminus (C_{j_0} \cup B)$, then by (1), (4) and (11)

$$\begin{aligned} \|f(x) - g(x)\| &\leqslant \|f(x) - f_{j_0}(x)\| + \|f_{j_0}(x) - g(x)\| \\ &= \|f(x) - f(x+\tau_{j_0,x})\| + \|f_{j_0}(x) - g(x)\| \\ &< \varepsilon. \end{aligned}$$

Set $M_0 = \sup_{s\in\mathbb{J}_a} \|f(s) - g(s)\|$, it follows from the inequality above that when t is sufficiently large

$$\begin{aligned} \frac{1}{t}\int_a^t \|f(x) - g(x)\|\,\mathrm{d}x &\leqslant \frac{1}{t}\Big\{(t-a)\varepsilon + \int_{C_{j_0}\cup B\cap[a,t]} \|f(x) - g(x)\|\,\mathrm{d}x\Big\} \\ &\leqslant \frac{1}{t}\big\{(t-a)\varepsilon + M_0 m(C_{j_0} \cup B \cap [a,t])\big\} < 2\varepsilon, \end{aligned}$$

because $m(C_{j_0} \cap [a,t])/t \to 0$ as $t \to \infty$ and $m(B) = 0$.

Finally, the decomposition is unique. Note that, for $g \in \mathcal{AP}(\mathbb{R}, X)$, $g|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X) \Leftrightarrow \| g|_{\mathbb{J}_a}(\cdot)\| \in \mathcal{PAP}_0(\mathbb{J}_a, \mathbb{C}) \Leftrightarrow \|g(\cdot)\| \in \mathcal{PAP}_0(\mathbb{R}) \Leftrightarrow g = 0$, where $\|g(\cdot)\|$ is the function $t \in \mathbb{R} \to \|g(t)\|$. Therefore if there are two functions g_1, $g_2 \in \mathcal{AP}(\mathbb{R}, X)$ such that $f - g_i|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$, $i = 1, 2$, then $g_1|_{\mathbb{J}_a} - g_2|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$. So $g_1 = g_2$.

The proof is complete. □

As a consequence of Theorem 11, we have

$$\mathcal{PAP}(\mathbb{J}_a, X) = \mathcal{AP}(\mathbb{R}, X) \oplus \mathcal{PAP}_0(\mathbb{J}_a, X).$$

In case $X = \mathbb{C}$, we will omit X from our notation and write, for example, $\mathcal{PAP}(\mathbb{J}_a)$ for $\mathcal{PAP}(\mathbb{J}_a, X)$.

Remarks 12. (1) and (2) are known decomposition theorems; we have them as corollaries of Theorem 11.

(1) For a function $f \in \mathcal{AP}(\mathbb{J}_0)$ (as in Remark 6 (3)), it is known that $f = g|_{\mathbb{J}_0} + \varphi$, where $g \in \mathcal{AP}(\mathbb{R})$ and $\varphi\colon \mathbb{J}_a \to \mathbb{C}$ is continuous and has limit of zero when $t \to \infty$; see, for example, [2, 4.3.14].
(2) For a function $f \in \mathcal{AAP}(\mathbb{J}_a, X)$ (as in Remark 6 (4)), it is shown in [10, Theorem 3.4] and [12] that $f = g|_{\mathbb{J}_a} + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and φ: $\mathbb{J}_a \to X$ is continuous and vanishes at ∞.
(3) When the functions of (1) and (2) in Remarks 6 are scalar-valued, there is no essential difference between them because by Theorem 11 each function of type (2) has a unique extension a function of type (1).

Remark 13. Let $\mathcal{WRC}(\mathbb{J}_0, X)$ be the space of vector-valued weakly almost periodic functions with totally bounded ranges. It follows from [7, Theorem 4.17] and [9, Theorem 7] that f is in $\mathcal{WRC}(\mathbb{J}_0, X)$ if and only if $f = g|_{\mathbb{J}_0} + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{WRC}_0(\mathbb{J}_0, X)$, the space of 'flight vectors', those members of $\mathcal{WRC}(\mathbb{J}_0, X)$ that have 0 in the weak closure of the set of translates. With a proof similar to that of Corollary 4.19 in [7], one can show that $\mathcal{WRC}_0(\mathbb{J}_0, X) \subset \mathcal{PAP}_0(\mathbb{J}_0, X)$. Thus, $\mathcal{WRC}(\mathbb{J}_0, X) \subset \mathcal{PAP}(\mathbb{J}_0, X)$.

Now we give an example to show that (2) is independent of (1) in Definition 5.

Example 14. For $n \geqslant 4$, define a function f on $[n, n+1)$ as follows:

$$f(t) = \begin{cases} -\frac{1/2+1/n}{1/n}(t-n) + 1, & t \in [n, n+\frac{1}{n}), \\ -(t-n) + 1/2, & t \in [n+\frac{1}{n}, n+\frac{1}{2}), \\ 0, & t \in [n+\frac{1}{2}, n+1-\frac{1}{n+1}), \\ (n+1)[t-(n+1-\frac{1}{n+1})], & t \in [n+1-\frac{1}{n+1}, n+1). \end{cases}$$

The graph of the function f in each interval $[n, n+1)$ consists of four segments, and $f\colon [4,\infty) \to [0,1]$ is continuous. For each $\varepsilon > 0$, set $P(\varepsilon) = \{n\colon n = 4, 5, \ldots\}$ and $C_\varepsilon = [4, 4+1/4] \cup \left\{ \bigcup_{n=5}^{\infty} [n - 1/n, n + 1/n] \right\}$; then the function f satisfies all the conditions in Definition 5 except (2) since

$$\lim_{n\to\infty} f(n+1/n) = \frac{1}{2}, \quad \text{while} \quad f(n-1/n) = 0, \quad n = 5, 6, \ldots. \tag{12}$$

(12) also shows that the function f can not have a decomposition as in Theorem 11.

The following theorem comes directly from Definition 5

Theorem 15. *$\mathcal{PAP}(\mathbb{J}_a, X)$ is a Banach space.*

Theorem 16. *Let $f_i \in \mathcal{PAP}(\mathbb{J}_a, X)$, $i = 1, 2, \ldots, n$. For each $\varepsilon > 0$, there are a $\delta > 0$, a relatively dense subset $P(\varepsilon)$ of $\mathbb{J}_a$, and an ergodic zero subset C_ε of $\mathbb{J}_a$ such that for $i = 1, 2, \ldots, n$,*

$$\|f_i(t) - f_i(t+\tau)\| < \varepsilon \qquad (\tau \in P(\varepsilon),\ t, t+\tau \in \mathbb{J}_a \setminus C_\varepsilon), \tag{13}$$

and

$$\|f_i(t') - f_i(t'')\| < \varepsilon \qquad (t', t'' \in \mathbb{J}_a \setminus C_\varepsilon,\ |t' - t''| < \delta). \tag{14}$$

Proof. We know from Theorem 11 that $f_i = g_i\big|_{\mathbb{J}_a} + \varphi_i$, where $g_i \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi_i \in \mathcal{PAP}_0(\mathbb{J}_a, X)$, $i = 1, 2, \ldots, n$. Therefore there exists a relatively dense subset $P(\varepsilon)$ of $\mathbb{J}_a$ from [5, Proof of Theorem 6.9] such that for $i = 1, 2, \ldots, n$

$$\|g_i(t) - g_i(t+\tau)\| < \varepsilon/3 \qquad (t \in \mathbb{R},\ \tau \in P(\varepsilon)).$$

Since an almost periodic function is uniformly continuous on $\mathbb{R}$ [5, Theorem 6.2], there exists a $\delta > 0$ such that

$$\|g_i(t') - g_i(t'')\| < \varepsilon/3 \qquad (i = 1, 2, \ldots, n,\ |t' - t''| < \delta).$$

Set $C_i = \{t \in \mathbb{J}_a\colon \|\varphi_i(t)\| \geqslant \varepsilon/3\}$, $i = 1, 2, \ldots, n$ and

$$C_\varepsilon = \bigcup_{i=1}^{n} C_i.$$

By Propositions 7 and 8, C_ε is an ergodic zero set in $\mathbb{J}_a$. The proof is finished. □

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