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NEW EDGE NEIGHBORHOOD GRAPHS

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Abstract. Let G be an undirected simple connected graph, and $e = uv$ be an edge of G. Let $N_G(e)$ be the subgraph of G induced by the set of all vertices of G which are not incident to e but are adjacent to u or v. Let $\mathcal{N}_e$ be the class of all graphs H such that, for some graph G, $N_G(e) \cong H$ for every edge e of G. Zelinka [3] studied edge neighborhood graphs and obtained some special graphs in $\mathcal{N}_e$. Balasubramanian and Alsardary [1] obtained some other graphs in $\mathcal{N}_e$. In this paper we given some new graphs in $\mathcal{N}_e$.

1. INTRODUCTION

A problem concerning the neighborhood graphs of vertices of undirected graphs was proposed by Zykov in 1963. A problem analogous to that of Zykov, but concerning edge neighborhood graphs was studied by Zelinka [3].

We follow the notation and terminology of Harary [2]. Let G be an undirected simple connected graph, and let $e = uv$ be an edge of G. Let U be the set of all vertices of G that are adjacent to at least one of the two vertices u and v, and let $U_e = U - \{u, v\}$. Then, the induced subgraph $\langle U_e \rangle$ of G is called *edge neighborhood graph of e in G* and is denoted $N_G(e)$.

The edge neighborhood version of the problem of Zykov is the following. Characterize the graphs H with the property that there exists a graph G such that $N_G(e)$ is isomorphic to H, (i.e., $N_G(e) \cong H$) for each edge e of G.

Let $\mathcal{N}_e$ be the class of all graphs H such that, for some graph G, $N_G(e) \cong H$ for every edge e of G. Such graph G is called a *city* [1] (or *required* [3]) graph containing H, and denoted by C_H.

Zelinka [3] has proved that $\mathcal{N}_e$ includes the following graphs:

(i) K_n, for every positive integer n,

(ii) $K_{m,n}$, for every pair of positive integers m, n,

(iii) cycles C_4, C_6, C_8,

(iv) cubes Q_1, Q_2, Q_3,

(v) $K^*_{n,n}$, $n \geqslant 2$, where $K^*_{n,n}$ is obtained from $K_{n,n}$ by deleting edges a maximum matching.

Moreover, Balasubramanian and Alsardary [1] proved that $\mathcal{N}_e$ also includes the following graphs:

(vi) nK_2, (n copies of K_2),

(vii) the complete k-partite graph $K_{m-1,m-1,m,\ldots,m}$, $m \geqslant 2$,

(viii) $4K_1$ and $2K_1 \cup 2K_2$.

In the present work, we obtain new edge neighborhood graphs.

2. New edge neighborhood graphs

First we shall present some simple propositions.

Proposition 1. $nK_1 \in \mathcal{N}_e$.

Proof. The star S_{n+2} of $n+2$ vertices has the property that $N_{S_{n+2}}(e) \cong nK_1$ for each edge e of S_{n+2}. □

Proposition 2. $K_1 \cup 2K_2 \in \mathcal{N}_e$.

Proof. Let G be the covering of the plane by identical hexagons surrounded by six triangles. (See Figure 1.) It is clear that G is a city graph of $K_1 \cup 2K_2$. □

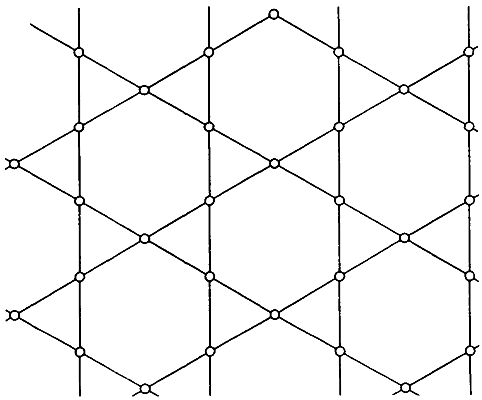

Fig. 1

Remark. In view of (vi), (viii) and Propositions 1 and 2 we may propose the following conjecture.

Conjecture. $nK_1 \cup mK_2 \in \mathcal{N}_e$.

Let V_1 and V_2 be the partition of $V(K_{3,m})$ into the independent subsets with $[V_1] = 3$ and $[V_2] = m$. Let $K_{3,m}^+$ be the graph obtained from $K_{3,m}$ by joining two vertices of V_1.

Theorem 1. *The line graph $L(K_{3,m}^+)$ belongs to $\mathcal{N}_e$.*

Proof. We show that $L(K_{m+3})$ is a city graph containing $L(K_{3,m}^+)$. Let $e = uv$ be an edge of $L(K_{m+3})$. Label the vertices of K_{m+3} by x_1, x_2, ..., x_{m+3} so that the edge x_1x_2 corresponds to the vertex u and the edge x_2x_3 corresponds to the vertex v of $L(K_{m+3})$. It is clear that the set of edges adjacent with x_1x_2 or x_2x_3 in K_{m+3} is

$$\{x_1x_3\} \cup \{x_1x_i, x_2x_i, x_3x_i : i = 4, 5, \dots, m+3\}.$$

Thus, the set of all vertices, other than u and v, which are adjacent with u or v in $L(K_{m+3})$ is

$$U_e = \{f(x_1x_3), f(x_1x_i), f(x_2x_i), f(x_3x_i) : i = 4, 5, \dots, m+3\},$$

where $f(x_ix_j)$, $i \neq j$, is the vertex of $L(K_{m+3})$ which corresponds to the edge x_ix_j of K_{m+3}. It is clear that

$$\{x_1x_3, x_1x_i, x_2x_i, x_3x_i : i = 4, 5, \dots, m+3\}$$

is the edge set of $K_{3,m}^+$ whose vertex set is partitioned into $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, \dots, x_m + 3\}$. Hence, the induced subgraph $\langle U_e \rangle$ of $L(K_{m+3})$ is isomorphic to $L(K_{3,m}^+)$. Therefore, $L(K_{m+3})$ is a city graph containing $L(K_{3,m}^+)$.

□

Theorem 2. *$K_n \cup (K_2 \times K_m)$ belongs to $\mathcal{N}_e$ for any positive integers m, n, where K_n is disjoint from $K_2 \times K_m$, and $K_2 \times K_m$ is the cartesian product of K_2 and K_m.*

Proof. We shall prove that $L(K_{m+1,n+2})$ is a city graph containing $K_m \cup (K_2 \times K_n)$. Let the vertex set of $K_{m+1,n+2}$ be partitioned into the independent subsets $\{x_1, x_2, \dots, x_{m+1}\}$ and $\{y_1, y_2, \dots, y_{n+2}\}$. Let e be any edge of $L(K_{m+1,n+2})$. We may assume, with loss of generality, that $e = f(x_1y_1)f(x_1y_2)$, where $f(x_1y_j)$ is the vertex of $L(k_{m+1,n+2})$ which corresponds to the edge x_iy_j of $K_{m+1,n+2}$. The set of edges adjacent with x_1y_1 is

$$\{x_iy_1, x_1y_j : i = 2, 3, \dots, m+1, j = 2, 3, \dots, n+2\},$$

and the set of edges adjacent with x_1y_2 is

$$\{x_iy_2, x_1y_j\colon i = 2,3,\dots,m+1, j = 1,3,4,\dots,n+2\},$$

Thus, the set of vertices, other than $f(x_1y_1)$, $f(x_1y_2)$, which are adjacent with $f(x_1y_1)$ or $f(x_1y_2)$ in $L(K_{m+1,n+2})$ is

$$U_e = \{f(x_iy_1), f(x_iy_2), f(x_1y_j)\colon i = 2,3,\dots,m+1, j = 3,4,\dots,n+2\}.$$

Let us partition U_e into S_1, S_2 and S_3 such that

$$S_1 = \{f(x_1y_j)\colon j = 3,4,\dots,n+2\},$$
$$S_2 = \{f(x_iy_1)\colon i = 2,3,\dots,m+1\},$$

and

$$S_3 = \{f(x_iy_2)\colon i = 2,3,\dots,m+1\}.$$

It is clear that the induced subgraphs $\langle S_1\rangle$, $\langle S_2\rangle$ and $\langle S_3\rangle$ of $L(K_{m+2,n+1})$ are complete graphs of orders n, m and m, respectively. For each $i = 2,3,\dots,m+1$, $f(x_iy_1)$ is adjacent with $f(x_iy_2)$. Thus, $\langle S_2 \cup S_3\rangle \cong K_2 \times K_m$.

Moreover, no vertex of S_1 is adjacent with a vertex of $S_2 \cup S_3$. Therefore

$$\langle U_e\rangle \cong K_n \cup (K_2 \times K_m).$$

□

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