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LATERAL AND DEDEKIND COMPLETIONS OF STRONGLY PROJECTABLE LATTICE ORDERED GROUPS

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For a lattice ordered group G we denote by G^L and G^D the lateral completion or the Dedekind completion of G, respectively. (For definitions, cf. Section 1 below.)

The main result of [2] is the following theorem:

$(*)$ (Bernau) Let G be an archimedean lattice ordered group. Then the relation

(1) $$G^{DL} = G^{LD}$$

is valid.

This solved a problem proposed by Conrad [4].

In the present paper the validity of (1) for strongly projectable lattice ordered groups will be proved.

Let us remark that an archimedean lattice ordered group need not be strongly projectable; also, a strongly projectable lattice ordered group need not be archimedean. Thus our result neither implies $(*)$ nor is implied by $(*)$.

For each lattice ordered group G the lateral completion G^L is defined uniquely up to isomorphism (cf. Conrad [4], Bernau [1]). Hence, in fact, the relation (1) is to be considered in the sense of isomorphism (leaving all the elements of G fixed).

1. PRELIMINARIES

In the whole paper G denotes a lattice ordered group.

An indexed system $(g_i)_{i\in I}$ $(I \neq \emptyset)$ of elements of G is called disjoint if $g_i \geqslant 0$ for each $i \in I$, and $g_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

G is said to be laterally complete if each indexed disjoint system in G has the supremum in G.

If G is an ℓ-subgroup of a lattice ordered group H such that for each $h \in H$ with $0 < h$ there exists $g \in G$ with $0 < g \leqslant h$, then G is called a dense ℓ-subgroup of H.

1.1. Definition. (Cf. Conrad [4].) A lattice ordered group H is said to be a *lateral completion* of G if the following conditions are satisfied:

(i) H is laterally complete.
(ii) G is a dense ℓ-subgroup of H.
(iii) If H_1 is an ℓ-subgroup of H such that $G \subseteq H_1$ and H_1 is laterally complete, then $H_1 = H$.

1.2. Theorem. (Bernau [1].) *Each lattice ordered group possesses a lateral completion. If H and H' are lateral completions of G, then there exists an isomorphism φ of H onto H' such that $\varphi(g) = g$ for each $g \in G$.*

Thus, up to isomorphism, the lateral completion of G is uniquely determined; we denote it by G^L.

Let $X \subseteq G$. The system of all upper bounds (or lower bounds, respectively) of X in G will be denoted by $U(X)$ (or $(L(X))$. A pair (A, B) of nonempty subsets A and B of G will be said to be a cut in G if $A = L(B)$ and $B = U(A)$. A cut (A, B) will be called a D-cut if the relations

$$\bigwedge_{a \in A, b \in B} (b - a) = 0,$$
$$\bigwedge_{a \in A, b \in B} (-a + b) = 0$$

are valid in G.

1.3. Definition. A lattice ordered group G is said to be *D-complete* if for each D-cut (A, B) in G there exists $g \in G$ such that the relation

$$\sup A = g = \inf B$$

is valid.

1.4. Definition. A lattice ordered group H is called a *Dedekind completion* of G if the following conditions are satisfied:

(i) H is D-complete.
(ii) G is an ℓ-subgroup of H.
(iii) For each $h \in H$ there are subsets X and Y of G such that the relations

$$\sup X = h = \inf Y$$

are valid in H.

From the results of Everett [5] (cf. also Fuchs [5], Chap. V, §10) we obtain

1.5. Theorem. *Each lattice ordered group possesses a Dedekind completion. If H and H' are Dedekind completions of G, then there exists an isomorphism φ of H onto H' such that $\varphi(g) = g$ for each $g \in G$.*

1.6. Theorem. (Conrad [4].) *Let G be a dense ℓ-subgroup of a laterally complete lattice ordered group H. Next, let H_0 be the intersection of all ℓ-subgroups H_i of H such that $G \subseteq H_i$ and H_i is laterally complete. Then H_0 is a lateral completion of G.*

2. Auxiliary results

If G is a dense ℓ-subgroup of a lattice ordered group H, then we express this fact by writing $G \subseteq_d H$.

It is obvious that if H' is a Dedekind completion of G, then $G \subseteq_d H'$.

2.1. Lemma. *Let $G \subseteq_d G'$. Suppose that H' is a lateral completion of G'. Then there is a lateral completion H of G such that $H \subseteq_d H'$.*

Proof. We have $G' \subseteq_d H'$, hence $G \subseteq_d H'$. Now it suffices to apply 1.6. □

2.2. Lemma. *Let H be a lattice ordered group such that $G \subseteq_d H$. Next, let H_0 be the set of all $h \in H$ such that there exist $X, Y \subseteq G$ having the property that the relation*

$$\sup X = h = \inf Y$$

is valid in H. Then H_0 is an ℓ-subgroup of H.

Proof. Let $h \in H$ and let X, Y be as above. Further, let $h' \in H$, $X' \subseteq G$, $Y' \subseteq G$ be such that $\sup X' = h' = \inf Y'$ is valid in H. Then we have

$$\sup\{x + x'\}_{x \in X, x' \in X'} = h + h' = \inf\{y + y'\}_{y \in Y, y' \in Y'}$$

in H. Analogous relations remain valid if $+$ is replaced by $\vee$ or by $\wedge$. Also,

$$\sup\{-y\}_{y \in Y} = -h = \inf\{-x\}_{x \in X}.$$

Hence H_0 is an ℓ-subgroup of H. □

2.3. Lemma. *Let H be a lattice ordered group, $\{x_i\}_{i\in I} \subseteq H$, $\{y_j\}_{j\in J} \subseteq H$, $h \in H$,*

$$\sup\{x_i\}_{i\in I} = h = \inf\{y_j\}_{j\in J}.$$

Then

$$\bigwedge_{i\in I, j\in J} (y_j - x_i) = 0 = \bigwedge_{i\in I, j\in J} (-x_i + y_j).$$

Proof. We have

$$0 = \bigwedge_{j\in J} y_j - \vee_{i\in I} x_i = \bigwedge_{j\in J} y_j + \bigwedge_{i\in I}(-x_i) = \\ = \bigwedge_{j\in J, i\in I} (y_j - x_i).$$

The other relation can be verified analogously. □

An ℓ-subgroup H_1 of a lattice ordered group H_2 will be called regular if, whenever $X \subseteq H_1$, $Y \subseteq H_1$, $x \in H_1$, $y \in H_1$ and the relations

$$\sup X = x, \quad \inf Y = y$$

are valid in H_1, then these relations are valid also in H_2.

2.4. Lemma. (Bernau [1].) *Let H_1 be a dense ℓ-subgroup of H_2. Then H_1 is a regular ℓ-subgroup of H_2.*

2.5. Lemma. *Let G, H and H_0 be as in 2.2. Assume that H is D-complete. Then H_0 is D-complete as well.*

Proof. Let (A, B) be a D-cut in H_0. We denote by B_1 the set of all upper bounds of A in H, and by A_1 the set of all lower bounds of B_1 in H. Then (A_1, B_1) is a cut in H and $A \subseteq A_1$, $B \subseteq B_1$. The relations

$$\bigwedge_{a\in A, b\in B} (b - a) = 0 = \bigwedge_{a\in A, b\in B} (-a + b)$$

are valid in H_0. In view of 2.4, these relations are valid also in H (since, obviously, H_0 is a dense ℓ-subgroup of H). Then the inclusions $A \subseteq A_1$, $B \subseteq B_1$ imply

$$\bigwedge_{a_1\in A_1, b_1\in B_1} (b_1 - a_1) = \bigwedge_{a_1\in A_1, b_1\in B_1} (-a_1 + b_1) = 0.$$

Thus (A_1, B_1) is a D-cut in H. Since H is D-complete, there exists $h \in H$ with

$$\sup A_1 = h = \inf B_1.$$

From the definition of B_1 and from $h = \inf B_1$ we see that the relation

$$h = \sup A$$

is valid in H. Since $A \subseteq H_0$, for each $a \in A$ there exists a subset $X(a)$ of G such that the relation

$$a = \sup X(a)$$

holds in H_0. Thus according to the definition of H_0 this relation holds also in H. Similarly, there exists $Y \subseteq G$ such that $h = \inf Y$ is valid in H. Denote $X = \bigcup_{a \in A} X(a)$. Then we have

$$h = \sup A = \sup\{\sup X(a)\}_{a \in A} = \sup X$$

in H. This yields that $h \in H_0$. Now, since (A, B) is a cut in H_0, we obtain that $h \in A \cap B$ and

$$h = \inf B$$

in H_0. Therefore in view of 1.3, H_0 is D-complete. □

2.6. Lemma. *Let G, H and H_0 be as in 2.2. Then H_0 is a Dedekind completion of G.*

P r o o f. We apply the conditions (i), (ii) and (iii) from 1.4. In view of 2.2, H_0 is an ℓ-subgroup of H and clearly $G \subseteq H_0$; thus G is an ℓ-subgroup of H_0. According to 2.5, H_0 is D-complete. Hence (i) and (ii) from 1.4 are satisfied. The definition of H_0 yields that (iii) from 1.4 also holds. □

2.7. Corollary. *Let G be a dense ℓ-subgroup of a lattice ordered group H and let H' be a Dedekind completion of H. Then there exists a Dedekind completion H_0 of G such that*

(i) *$H_0 \subseteq_d H'$;*
(ii) *if H_0^1 is a dense ℓ-subgroup of H' such that $G \subseteq H_0^1$ and if H_0^1 is D-complete, then $H_0 \subseteq H_0^1$.*

3. Strong projectability

For $X \subseteq G$ we denote by X^δ the polar of X in G; i.e.,

$$X^\delta = \{g \in G | g| \wedge |x| = 0 \text{ for each } x \in X\}.$$

A lattice ordered group H is said to be strongly projectable if each polar of H is a direct factor of H.

If we have a direct product decomposition

$$G = \prod_{i \in I} G_i$$

and if $g \in G$, $i \in I$, then the component of g in G_i will be denoted by $g(G_i)$ or by $g(i)$. We identify the element $g(G_i)$ with the element g' of G such that $g'(G_i) = g(G_i)$ and $g'(G_{i(1)}) = 0$ for each $i(1) \in I$ with $i(1) \neq i$.

It is well-known that if $0 < g \in G$, then $g(i)$ is the greatest element of the set $G_i \cap [0, g]$.

3.1. Lemma. *Let G be laterally complete and strongly projectable. Let $(A_i)_{i \in I}$ be an indexed system of direct factors of G such that $A_{i(1)} \cap A_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of I. Put*

$$B = \Big(\bigcup_{i \in I} A_i\Big)^\delta.$$

Then $G = B \times \prod_{i \in I} A_i$.

Proof. G is strongly projectable and hence B is a direct factor of G. Consider the mapping

$$\varphi : G \to B \times \prod_{i \in I} A_i$$

such that

$$\varphi(x)(A_i) = x(A_i) \quad \text{for each } i \in I,$$

$$\varphi(x)(B) = x(B).$$

Then φ is a homomorphism of G into $B \times \prod_{i \in I} A_i$.

Let $x \in G$, $\varphi(x) = 0$. Then $\varphi(|x|) = 0$. Thus $|x| \wedge a_i = 0$ whenever $i \in I$ and $0 \leqslant a_i \in A_i$. Hence $|x| \in B$ yielding that $|x|(B) = |x|$. But $|x|(B) = 0$ and therefore $|x| = 0 = x$. Hence φ is an isomorphism of G into $B \times \prod_{i \in I} A_i$.

For proving that φ is surjective it suffices to verify that if $0 \leqslant x^i \in A_i$ for $i \in I$ and $0 \leqslant b \in B$, then there exists $g \in G$ such that $\varphi(g)(i) = x^i$ for $i \in I$ and $\varphi(g)(B) = b$.

Choose $0 \leqslant x^i \in A_i$ $(i \in I)$ and $0 \leqslant b \in B$. Since G is laterally complete there exists $g \in G$ such that

$$g = b \vee \left(\bigvee_{i \in I} x^i\right).$$

It is easy to verify that

$$x^i = \max([0, g] \cap A_i)$$

for $i \in I$ and that

$$b = \max([0, g] \cap B).$$

Hence $\varphi(g)(i) = x^i$ for $i \in I$ and $\varphi(g)(B) = b$. Therefore φ is an isomorphism of G onto $B \times \prod_{i \in I} A_i$, which completes the proof. □

3.2. Lemma. *Let G be laterally complete and strongly projectable. Next, let H be a Dedekind completion of G. Then H is laterally complete.*

Proof. Let $(h_i)_{i \in I}$ be a disjoint indexed system of elements of H. Let $i \in I$. There exists $X_i \subseteq G^+$ such that

$$h_i = \sup X_i$$

is valid in H. Then

$$x_{i(1)} \wedge x_{i(2)} = 0$$

whenever $x_{i(1)} \in X_{i(1)}$, $x_{i(2)} \in X_{i(2)}$ and $i(1), i(2)$ are distinct elements of I. Put

$$A_i = X_i^{\delta\delta} \quad \text{for} \quad i \in I,$$

$$B = \left(\bigcup_{i \in I} A_i\right)^{\delta}.$$

We have

$$A_{i(1)} \cap A_{i(2)} = \{0\}$$

if $i(1)$ and $i(2)$ are distinct elements of I. Thus according to 3.1 we obtain

$$G = B \times \prod_{i \in I} A_i.$$

In [10] it was proved that if an abelian lattice ordered group G^1 is a direct product of lattice ordered groups G_i^1 $(i \in I)$ and if G^2 is a Dedekind completion of G^1, then there are Dedekind completions G_i^2 of G_i^1 ($i \in I$) such that G^2 is a direct product of G_i^2 ($i \in I$). It is easy to verify that this result remains valid for the non-abelian case as well.

Hence there exists a direct product decomposition

$$H = B^0 \times \prod_{i \in I} A_i^0$$

such that B^0 is a Dedekind completion of B and A_i^0 is a Dedekind completion of A_i ($i \in I$).

Since $h_i \in A_i^0$ for $i \in I$, we infer that there exists $h \in H$ such that

$$h(B^0) = 0, \quad h(A_i^0) = h_i \quad \text{for } i \in I.$$

Then the relation $h = \bigvee_{i \in I} h_i$ is valid in H and therefore H is laterally complete. □

3.3. Lemma. *Let G be strongly projectable and let H be a lateral completion of G, $0 \leqslant h \in H$. Then there exists a disjoint indexed system $(x_i)_{i \in I}$ in G such that the relation $h = \bigvee_{i \in I} x_i$ is valid in H.*

This was proved in [8].

3.4.1. Lemma. *Let G be strongly projectable and let H be a lateral completion of G. Then H is strongly projectable.*

Proof. Let $\emptyset \neq X \subseteq H$. The polar of X in H will be denoted by $X^\perp$. There exists $X_1 \subseteq H^+$ such that

$$X^\perp = X_1^\perp, \quad X^{\perp\perp} = X_1^{\perp\perp}.$$

In view of 3.3, for each $x_1 \in X_1$ there exists a subset $Y(x_1)$ of G^+ such that the relation

$$x_1 = \sup Y(x_1)$$

is valid in H. Put

$$Y = \bigcup_{x_1 \in X_1} Y(x_1).$$

Then we have

$$Y^\perp = X_1^\perp$$

and hence $Y^{\perp\perp} = X_1^{\perp\perp}$. Since G is strongly projectable, we obtain

$$G = Y^{\delta\delta} \times Y^\delta.$$

It is easy to verify that $Y^{\perp}$ and $Y^{\perp\perp}$ are Dedekind completions of Y^{δ} or of $Y^{\delta\delta}$, respectively. Then according to [9] (cf. the quotation in the proof of 3.2) we get

$$H = Y^{\perp\perp} \times Y^{\perp}.$$

Hence H is strongly projectable. □

3.4.2. Lemma. *Let G be strongly projectable and let H be a Dedekind completion of G. Then H is strongly projectable.*

The proof is as in 3.4.1 with the following distinction: the existence of $Y(x_1)$ with the desired properties is a consequence of the definition of the Dedekind completion (i.e., we need not apply 3.3).

3.5. Lemma. *Suppose that G is strongly projectable and D-complete. Let H be a lateral completion of G. Assume that $0 < h \in H$, $b \in G$, $h \leqslant b$. Then $h \in G$.*

Proof. We have $0 \leqslant -h + b$. Since G is strongly projectable, according to 3.3 there are disjoint indexed systems $(g_i^1)_{i \in I}$ and $(g_j^2)_{j \in J}$ in G such that the relations

$$h = \bigvee_{i \in I} g_i^1, \tag{1}$$

$$-h + b = \bigvee_{j \in J} g_j^2 \tag{2}$$

are valid in H. From (2) we infer that the following relations hold in H:

$$-h = \bigvee_{j \in J} (g_j^2 - b),$$

$$h = \bigwedge_{j \in J} g_j^3, \tag{3}$$

where $g_j^3 = b - g_j^2$. Hence $g_j^3 \in G$ for each $j \in J$. Next, (1) and (3) yield by simple calculation that the relations

$$\bigwedge_{i \in I, j \in J} (g_j^3 - g_i^1) = 0 = \bigwedge_{i \in I, j \in J} (-g_i^1 + g_j^3) \tag{4}$$

are valid in H. Hence these relations hold in G as well.

Denote

$$B_1 = U(\{g_i^1\}_{i\in I}), \quad A_1 = L(B_1),$$

where the symbols U and L are taken with respect to G. Then (A_1, B_1) is a cut in G. Clearly

$$\{g_i^1\}_{i\in I} \subseteq A_1, \tag{5.1}$$

$$\{g_j^3\}_{j\in J} \subseteq B_1. \tag{5.2}$$

From (4) we obtain that the relations

$$\bigwedge_{a_1\in A_1, b_1\in B_1} (b_1 - a_1) = 0 = \bigwedge_{a_1\in A_1, b_1\in B_1} (-a_1 + b_1)$$

hold in G. Hence (A_1, B_1) is a D-cut in G. Now we apply the assumption that G is D-complete. Thus there is $g^0 \in G$ such that

$$\sup A_1 = g^0 = \inf B_1 \tag{6}$$

is valid in G. Since G is dense in H (this is a consequence of 3.3), the relations (6) hold also in H. Then from (5.1) we get $h \leqslant g^0$ and from (5.2) we obtain $h \geqslant g^0$. Therefore $h = g^0$, which completes the proof. □

3.6. Lemma. *Let G be strongly projectable and D-complete. Suppose that H is a lateral completion of G. Then H is D-complete.*

Proof. Let H_1 be a Dedekind completion of H. We have to show that $H_1 = H$. It suffices to verify that $H_1^+ \subseteq H$.

Let $0 \leqslant h_1 \in H_1$. There exist subsets A_1 and B_1 of H such that the relations

$$\sup A_1 = h_1 = \inf B_1 \tag{1}$$

are valid in H_1. Choose $b_1 \in B_1$. There exists a disjoint indexed system $(b_i)_{i\in I}$ of elements of G such that the relation

$$b_! = \vee_{i\in I} b_i \tag{2}$$

holds in H.

It follows from the Axiom of Choice that there exists a disjoint indexed system $(b_i)_{i\in I'}$ of elements of G such that $I' \subseteq I$ and, whenever $0 < g \in G$, then $g \wedge b_i > 0$ for some $i \in I'$.

Let the symbol $\perp$ have the same meaning as above (i.e., it is applied for denoting polars in H). For each $i \in I'$ we put

$$C_i = \{b_i\}^{\perp\perp}.$$

Since H is laterally complete it is strongly projectable and hence in view of 3.1 we have

$$H = \prod_{i \in I'} C_i. \tag{3.1}$$

Hence according to [9] (cf. the quotation in the proof of 3.2),

$$H_1 = \prod_{i \in I'} C_i^1, \tag{3.2}$$

where C_i^1 is a Dedekind completion of C_i ($i \in I'$).

From (2) we obtain that if $i \in I$, then

$$b_i(C_i) = b_i, \quad b_{i(1)}(C_i) = 0 \quad \text{for } i(1) \in I' \setminus \{i\}.$$

These relations remain valid if C_i is replaced by C_i^1 ($i \in I$).

Since $H \subseteq_d H_1$, the relation (2) holds in H_1 as well. Then

$$h_1 = h_1 \wedge b_1 = \vee_{i \in I}(h_1 \wedge b_i) \tag{4}$$

is valid in H_1.

Let $i \in I$ be fixed. From (4) we obtain that

$$h_1(C_i^1) = h_1 \wedge b_i \geqslant 0.$$

Thus $h_1 \wedge b_i \in C_i^1$.

There exist $A_i, B_i \subseteq C_i$ such that the relations

$$\sup A_i = h_1 \wedge b_i = \inf B_i$$

are valid in C_i^1. Denote

$$A_i^* = A_i \cap G, \quad B_i^* = B_i \cap G.$$

Since $b_i \in A$, in view of 3.5 we have $h \in H$ for each $h \in H$ with $0 \leqslant h \leqslant b_i$. This yields that the relations

$$\sup A_i^* = h_1 \wedge b_i = \inf B_i^*$$

hold in C_i^1. Thus

(5) $$\bigwedge_{x\in A_i^*, y\in B_i^*} (y-x) = 0 = \bigwedge_{x\in A_i^*, y\in B_i^*} (-x+y).$$

However, $A_i^*, B_i^* \subseteq G$ and thus, since G is D-complete, we infer from (5) in the obvious way that there exists $z \in G$ such that

(6) $$\sup A_i^* = z = \inf B_i^*$$

is valid in G.

Since $G \subseteq_d H$ (cf. 3.3) we get $G \subseteq_d H_1$ and thus (6) holds also in H_1. We obtain $z = h_1 \wedge b_i$. Therefore $h_1 \wedge b_i \in G$ for each $i \in I$.

The indexed system $(h_1 \wedge b_i)_{i\in I}$ of elements of G is disjoint, hence there exist $h_0 \in H$ such that

(7) $$h_0 = \vee_{i\in I}(h \wedge b_i)$$

is valid in H. Since $H \subseteq_d H_1$, the relation (7) holds also in H_1. Then (4) yields that $h_1 = h_0 \in H$, which completes the proof. □

4. Isomorphisms of G^{DL} and G^{LD}

Let G be a lattice ordered group. We denote by
H—a lateral completion of G;
H_1—a Dedekind completion of H;
K—a Dedekind completion of G;
K_1—a lateral completion of K.

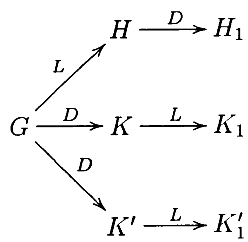

Figure 1

4.1. Theorem. *Let G be a strongly projectable lattice ordered group and let H, H_1, K, K_1 be as above. Then there exists an isomorphism φ of K_1 onto H_1 such that $\varphi(g) = g$ for each $g \in G$.*

Proof. (Cf. Fig. 1.) Since $G \subseteq_d H$ and $H \subseteq_d H_1$, in view of 2.7 there exists a Dedekind completion K' of G such that $K' \subseteq_d H_1$. In view of 3.4.2, K' is strongly projectable.

According to 3.2 and 3.4.1, H_1 is laterally complete. Then 2.1 yields that there is a lateral completion K'_1 of K' such that $K'_1 \subseteq_d H_1$.

Since $G \subseteq_d K' \subseteq_d H_1$ we get $G \subseteq_d K' \subseteq_d K'_1$. This and the lateral completeness of K'_1 yield (cf. 2.1) that $H \subseteq_d K'_1$.

By applying the definition of K'_1 and 3.6 we obtain that $H_1 \subseteq K'_1$. Therefore we have

$$H_1 = K'_1. \tag{1}$$

There exists an isomorphism φ_1 of K onto K' such that $\varphi(g) = g$ for each $g \in G$. Next, there exists an isomorphism φ of K_1 onto K'_1 such that $\varphi(x) = x$ for each $x \in K$. In particular, $\varphi(g) = g$ for each $g \in G$. To complete the proof it suffices to apply the relation (1). □

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