Czechoslovak Mathematical Journal

Jiří Jarník; Jaroslav Kurzweil

Another Perron type integration in n dimensions as an extension of integration of stepfunctions

Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 3, 557–575

Persistent URL: http://dml.cz/dmlcz/127377

Czechoslovak Mathematical Journal, 47 (122) (1997), Praha

ANOTHER PERRON TYPE INTEGRATION IN n DIMENSIONS AS AN EXTENSION OF INTEGRATION OF STEPFUNCTIONS

JIŘÍ JARNÍK and JAROSLAV KURZWEIL,[1] Praha

(Received September 18, 1995)

Abstract. For a new Perron-type integral a concept of convergence is introduced such that the limit f of a sequence of integrable functions f_k, $k \in \mathbb{N}$ is integrable and any integrable f is the limit of a sequence of stepfunctions g_k, $k \in \mathbb{N}$.

0. Introduction

The density of the set of stepfunctions in a convergence space of Perron-type integrable functions is proved for a new Perron-type integration on n-dimensional intervals. The integration involved is strong in the sense that the set of integrable functions is rather restricted; on the other hand partial derivatives of differentiable functions are integrable.

In Section 1 the integration is introduced, its basic properties are presented (the proofs are standard and are omitted or indicated). Moreover, the *equiconvergence is introduced and the main result is stated. In Section 2 two lemmas are proved and in Section 3 the proof of the main result is given; with some modifications it runs along the same lines as the proof of an analogous result from the preceding paper of the authors.

[1] This paper was supported by grant No 201/94/1068 of the GA of the Czech Republic.

The notation and concepts used are analogous to those in [1], [2]. Let

$$I = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n. \tag{1.1}$$

A finite set $\Xi = \{(s, K)\}$ is an *L-system* (on I) if $s \in I$, K is an interval of the form

$$K = [c_1, d_1] \times \ldots \times [c_n, d_n] \subset I \tag{1.2}$$

for every couple $(s, K) \in \Xi$ and if the intervals K are nonoverlapping (i.e. $\operatorname{Int} K_1 \cap \operatorname{Int} K_2 = \emptyset$ provided $(s_1, K_1), (s_2, K_2) \in \Xi$, $(s_1, K_1) \neq (s_2, K_2)$, $s_1 = s_2$ being admitted). If in addition, $\bigcup_{\Xi} K = I$ then Ξ is an *L-partition* (of I). $\|t\|$ is the maximum norm of $t \in \mathbb{R}^n$. For $t \in \mathbb{R}^n$, $\nu > 0$ put $V(t, \nu) = \{x \in \mathbb{R}^n ; \|x - t\| \leqslant \nu\}$. $\partial K, \operatorname{Int} K$ and $m(K)$ respectively denote the boundary, the interior and the Lebesgue measure of an interval K. If $s \in \mathbb{R}^n$ and if K is an interval of the form (1.2), then the diameters $d(K), d(s, K)$ and the regularities $\operatorname{reg} K$, $^*\operatorname{reg}(s, K)$ are defined as follows:

$$\begin{aligned} d(K) &= \max\{\|x - y\| ; x, y \in K\}, \\ d(s, K) &= \max\{\|x - y\| ; x, y \in K \cup \{s\}\}, \\ \operatorname{reg} K &= \min\{d_i - c_i ; i = 1, 2, \ldots, n\}/d(K), \\ ^*\operatorname{reg}(s, K) &= \min\{d_i - c_i ; i = 1, 2, \ldots, n\}/d(s, K). \end{aligned}$$

Let $\Xi = \{(s, K)\}$ be an L-system or L-partition, $\varrho \in (0, 1), A \subset I$. Ξ is called *ϱ-*regular* (*A-tagged*) if $^*\operatorname{reg}(s, K) > \varrho$ $(s \in A)$ for $(s, K) \in \Xi$. Let $\delta \colon A \to (0, 1]$; δ is called a *gauge*. Let Ξ be A-tagged; Ξ is called *δ-fine* if $K \subset V(s, \delta(s))$ for $(s, K) \in \Xi$.

1.1 Definition. A function $f \colon I \to \mathbb{R}$ is **integrable* (over I) if for every $\varepsilon > 0$ and every $\varrho \in (0, 1)$ there exists a gauge $\delta \colon I \to (0, 1]$ such that

$$\left| \sum_{\Delta} f(t) m(J) - \sum_{\Xi} f(s) m(K) \right| \leqslant \varepsilon$$

provided $\Delta = \{(t, J)\}, \Xi = \{(s, K)\}$ are δ-fine *ϱ-regular* L-partitions of I.

1.2 Note. The concept of an *integrable function f does not change if ϱ is replaced by ε in Definition 1.1.

1.3 Note. If f is *integrable over I then there exists a unique $^*\int_I f \in \mathbb{R}$ such that for every $\varepsilon > 0, \varrho \in (0, 1)$ there exists a gauge $\delta \colon I \to (0, 1]$ such that

$$\left| \sum_{\Delta} f(t) m(J) - {}^*\!\int_I f \right| \leqslant \varepsilon$$

provided $\Delta = \{(t, J)\}$ is a δ-fine *ϱ-regular* L-partition of I.

1.4 Note. Let f be *integrable over I. Then for any interval $J \subset I$ the restriction $f\big|_J$ is *integrable over J; put $F(J) = {}^*\int_J f\big|_J$. F is an additive interval function on I; it is called *the primitive* of f.

1.5 Note. Let $h\colon I \to \mathbb{R}^n$ be differentiable at every $t \in I$. Then $\partial h/\partial t_1$ is *integrable.

Observe that

$$(1.3) \qquad \varrho d(u,L) < d(L), \quad \operatorname{reg} L > \varrho, \quad \varrho^{n-1}(d(L))^n < m(L)$$

if ${}^*\operatorname{reg}(u,L) > \varrho$. The above result can be proved in the same way as the corresponding result in [5] since for any ϱ-regular L-partition $\theta = \{(u,L)\}$ of I we have

$$\sum_\theta \mathcal{H}(\partial L)d(u,L) \leqslant \sum_\theta 2n(d(L))^{n-1}\varrho^{-1}d(L) \leqslant 2n\varrho^{-n}\sum_\theta m(L) \leqslant 2n\varrho^{-n}m(I),$$

$\mathcal{H}(\partial J)$ denoting the $(n-1)$-dimensional measure of the boundary of J, $\mathcal{H}(\partial J) \leqslant 2n(d(J))^{n-1}$.

On the other hand, let $p\colon [0,1]\times[0,1] \to \mathbb{R}$, $p(t) = (-1)^i 4^i/i$ for $t \in [2^{-i}, 2^{-i+1}) \times [2^{-i}, 2^{-i+1})$, $p(t) = 0$ otherwise; it can be proved directly from the definitions that p is ϱ-integrable for every $\varrho \in (0,1)$, but p is not *integrable.

1.6 Note. The *integration is an extension of the Lebesgue integration. This follows immediately from the fact that $f\colon I \to \mathbb{R}$ is Lebesgue integrable iff for every $\varepsilon > 0$ there exists a gauge $\delta\colon I \to (0,1]$ such that

$$\left|\sum_\Delta f(t)m(J) - \sum_\Xi f(s)m(K)\right| \leqslant \varepsilon$$

provided $\Delta = \{(t,j)\}, \Xi = \{(s,K)\}$ are δ-fine L-partitions of I.

This result goes back to E. J. McShane [4] (see also [3], Theorem 7.6 or [6], Chapter 4, Definition 1-1 and a comment before Corollary 6-5).

1.7 Lemma. *Let $f\colon I \to \mathbb{R}$ be *integrable and let F be its primitive, $N \subset I$, $m(N) = 0$. Then*

(1.4) *for every $\lambda > 0$, $\varrho \in (0,1)$ there exists a gauge $\gamma\colon N \to (0,1]$ such that*

$$\sum_\Xi |F(K)| \leqslant \lambda$$

*provided $\Xi = \{(s,K)\}$ is a γ-fine ϱ-*regular N-tagged L-system.*

Lemma 1.7 is a consequence of the Saks-Henstock Lemma for the *integration and of [2], Lemma 1.8.

For an additive interval function G on I let D_G be the set of $s \in I$ such that G is regularly differentiable to $G'(s)$ at s (cf. [2] Definition 2.6), $N_G = I \setminus D_G$.

1.8 Note. Let $\varrho \in (0,1)$ and let $g\colon I \to \mathbb{R}$ be *integrable, F being its primitive. Then g is ϱ-integrable and F is its primitive with respect to the ϱ-integration as well (cf. [2], Definition 1.2). This is an immediate consequence of the definitions.

1.9 Lemma. *Let g be *integrable over I and let F be its primitive. Then*

$$m(N_F) = 0, \quad F'(s) = g(s) \text{ at almost every } s \in I.$$

Lemma 1.9 follows immediately from Note 1.8 and [2], Theorem 2.8.

1.10 Theorem. *Let $f\colon I \to \mathbb{R}$ and let F be an additive interval function on I. The function f is *integrable and F is its primitive iff there exists $N \subset I$ such that $N_F \subset N$, $m(N) = 0, F'(t) = f(t)$ for $t \in I \setminus N$ and (1.4) holds.*

Proof. The *only if* part follows by Lemmas 1.7 and 1.9. The *if* part follows from Definition 1.1 and [2], Lemma 1.8. □

1.11 Definition. Let $f_k\colon I \to \mathbb{R}$ be *integrable, F_k being its primitive for $k \in \mathbb{N}, f\colon I \to \mathbb{R}$. The sequence f_k is said to be *equiconvergent to f if there exists $N \subset I, m(N) = 0$ such that

(1.5) $$f_k(t) \to f(t) \quad \text{for } k \to \infty, t \in I \setminus N,$$

(1.6) for every $\varepsilon, \varrho \in (0,1)$ there exists a gauge $\delta_1\colon I \setminus N \to (0,1]$ such that

$$\sum_{\Delta} |F_k(J) - f_k(t)m(J)| \leqslant \varepsilon$$

for every system $\Delta = \{(t, J)\}$ which is δ_1-fine, ϱ-*regular and $I \setminus N$ tagged, and for every $k \in \mathbb{N}$,

(1.7) for every $\varepsilon, \varrho \in (0,1)$ there exists a gauge $\delta_2\colon N \to (0,1]$ such that

$$\sum_{\Delta} |F_k(J)| \leqslant \varepsilon$$

for every system Δ which is δ_2-fine, ϱ-*regular and N-tagged, and for every $k \in \mathbb{N}$.

1.12 Theorem. *Let $f_k\colon I \to \mathbb{R}$ be *integrable for $k \in N$ and *equiconvergent to $f\colon I \to \mathbb{R}$. Then f is *integrable. Moreover, if F_k is the primitive of f_k and F is the primitive of f, then*

$$F_k(L) \to F(L) \quad \text{for } k \to \infty \text{ and every interval } L \subset I. \tag{1.8}$$

Proof. Since the sequence f_k is *equiconvergent to f it may be assumed without loss of generality that $f_k(t) = 0$ for $t \in N$, $k \in \mathbb{N}$. Let $\varepsilon > 0$, $\varrho \in (0,1)$ and let δ_1 and δ_2 fulfil respectively (1.6) and (1.7). Put

$$\delta(t) = \begin{cases} \delta_1(t) & \text{for } t \in I \setminus N, \\ \delta_2(t) & \text{for } t \in N. \end{cases}$$

Let $\Delta = \{(t,J)\}, \Xi = \{(s,K)\}$ be δ-fine ϱ-*regular L-partitions of I. Since $F_k(I) = \sum\limits_{\Delta} F_k(J) = \sum\limits_{\Xi} F_k(K)$ for $k \in \mathbb{N}$, we have

$$\begin{aligned}\Big|\sum_{\Delta} f_k(t)m(J) - \sum_{\Xi} f_k(s)m(K)\Big| &\leqslant \sum_{\Delta, t\in I\setminus N} |f_k(t)m(J) - F_k(J)| + \sum_{\Delta, t \in N} |F_k(J)| \\ &\quad + \sum_{\Xi, s\in I\setminus N} |f_k(s)m(K) - F_k(K)| + \sum_{\Xi, s\in N} |F_k(K)| \\ &\leqslant 4\varepsilon\end{aligned}$$

and the *integrability of f is obtained by passing to the limit for $k \to \infty$. The proof of (1.8) is standard. □

A function $g\colon I \to \mathbb{R}$ is called a stepfunction, if there exists a partition $\Theta = \{(u,L)\}$ of I such that g is constant on $\operatorname{Int} L$ for any $(u,L) \in \Theta$.

1.13 Theorem (Main Result). *Let $g\colon I \to \mathbb{R}$ be *integrable. Then there exists a sequence of stepfunctions g_k, $k \in \mathbb{N}$ which is *equiconvergent to g.*

2.1 Lemma. *Let $J, K \subset \mathbb{R}^n$ be intervals, K being of the form (1.2), $s \in \mathbb{R}^n$, $\varrho \in (0,1)$, $K \subset J$, $^*\mathrm{reg}(s,K) > \varrho$, $\mathrm{reg}\, J > 1/2$. Then*

$$d(s,J) \leqslant \Big(\frac{1}{\varrho} + 1\Big) d(J), \tag{2.1}$$

$$^*\mathrm{reg}(s,J) > \frac{\varrho}{2(\varrho+1)}. \tag{2.2}$$

Proof. Since $^*\mathrm{reg}(s,K) > \varrho, K \subset J$, we have $\varrho d(s,K) < d(K) \leqslant d(J)$. Obviously $d(s,J) \leqslant d(s,K) + d(J) \leqslant \big(\frac{1}{\varrho} + 1\big) d(J)$ and (2.1) holds. Since $\mathrm{reg}\, J > \frac{1}{2}$ we have $^*\mathrm{reg}(s,J) > \frac{1}{2} d(J)/d(s,J)$ and (2.2) follows from (2.1). □

For $W \subset \mathbb{R}^n$ let $\chi(W)\colon \mathbb{R}^n \to \{0,1\}$ be the characteristic function of W. Similarly for $C \subset \mathbb{R}$ let $\chi(C)\colon \mathbb{R} \to \{0,1\}$ be the characteristic function of C. Let I and $K \subset I$ be intervals of the form (1.1) and (1.2), respectively. Put

$$\big(K(i)\big)^0 = \begin{cases} [c_i, d_i) & \text{if } d_i < b_i, \\ [c_i, d_i] & \text{if } d_i = b_i, \end{cases}$$

and

$$K^0 = \big(K(1)\big)^0 \times \ldots \times \big(K(n)\big)^0 \tag{2.3}$$

(if L, M are nonoverlapping intervals then L^0 and M^0 are disjoint).

2.2 Lemma. *Let S, A be intervals, $A \subset S \subset I$, $\varrho \in (0,1)$, $^*\mathrm{reg}(s,S) > \varrho$. Let G be an additive interval function on I. Then there exist intervals $Z_j \subset I$ and numbers $\zeta_j \in \{-1,0,1\}$ for $j \in \{1,2,\ldots,3^n\}$ such that*

$$^*\mathrm{reg}(s,Z_j) > \varrho/2, \tag{2.4}$$

$$\chi(A^0) = \sum_{j=1}^{3^n} \zeta_j \chi(Z_j^0), \tag{2.5}$$

$$G(A) = \sum_{j=1}^{3^n} \zeta_j G(Z_j). \tag{2.6}$$

Proof. Let S and A be of the forms

$$S = S(1) \times \ldots \times S(n) = [\sigma_1, \tau_1] \times \ldots \times [\sigma_n, \tau_n],$$
$$A = A(1) \times \ldots \times A(n) = [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n].$$

If $\sigma_i \leqslant \alpha_i < \frac{1}{2}(\sigma_i + \tau_i) \leqslant \beta_i \leqslant \tau_i$, put $Q_i = \{1,2,3\}$, $Y^1(i) = [\sigma_i, \beta_i], Y^2(i) = [\alpha_i, \tau_i]$, $Y^3(i) = [\sigma_i, \tau_i]$, $\zeta_i^1 = 1, \zeta_i^2 = 1$, $\zeta_i^3 = -1$, so that

$$\chi\big((A(i))^0\big) = \sum_{q_i \in Q_i} \zeta_i^{q_i} \chi\big((Y^{q_i}(i))^0\big). \tag{2.7}$$

If $\sigma_i \leqslant \alpha_i < \beta_i < \frac{1}{2}(\sigma_i + \tau_i)$ put $Q_i = \{1,2\}$, $Y^1(i) = [\alpha_i, \tau_i]$, $Y^2(i) = [\beta_i, \tau_i]$, $\zeta_i^1 = 1$, $\zeta_i^2 = -1$. Then (2.7) holds.

If $\frac{1}{2}(\sigma_i + \tau_i) \leqslant \alpha_i < \beta_i \leqslant \tau_i$, put $Q_i = \{1,2\}$, $Y^1(i) = [\sigma_i, \beta_i]$, $Y^2(i) = [\sigma_i, \alpha_i]$, $\zeta_i^1 = 1$, $\zeta_i^2 = -1$, $i \in \{1,2,\ldots,n\}$. Then (2.7) holds again.

For $q = (q_1, \ldots, q_n) \in Q = Q_1 \times \ldots \times Q_n$ put $Y^q = Y^{q_1}(1) \times \ldots \times Y^{q_n}(n)$, $\zeta^q = \zeta_1^{q_1} \cdot \zeta_2^{q_2} \cdot \ldots \cdot \zeta_n^{q_n}$. It follows from (2.7) that

$$\chi(A^0) = \sum_{q \in Q} \zeta^q \chi\big((Y^q)^0\big).$$

Put $\gamma = \#Q$. Let φ be a bijection of Q onto $\{1,2,\ldots,\gamma\}$ and put $Z_{\varphi(q)} = Y^q$, $\zeta_{\varphi(q)} = \zeta^q$. For $j \in \{\gamma+1, \gamma+2, \ldots, 3^n\}$ put $\zeta_j = 0$, $Z_j = S$. Then (2.5) holds and (2.6) follows from (2.5).

Finally,

$$\begin{aligned} {}^*\mathrm{reg}(s, Y^q) = \frac{\min\{d(Y^{q_i}(i));\, i = 1,2,\ldots,n\}}{d(s, Y^q)} &\geqslant \frac{\frac{1}{2}\min\{\tau_i - \sigma_i;\, i = 1,2,\ldots,n\}}{d(s,S)} \\ &\geqslant \frac{1}{2}\varrho. \end{aligned}$$

It follows that (2.4) holds. □

3. Proof of main result

Let $g\colon I \to \mathbb{R}$ be *integrable and let F be its primitive. F is regularly differentiable almost everywhere and (1.4) holds. Let $\varrho \in (0,1)$. Since g is ϱ-integrable and F is its primitive with respect to the ϱ-integration (cf. Note 1.8), F is continuous at any interval $L \subset \operatorname{Int} I$, i.e. for every $\sigma > 0$ there is a $\tau > 0$ such that $|F(K) - F(L)| \leqslant \sigma$ for every interval $K \subset I$ satisfying $m(K \setminus L) + m(L \setminus K) \leqslant \tau$ (cf. [2], Theorem 2.1 and the comment at the beginning of Section 3 of [1]). All assumptions of [1], Lemma 2.6 being fulfilled (cf. (1.4)) it may be concluded that g is measurable and there exist

$$N \subset I,\ N \supset N_F \cup \partial I, \quad \xi \in \Big(0, \frac{1}{4}\Big),$$

$\eta\colon [0,\xi] \to [0,1)$ increasing, $\eta(\sigma) > \sigma$ for $\sigma \in (0,\xi)$, $\lim_{\sigma\to 0+} \eta(\sigma) = 0$,
$\omega\colon I \setminus N \to (0,\xi]$ measurable, $V(t,\omega(t)) \subset I$ for $t \in I \setminus N$ such that

$$|F(K) - g(t)m(K)| \leqslant \eta(\nu)\nu^n \tag{3.1}$$

for every $t \in I \setminus N$, $\nu \in (0,\omega(t)]$, $K \subset \operatorname{Int} V(t,\nu)$ (K being an interval).

Observe that (3.1) implies that

$$F'(t) = g(t) \quad \text{for } t \in I \setminus N.$$

Moreover, (1.3) holds. Let us choose sequences

$$\frac{1}{2} > \tau_1 > \tau_2 > \ldots > 0, \quad 0 < \tau_{i+1} < \frac{\tau_i}{2(1+\tau_i)} \quad \text{for } i \in \mathbb{N}, \tag{3.2}$$

$$\xi \geqslant \xi_1 > \xi_2 > \ldots, \ \lim_{i\to\infty} \xi_i = 0, \quad ([0,\xi] \text{ being the domain of } \eta). \tag{3.3}$$

There is a measurable $\omega_1\colon I \setminus N \to (0,1]$ such that

$$|g(t)| \leqslant \left[\eta\big(2\omega_1(t)\big)\right]^{-\frac{1}{4n}} \tag{3.4}$$

for $t \in I \setminus N$. Let us set

$$\delta_k(t) = \min\left\{\frac{1}{2}\xi_k, \omega_1(t), \omega(t)\right\} \tag{3.5}$$

for $t \in I \setminus N$, $k \in \mathbb{N}$. Referring to (1.4) let us choose $\delta_k(t)$ for $t \in N$ such that

$$\delta_k(t) \leqslant \frac{1}{2}\xi_k \tag{3.6}$$

and

$$\sum_{\Xi} |F(K)| \leqslant \xi_k \tag{3.7}$$

provided $\Xi = \{(s,K)\}$ is a δ_k-fine τ_{k+1}-*regular N-tagged L-system, $k \in \mathbb{N}$. The desired sequence of stepfunctions g_k is defined as follows: For $k \in \mathbb{N}$ let us choose a δ_k-fine $\frac{1}{2}$-*regular partition $\Delta_k = \{(t,J)\}$ of I with $t \in J$ for $(t,J) \in \Delta_k$ (cf. [2], Lemma 1.1) and for $s \in I$ let us set

$$g_k(s) = \frac{F(J)}{m(J)} \tag{3.8}$$

where J is such that $(t, J) \in \Delta_k$ for some $t \in I$ and $s \in J^0$ (cf. (2.3)); evidently there is a unique J with the property. The function g_k is *integrable (see Note 1.6); let G_k be its primitive function, $k \in \mathbb{N}$. For any interval $M \subset I$ we have

$$G_k(M) = \sum_{(t,J)\in\Delta_k} \frac{F(J)}{m(J)} m(J \cap M). \tag{3.9}$$

The result to be established can be formulated as follows.

3.1. Theorem. *The sequence $\{g_k\}$ is *equiconvergent to g.*

It is a consequence of the following two propositions.

3.2. Proposition. *For every $\varepsilon > 0$ and $\varrho \in (0, 1)$ there are $l_1 \in \mathbb{N}$ and $\vartheta_1 \colon N \to (0, 1]$ such that*

$$\Sigma_1 = \sum_{\Theta} |G_k(L)| \leqslant \varepsilon \tag{3.10}$$

*for every ϑ_1-fine ϱ-*regular N-tagged L-system $\Theta = \{(u, L)\}$ and every $k \geqslant l_1$.*

3.3. Proposition. *For every $\varepsilon > 0$ and $\varrho \in (0, 1)$ there are $l_2 \in \mathbb{N}$ and $\vartheta_2 \colon I \setminus N \to (0, 1]$ such that*

$$\Sigma_2 = \sum_{\Theta} |G_k(L) - g_k(u)m(L)| \leqslant \varepsilon \tag{3.11}$$

*for every ϑ_2-fine ϱ-*regular $I \setminus N$-tagged L-system $\Theta = \{(u, L)\}$ and every $k \geqslant l_2$. Moreover,*

$$g_k(s) \to g(s) \quad \text{for } s \in I \setminus N,\ k \to \infty. \tag{3.12}$$

3.4. Convention. To simplify the formulas we will assume (without loss of generality) that $m(I) \leqslant 1$.

3.5. Lemma. *Let $j \in \mathbb{N}$, and let $\Theta = \{(u, L)\}$ be a δ_j-fine τ_j-*regular N-tagged L-system. Then*

$$\sum_{\Theta} \sup\{|F(K)|;\ K \subset L\} \leqslant 3^n \xi_j; \tag{3.13}$$

for the partition Δ_k we have

$$\sum_{\Delta_k, t\in N} \sup\{|F(K)|;\ K \subset J\} \leqslant 3^n \xi_k \tag{3.14}$$

(K denoting an interval in (3.13) *and* (3.14) *and the summation in* (3.14) *being restricted to (t, J) such that $t \in N$).*

Proof. For every $(u, L) \in \Theta$ let $X(u, L) \subset L$ be an interval. By Lemma 2.2 there exist intervals $Z_i(u, L) \subset L$ and numbers $\zeta_i(u, L) \in \{-1, 0, 1\}$, $i \in \{1, 2, \ldots, 3^n\}$ such that $^*\mathrm{reg}(u, Z_i(u, L)) > \tau_{j+1}$ and

$$F\big(X(u, L)\big) = \sum_{i=1}^{3^n} \zeta_i(u, L) F(Z_i(u, L)). \tag{3.15}$$

Now $\Phi_i = \{(u, Z_i(u, L);\ (u, L) \in \Theta\}$ is a δ_j-fine τ_{j+1}-regular N-tagged L-system so that

$$\sum_{\Phi_i} |F(Z_i(u, L))| \leqslant \xi_j$$

(cf. (3.7)) and (3.13) holds by (3.15). The proof of (3.14) is quite analogous since Δ_k is $\frac{1}{2}$-regular and $\tau_{k+1} \leqslant \frac{1}{4}$ (cf. (3.2) and (3.7)). □

Proof of Proposition 3.2. Given $\varepsilon > 0$ and $\varrho \in (0, 1)$, let us choose $j \in \mathbb{N}$ such that

$$\tau_j \leqslant \varrho,\ (3 + 2 \cdot 18^n)\xi_j < \frac{\varepsilon}{2} \tag{3.16}$$

(cf. (3.2) and (3.3)) and denote

$$r(u) = \min\{k \in \mathbb{N};\ \xi_k < \tau_{j+1}\delta_j(u)\} \quad \text{for } u \in N. \tag{3.17}$$

For every $k \in \mathbb{N}$ there is an open set $U_k \subset \mathbb{R}^n$ such that $N \subset U_k$ and

$$m\big(U_k\big) \leqslant \xi_j \beta_k, \quad \beta_k = \frac{\min\{m(J);\ (t, J) \in \Delta_k\}}{\max\{1 + |F(J)|;\ (t, J) \in \Delta_k\}}. \tag{3.18}$$

For every $k \in \mathbb{N}$ there is a gauge $\mu_k \colon N \to (0, 1]$ such that

$$V\big(u, \mu_k(u)\big) \subset U_k \quad \text{for } u \in N. \tag{3.19}$$

We choose a gauge $\vartheta_1 \colon N \to (0, 1]$ satisfying the condition

$$\begin{aligned} &\vartheta_1(u) \leqslant \mu_k(u) \quad \text{for } k < r(u), \\ &\vartheta_1(u) \leqslant \delta_j(u) \quad \text{for } u \in N. \end{aligned} \tag{3.20}$$

Now we seek estimates leading to (3.10). Let $\Theta = \{(u, L)\}$ be a ϑ_1-fine ϱ-*regular N-tagged L-system. For $k \in \mathbb{N}$ we have

$$\Sigma_1 \leqslant \Gamma_1 + \Gamma_2 = \sum_{\substack{\Theta \\ \exists (t,J)\in\Delta_k, L\subset J}} |G_k(L)| + \sum_{\substack{\Theta \\ L\setminus J\neq\emptyset, \forall (t,J)\in\Delta_k}} |G_k(L)|.$$

By virtue of (3.9) we obtain

$$\Gamma_1 \leqslant \Gamma_3 + \Gamma_4 = \sum_{\Delta_k} \sum_{\substack{\Theta \\ \exists (t,J)\in\Delta_k, L\subset J \\ k<r(u)}} |F(J)| \frac{m(L\cap J)}{m(J)}$$

$$+ \sum_{\Delta_k} \sum_{\substack{\Theta \\ \exists (t,J)\in\Delta_k, L\subset J \\ k\geqslant r(u)}} |F(J)| \frac{m(L\cap J)}{m(J)}.$$

If $(t, J) \in \Delta_k$, $(u, L) \in \Theta$, $k < r(u)$, $L \subset J$ then $L \subset U_k$ since $u \in N$ (cf. (3.19), (3.20)), and consequently (cf. (3.18))

$$\Gamma_3 \leqslant \beta_k^{-1} \sum_{\Delta_k} \sum_{\substack{\Theta \\ \exists (t,J)\in\Delta_k, L\subset J \\ k<r(u)}} m(L) \leqslant \beta_k^{-1} \sum_{\Delta_k} m(J\cap U_k) \leqslant \xi_j. \tag{3.21}$$

We proceed to Γ_4. For $(t, J) \in \Delta_k$ let $\Omega(t, J)$ be the set of $(u, L) \in \Theta$ such that $L \subset J$, $k \geqslant r(u)$. We have

$$\Gamma_4 \leqslant \sum_{\Delta_k} |F(J)| \sum_{\Omega(t,J)} \frac{m(J\cap L)}{m(J)} \leqslant \sum_{\substack{\Delta_k \\ \exists (u,L)\in\Theta, L\subset J \\ k\geqslant r(u)}} |F(J)|. \tag{3.22}$$

Since $L \subset J$, $^*\mathrm{reg}(u, L) \geqslant \varrho \geqslant \tau_j, \mathrm{reg}\, J \geqslant \frac{1}{2}$, we have by (2.1) and (3.2)

$$d(u, J) \leqslant \Big(\frac{1}{\tau_j} + 1\Big) d(J) < \frac{1}{\tau_{j+1}} d(J).$$

Moreover, for $(t, J) \in \Delta_k$ and $k \geqslant r(u)$ we have (see (3.6), (3.17))

$$d(u, J) \leqslant \xi_k < \tau_{j+1}\delta_j(u)$$

so that

$$d(u, J) < \delta_j(u), \quad J \subset V(u, \delta_j(u))$$

and by (2.2) and (3.2)

$$^*\mathrm{reg}(u, J) \geqslant \tau_{j+1}.$$

Since $u \in N$, we obtain from (3.22) and (3.7)

$$\Gamma_4 \leqslant \xi_j. \tag{3.23}$$

Now we shall estimate Γ_2. Using (3.9) we obtain

$$\Gamma_2 \leqslant \Gamma_5 + \Gamma_6 = \sum_{\Theta} |F(L)| + \sum_{\Theta} \Big| \sum_{\substack{\Delta_k \\ L \setminus J \neq \emptyset, \forall (t,J) \in \Delta_k}} \Big(\frac{F(J)}{m(J)} m(L \cap J) - F(L \cap J)\Big)\Big|.$$

Θ is δ_j-fine and τ_j-*regular (cf. (3.20) and (3.16)). Therefore (cf. (3.7))

$$\Gamma_5 \leqslant \xi_j. \tag{3.24}$$

Further, we can write

$$\Gamma_6 \leqslant \Gamma_7 + \Gamma_8 = \sum_{\substack{\Theta \\ L \setminus J \neq \emptyset, \forall (t,J) \in \Delta_k \\ t \in N}} \Big| \sum_{\Delta_k} \Big(\frac{F(J)}{m(J)} m(L \cap J) - F(L \cap J)\Big)\Big|$$
$$+ \sum_{\substack{\Theta \\ L \setminus J \neq \emptyset, \forall (t,J) \in \Delta_k \\ t \in I \setminus N}} \Big| \sum_{\Delta_k} \Big(\frac{F(J)}{m(J)} m(L \cap J) - F(L \cap J)\Big)\Big|.$$

The first sum can be divided into three terms:

$$\Gamma_7 \leqslant \Gamma_9 + \Gamma_{10} + \Gamma_{11} = \sum_{\substack{\Delta_k \\ t \in N}} \frac{|F(J)|}{m(J)} \sum_{\Theta} m(L \cap J)$$
$$+ \sum_{\Theta} \sum_{\substack{\Delta_k \\ d(J) \geqslant d(L)}} |F(L \cap J)| + \sum_{\substack{\Delta_k \\ t \in N}} \sum_{\substack{\Theta \\ d(L) > d(J)}} |F(L \cap J)|.$$

By (3.7) we obtain

$$\Gamma_9 \leqslant \xi_k \tag{3.25}$$

since the inner sum does not exceed $m(J)$. Further,

$$\Gamma_{10} \leqslant \sum_{\Theta} \max\{|F(K)|;\, K \subset L\} \cdot \#\{(t,J) \in \Delta_k;\, J \cap L \neq \emptyset, d(J) \geqslant d(L)\}.$$

By [1], Lemma 2.5 the number of elements of Δ_k in the summands on the righthand side of the inequality has the upper bound $3^n 2^{n-1}$ which together with (3.13) yields

$$\Gamma_{10} \leqslant (18)^n \xi_j. \tag{3.26}$$

In a similar manner, with the role of Δ_k and Θ interchanged, taking into account that $\operatorname{reg} L \geqslant \varrho$ for $(u,L) \in \Theta$ and making use of (3.14) and of [1], Lemma 2.5 again, we obtain

$$\begin{aligned}(3.27)\ \Gamma_{11} &\leqslant \sum_{\Delta_k;t\in N} \sup\{|F(H);\, H \subset J\} \cdot \#\{(u,L) \in \Theta;\, L \cap J \neq \emptyset, d(L) > d(J)\}\\ &\leqslant 3^n \varrho^{1-n} \cdot 3^n \xi_k \leqslant 9^n \varrho^{1-n} \xi_k.\end{aligned}$$

Returning to Γ_8, note that $t \in J$ and $\operatorname{reg} J \geqslant \frac{1}{2}$ for $(t,J) \in \Delta_k, k \in \mathbb{N}$ so that (3.1) and (3.5) yield

$$\begin{aligned}(3.28)\qquad |F(J) - g(t)m(J)| &\leqslant \eta\big(d(J)\big)\big(d(J)\big)^n \leqslant 2^{n-1}\eta\big(d(J)\big)m(J),\\ |F(L\cap J) - g(t)m(L\cap J)| &\leqslant 2^{n-1}\eta\big(d(J)\big)m(J)\end{aligned}$$

provided $t \in I \setminus N, L$ being any interval. Hence

$$\left|\frac{F(J)}{m(J)} m(L\cap J) - F(L\cap J)\right| \leqslant 2^n \eta\big(d(J)\big)m(J). \tag{3.29}$$

Now we can write

$$\begin{aligned}\Gamma_8 \leqslant \Gamma_{12} + \Gamma_{13} &= \sum_{\substack{\Delta_k\\ t\in I\setminus N}} \sum_{\substack{\Theta\\ L\cap J\neq\emptyset\\ d(L)\geqslant[\eta(d(J))]^{\frac{3}{4n}} d(J)}} \left|\frac{F(J)}{m(J)} m(L\cap J) - F(L\cap J)\right|\\ &+ \sum_{\Theta} \sum_{\substack{\Delta_k;t\in I\setminus N\\ d(L)<[\eta(d(J))]^{\frac{3}{4n}} d(J)}} \left|\frac{F(J)}{m(J)} m(L\cap J) - F(L\cap J)\right|.\end{aligned}$$

Estimating Γ_{12} with help of (3.29) and [1], Lemma 2.5 we arrive at

$$\begin{aligned}\Gamma_{12} &\leqslant \sum_{\Delta_k;t\in I\setminus N} 2^n\eta\big(d(J)\big)m(J) \cdot \#\big\{(u,L) \in \Theta;\, L \cap J \neq \emptyset, d(L) \geqslant \big[\eta\big(d(J)\big)\big]^{\frac{3}{4n}} d(J)\big\}\\ &\leqslant \sum_{\Delta_k;t\in I\setminus N} 2^n\eta\big(d(J)\big)m(J)3^n\varrho^{1-n}\big[\eta\big(d(J)\big)\big]^{-\frac{3}{4}}.\end{aligned}$$

By (3.5) and Convention 3.4 we obtain

$$\Gamma_{12} \leqslant 6^n \varrho^{1-n} [\eta(\xi_k)]^{\frac{1}{4}}. \tag{3.30}$$

In order to estimate Γ_{13} we use the first inequality (3.28):

$$\Gamma_{13} \leqslant \Gamma_{14} + \Gamma_{15} + \Gamma_{16} = \sum_{\substack{\Delta_k \\ t\in I\setminus N}} |g(t)| \sum_{\substack{\Theta \\ L\setminus J\neq\emptyset\neq L\cap J \\ d(L)\leqslant[\eta(d(J))]^{\frac{3}{4n}} d(J)}} m(L\cap J)$$
$$+ 2^{n-1} \sum_{\Delta_k} \sum_{\Theta} \eta\big(d(J)\big) m(L\cap J) + \sum_{\Theta} \sum_{\substack{\Delta_k \\ d(J)>d(L)}} |F(L\cap J)|.$$

Now (3.4), (3.5) imply

$$\Gamma_{14} \leqslant \sum_{\Delta_k} \big[\eta\big(d(J)\big)\big]^{-\frac{1}{4n}} \sum_{\substack{\Theta \\ L\cap J\neq\emptyset\neq L\setminus J \\ d(L)\leqslant[\eta(d(J))]^{\frac{3}{4n}} d(J)}} m(L\cap J).$$

Taking into account that $\operatorname{reg} J \geqslant \frac{1}{2}$ and assuming

$$[\eta(\xi_k)]^{\frac{3}{4n}} < \frac{\varrho}{2} \tag{3.31}$$

we conclude by (3.5) and [1], Lemma 2.4 (cf. Convention 3.4) that

$$\begin{aligned} \Gamma_{14} &\leqslant \sum_{\Delta_k} \big[\eta\big(d(J)\big)\big]^{-\frac{1}{4n}} \kappa 2^{n-1} m(J) \big[\eta\big(d(J)\big)\big]^{\frac{3}{4n}} \\ &\leqslant \kappa 2^{n-1} [\eta(\xi_k)]^{\frac{1}{2n}}. \end{aligned} \tag{3.32}$$

Evidently,

$$\Gamma_{15} \leqslant 2^{n-1} \sum_{\Delta_k} \eta\big(d(J)\big) m(J) \leqslant 2^{n-1} \eta(\xi_k) \tag{3.33}$$

and finally, by [1], Lemma 2.5 and by (3.13),

$$\begin{aligned} \Gamma_{16} &\leqslant \sum_{\Theta} \sup\{|F(K)|;\, K\subset L\} \cdot \#\{(t,J)\in\Delta_k;\, J\cap L\neq\emptyset, d(J)>d(L)\} \\ &\leqslant 3^n 2^{n-1} 3^n \xi_j \leqslant (18)^n \xi_j. \end{aligned} \tag{3.34}$$

Putting together the estimates (3.21), (3.23)–(3.27), (3.30), (3.32)–(3.34) we obtain

$$\Sigma_1 \leqslant \big(3 + 2\cdot(18)^n\big)\xi_j + (1 + 9^n\varrho^{1-n})\xi_k + 6^n\varrho^{1-n}[\eta(\xi_k)]^{\frac{1}{4}} + \kappa 2^{n-1}[\eta(\xi_k)]^{\frac{1}{2n}} + 2^{n-1}\eta(\xi_k).$$

This together with (3.16) implies that Proposition 3.2 holds for $k \geqslant l_1$ where l_1 is such that (3.31) and

$$(1 + 9^n\varrho^{1-n})\xi_k + 6^n\varrho^{1-n}[\eta(\xi_k)]^{\frac{1}{4}} + \kappa 2^{n-1}[\eta(\xi_k)]^{\frac{1}{2n}} + 2^{n-1}\eta(\xi_k) < \frac{\varepsilon}{2}$$

hold for every $k \geqslant l_1$. □

Proof of Proposition 3.3. Given $\varepsilon > 0$ and $\varrho \in (0,1)$, let us choose $h \in \mathbb{N}$ such that

$$\xi_h + (1 + 6^n)\varrho^{1-2n}\eta(\xi_h) < \frac{\varepsilon}{2}, \quad \tau_h < \varrho \tag{3.35}$$

and denote

$$R(s) = \min\Big\{k \in \mathbb{N};\ \Big(1 + \frac{1}{\varrho}\Big)\xi_k < \delta_h(s)\Big\}. \tag{3.36}$$

For $k \in \mathbb{N}$ let a gauge $\gamma_k\colon I \setminus N \to (0,1]$ be such that

$$\sum_{\Xi} |G_k(K) - g_k(s)m(K)| \leqslant \xi_h \tag{3.37}$$

is satisfied provided $\Xi = \{(s,K)\}$ is a γ_k-fine ϱ-*regular $(I \setminus N)$-tagged L-system (cf. Note 1.6). We choose a gauge $\vartheta_2\colon I \setminus N \to (0,1]$ satisfying the condition

$$\begin{aligned} &\vartheta_2(s) \leqslant \gamma_k(s) \quad \text{for } k < R(s),\\ &\vartheta_2(s) \leqslant \frac{1}{4}\delta_h(s) \quad \text{for } s \in I \setminus N. \end{aligned} \tag{3.38}$$

According to the definition of the functions g_k we have $g_k(s) = F(K)/m(K)$ where $(z,K) \in \Delta_k$, $s \in K^0$. If, moreover, $s \in I \setminus N$, $k \geqslant R(s)$, then $K \subset V\big(z, \delta_k(z)\big)$, $d(K) \leqslant 2\delta_k(z) \leqslant \xi_k \leqslant \frac{1}{2}\delta_h(s) \leqslant \omega(s)$ (see (3.5) and (3.36)), hence $K \subset V\big(s, \delta_h(s)\big) \subset V\big(s, \omega(s)\big)$, and putting $t = s, \nu = d(K)$ in (3.1) and taking into account that $\operatorname{reg} K \geqslant \frac{1}{2}$, $m(K) \geqslant 2^{1-n}[d(K)]^n$ we obtain

$$\Big|F(K) - g(s)m(K)\Big| \leqslant 2^{n-1}\eta\big(d(K)\big)m(K)$$

and consequently,

$$(3.39) \qquad |g_k(s) - g(s)| \leqslant 2^{n-1}\eta(\xi_k).$$

Now we start estimates leading to (3.11). Let $\Theta = \{(u, L)\}$ be a ϑ_2-fine ϱ-*regular $(I \setminus N)$-tagged L-system. For $k \in \mathbb{N}$ we have

$$\Sigma_2 \leqslant \Gamma_{17} + \Gamma_{18} = \sum_{\substack{\Theta \\ k<R(u)}} |G_k(L) - g_k(u)m(L)| + \sum_{\substack{\Theta \\ k \geqslant R(u)}} |G_k(L) - g_k(u)m(L)|.$$

By (3.38) and (3.37) we have

$$(3.40) \qquad \Gamma_{17} \leqslant \xi_h.$$

Further, we can write

$$\Gamma_{18} \leqslant \Gamma_{19} + \Gamma_{20} = \sum_{\substack{\Theta \\ k \geqslant R(u)}} |g(u) - g_k(u)|m(L) + \sum_{\substack{\Theta \\ k \geqslant R(u)}} |G_k(L) - g(u)m(L)|$$

and by virtue of (3.39) we have

$$(3.41) \qquad \Gamma_{19} \leqslant 2^{n-1}\eta(\xi_k)$$

(cf. Convention 3.4). Proceeding to Γ_{20} we estimate it as

$$\Gamma_{20} \leqslant \Gamma_{21} + \Gamma_{22} = \sum_{\Theta} |F(L) - g(u)m(L)| + \sum_{\Theta} \sum_{\substack{\Delta_k \\ k \geqslant R(u)}} \left| \frac{F(J)}{m(J)} m(L \cap J) - F(L \cap J) \right|.$$

To estimate Γ_{21} observe that (cf. (1.3))

$$(3.42) \qquad m(L) \geqslant \varrho^{n-1}\big(d(L)\big)^n \geqslant \varrho^{2n-1}\big(d(u, L)\big)^n.$$

Moreover, $L \subset V(u, \vartheta_2(u))$ so that (cf. (3.5) and (3.38))

$$(3.43) \qquad d(u, L) \leqslant 2\vartheta_2(u) < 2\omega(u).$$

Obviously $L \subset V(u, d(u, L))$. Applying (3.1), (3.42) and (3.43) we have

$$(3.44) \qquad |F(L) - g(u)m(L)| \leqslant \eta(d(u, L))(d(u, L))^n \leqslant \eta(2\vartheta_2(u))\varrho^{1-2n}m(L)$$

and (cf. (3.38), (3.5) and Convention 3.4)

$$\Gamma_{21} \leqslant \varrho^{1-2n}\eta(\xi_h). \tag{3.45}$$

The term Γ_{22} is divided into three sums:

$$\begin{aligned}\Gamma_{22} \leqslant \Gamma_{23} + \Gamma_{24} + \Gamma_{25} &= \sum_{\Theta} \sum_{\substack{\Delta_k;k\geqslant R(u)\\ d(J)\geqslant d(L)}} \Big|\frac{F(J)}{m(J)}m(L\cap J) - F(L\cap J)\Big| \\ &+ \sum_{\Theta} \sum_{\substack{\Delta_k;k\geqslant R(u)\\ t\in I\setminus N, d(L)>d(J)}} \Big|\frac{F(J)}{m(J)}m(L\cap J) - F(L\cap J)\Big| \\ &+ \sum_{\Theta} \sum_{\substack{\Delta_k;k\geqslant R(u)\\ t\in N, d(L)>d(J)}} \Big|\frac{F(J)}{m(J)}m(L\cap J) - F(L\cap J)\Big|,\end{aligned}$$

where

$$\begin{aligned}\Gamma_{23} &\leqslant \Gamma_{26} + \Gamma_{27} \\ &= \sum_{\Theta} \sum_{\substack{\Delta_k;k\geqslant R(u)\\ d(J)\geqslant d(L)}} \Big|\frac{F(J)}{m(J)} - g(u)\Big| m(L\cap J) \\ &+ \sum_{\Theta} \sum_{\substack{\Delta_k\\ d(J)\geqslant d(L)}} |g(u)m(L\cap J) - F(L\cap J)|.\end{aligned}$$

Let us estimate Γ_{26}. The partition Δ_k is δ_k-fine so that $d(J) \leqslant 2\delta_k(t) \leqslant \xi_k$ by (3.5). If a summand in Γ_{26} is nonzero then necessarily $L \cap J \neq \emptyset$, which implies $J \subset V\big(u, d(u,L) + d(J)\big)$. Taking into account (1.3) and (3.36) together with $d(L) \leqslant d(J)$ and $k \geqslant R(u)$ we get $d(u,L) + d(J) \leqslant \big(1 + \frac{1}{\varrho}\big)d(J) \leqslant \big(1+\frac{1}{\varrho}\big)\xi_k < \delta_h(u) < \omega(u)$ so that by (3.1)

$$\begin{aligned}\Big|F(J) - g(u)m(J)\Big| &\leqslant \eta\Big(\Big(1+\frac{1}{\varrho}\Big)d(J)\Big)\Big[\Big(1+\frac{1}{\varrho}\Big)d(J)\Big]^n \\ &\leqslant 2^{n-1}\Big(1+\frac{1}{\varrho}\Big)^n \eta\Big(\Big(1+\frac{1}{\varrho}\Big)\xi_k\Big)m(J)\end{aligned}$$

since $\operatorname{reg} J \geqslant \frac{1}{2}, m(J) > 2^{1-n}(d(J))^n$. It follows that

$$\Gamma_{26} \leqslant 2^{n-1}\Big(1+\frac{1}{\varrho}\Big)^n \eta\Big(\Big(1+\frac{1}{\varrho}\Big)\xi_k\Big). \tag{3.46}$$

For the nonvanishing summands of Γ_{27} we have by (3.1) and (3.43)

$$|F(L\cap J)-g(u)m(L\cap J)|\leqslant \eta\big(d(u,L)\big)\varrho^{1-n}\big(d(u,L)\big)^n.$$

Moreover, $d(u,L)\leqslant 2\vartheta_2(u)\leqslant \delta_h(u)\leqslant \frac{1}{2}\xi_h$ (cf. (3.38) and (3.5)) so that (cf. (1.3))

$$\Gamma_{27}\leqslant \varrho^{1-2n}\sum_{\Theta}\eta(\xi_h)m(L)\#\{(t,J)\in\Delta_k\,;\ J\cap L\neq\emptyset, d(J)\geqslant d(L)\}.$$

Observe that $\operatorname{reg} J>\frac{1}{2}$. By [1], Lemma 2.5 for every $(u,L)\in\Theta$ the number of elements of Δ_k on the righthand side of the inequality does not exceed 3^n2^{n-1} and so

$$\Gamma_{27}\leqslant 6^n\varrho^{1-2n}\eta(\xi_h). \tag{3.47}$$

Returning to Γ_{24} and taking into account that $\operatorname{reg} J>\frac{1}{2}, m(J)>2^{1-n}(d(J))^n$ we get by (3.1)

$$|F(J)-g(t)m(J)|\leqslant 2^{n-1}\eta\big(d(J)\big)m(J),$$
$$|F(L\cap J)-g(t)m(L\cap J)|\leqslant 2^{n-1}\eta\big(d(J)\big)m(J),$$

which yields

$$\left|\frac{F(J)}{m(J)}m(L\cap J)-F(L\cap J)\right|\leqslant 2^n\eta\big(d(J)\big)m(J)$$

and

$$\Gamma_{24}\leqslant 2^n\sum_{\Delta_k}\eta\big(d(J)\big)m(J)\cdot\#\{(u,L)\in\Theta;\ L\cap J\neq\emptyset, d(L)>d(J)\}.$$

By [1], Lemma 2.5 for every $(t,J)\in\Delta_k$ the number of the elements of Θ on the righthand side of the inequality does not exceed $3^n\varrho^{1-n}$. It follows that

$$\Gamma_{24}\leqslant 6^n\varrho^{1-n}\eta(\xi_k). \tag{3.48}$$

Finally, we write

$$\Gamma_{25}\leqslant\Gamma_{28}+\Gamma_{29}=\sum_{\Theta}\sum_{\Delta_k;t\in N}\frac{|F(J)|}{m(J)}m(L\cap J)+\sum_{\Theta}\sum_{\substack{\Delta_k;t\in N\\ d(L)>d(J)}}|F(L\cap J)|.$$

By (3.7)

$$\Gamma_{28}\leqslant\sum_{\Delta_k,t\in N}|F(J)|\sum_{\Theta}\frac{m(L\cap J)}{m(J)}\leqslant\sum_{\Delta_k;t\in N}|F(J)|\leqslant\xi_k. \tag{3.49}$$

Finally,

$$\Gamma_{29} \leqslant \sum_{\Delta_k;t\in N} \max\{|F(K)|;\, K \subset J\} \cdot \#\{(u,L) \in \Theta;\, L \cap J \neq \emptyset, d(L) > d(J)\}.$$

As above, for every $(t,J) \in \Delta_k$ the number of elements of Θ on the righthand side of the inequality does not exceed $3^n \varrho^{1-n}$, which combined with (3.14) yields

$$\Gamma_{29} \leqslant 9^n \varrho^{1-n} \xi_k. \tag{3.50}$$

Putting together the estimates (3.40), (3.41), (3.45)–(3.50) we obtain that

$$\begin{aligned}\Sigma_2 \leqslant{}& \xi_h + 2^{n-1}\eta(\xi_k) + \varrho^{1-2n}\eta(\xi_h)\\ &+ 2^{n-1}\Big(1+\frac{1}{\varrho}\Big)^n \eta\Big(\Big(1+\frac{1}{\varrho}\Big)\xi_k\Big) + 6^n\varrho^{1-2n}\eta(\xi_h)\\ &+ 6^n\varrho^{1-n}\eta(\xi_k) + \xi_k + 9^n\varrho^{1-n}\xi_k.\end{aligned}$$

It follows by (3.35) that Proposition 3.3 holds provided l_2 is so large that

$$(2^{n-1} + 6^n\varrho^{1-n})\eta(\xi_k) + (1 + 9^n\varrho^{1-n})\xi_k + 2^{n-1}\Big(1+\frac{1}{\varrho}\Big)^n \eta\Big(\Big(1+\frac{1}{\varrho}\Big)\xi_k\Big) < \frac{\varepsilon}{2}.$$

□

References

[1] *J. Kurzweil and J. Jarník*: Perron-type integration on n-dimensional intervals as an extension of integration of stepfunctions by strong equiconvergence. Czechosl. Math. J. *46 (121)* (1996), 1–20.

[2] *J. Jarník and J. Kurzweil*: Perron-type integration on n-dimensional intervals and its properties. Czechosl. Math. J. *45* (1995), 79–106.

[3] *J. Kurzweil*: Nichtabsolut konvergente Integrale. Teubner, Leipzig, 1980.

[4] *E. J. McShane*: A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals. Mem. Amer. Math. Soc. *88* (1969).

[5] *J. Mawhin*: Generalized multiple Perron integrals and the Green-Goursat theorem for differentiable vector fields. Czechosl. Math. J. *31* (1981), 614–632.

[6] *E. J. McShane*: Unified Integration. Academic Press, 1983.

Authors' addresses: J. Jarník, M. D. Rettigové 4, 116 39 Praha 1, Pedagogická fakulta UK, Czech Republic; J. Kurzweil, Žitná 25, 115 67 Praha 1, Matematický ústav AV ČR, Czech Republic.