Mumtaz Ahmad Khan

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A STUDY OF q-LAGUERRE POLYNOMIALS THROUGH THE $T_{k,q,x}$-OPERATOR

M. A. KHAN, Aligarh

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Abstract. The present paper deals with certain generating functions and recurrence relations for q-Laguerre polynomials through the use of the $T_{k,q,x}$-operator introduced in an earlier paper [7].

1. INTRODUCTION

In [7], the present author introduced the $T_{k,q,x}$-operator by means of the relation

$$T_{k,q,x} \equiv x(1-q)\{[k]+q^k xD_{q,x}\}, \tag{1.1}$$

obtaining in [8] operational representations for various q-polynomials. The operational representations for q-Laguerre polynomials will be used in the present paper to establish certain generating functions and recurrence relations for q-Laguerre polynomials. The use of operational representations obtained in [8] for finding generating functions and recurrence relations of other important q-polynomials will be dealt with elsewhere. For definitions and notation one is referred to W. Hahn [3], M. A. Khan [4–6] and L. J. Slater [12].

2. GENERATING FUNCTIONS

In this section, some generating functions for q-Laguerre polynomials will be obtained from the operational representations established in [7–8].

To start with, consider the identity

$$e_q(-x)E_q(-xt) = e_q(-[1-t]x),$$

which we can write as

$$\sum_{r=0}^{\infty} \frac{q^{\frac{1}{2}r(r-1)}t^r}{(q)_r} x^{a+r} e_q(-x) = x^a e_q\big(-[1-t]x\big). \tag{2.1}$$

Now, operating on both sides of (2.1) with $T_{k,q}^m$ and then replacing x by $xq^{1-m-a-k}$ and t by t/x, we obtain the generating function

$$\begin{aligned} &E_q(-tq^{1-m-a-k})\,{}_qL_m^{(a+k-1)}([x-t],1) \\ &= \sum_{r=0}^{\infty} \frac{t^r q^{\frac{1}{2}r(r-1)+r(1-m-a-k)}}{(q)_r}\,{}_qL_m^{(a+r+k-1)}(xq^r,1). \end{aligned} \tag{2.2}$$

Similarly, considering the identity

$$\sum_{r=0}^{\infty} \frac{t^r}{(q)_r} x^{a+r} E_q(x) = x^a E_q([1-t]x) \tag{2.3}$$

and operating on both sides of (2.3) with $T_{k,q}^m$ and finally replacing x by xq^{-m} and t by t/x, we obtain the generating function

$$e_q(t)\,{}_qL_m^{(a+k-1)}([x-t]) = \sum_{r=0}^{\infty} \frac{t^r q^{-mr}}{(q)_r}\,{}_qL_m^{(a+r+k-1)}(x). \tag{2.4}$$

We next consider the operational formula

$$e_q(tT_{k,q})\{x^a e_q(-x)\} = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n} T_{k,q}^n\{x^a e_q(-x)\}.$$

Each term on the right hand side can be evaluated by means of [8, (4.9)] and on the left had side by means of [7, (3.18)]. This immediately yields the generating function

$$e_q\left(\frac{-x}{[1-tq^{k+a}]}\right) E_q(-x) = (1-t)_{k+a} \sum_{n=0}^{\infty} {}_qL_n^{(a+k-1)}(xq^{n+a+k-1},1)t^n. \tag{2.5}$$

Similarly, considering the expansion

$$e_q(tT_{k,q})\{x^a E_q(x)\} = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n} T_{k,q}^n\{x^\alpha E_q(x)\},$$

one gets

$$e_q(x) E_q\left(\frac{x}{[1-tq^{k+a}]}\right) = (1-t)_{k+a} \sum_{n=0}^{\infty} \frac{t^n}{(x)_n}\,{}_qL_n^{(a+k-1)}(xq^n). \tag{2.6}$$

Another way of deriving (2.5) is by means of the relation

$$\frac{x^a}{(1-xt)_{a+k}} = \sum_{n=0}^{\infty} \frac{(q^{a+k})_n t^n}{(q)_n} x^{a+n}.$$

Operating on both sides with ${}_0\Phi_1\left[-;q^{a+k};T_{k,q}\right]$, then evaluating the left hand side by means of [8, (4.7)] and the right hand side by means of [8, (4.17)] and finally replacing x by $-x$ and t by $-t/x$, we get (2.5).

Similarly, for another way of deriving (2.6), consider

$$\frac{x^a}{(1-qxt)_{k+a}} = \sum_{n=0}^{\infty} \frac{(q^{k+a})_n q^n t^n}{(q)_n} x^{a+n}.$$

Operating on both sides with

$${}_0\Phi_1\left[\begin{matrix} \ldots; & T_{k,q} \\ q^{k+a}; & q \end{matrix}\right],$$

then evaluating left had side by [8, (4.8)] and the right hand side by [8, (4.18)] and finally replacing x by $-x/q$ and t by $-t/q$, we get (2.6).

It may be remarked that formula [7, (3.2)], in particular, yields

$$\begin{aligned} &{}_0\Phi_1\left[\ldots;q^{k+a};tT_{k,q}\right]\left\{x^a e_q(-x)\right\} \\ &\quad = x^a e_q(-x) e_q(xt)\cdot{}_0\Phi_1\left[\begin{matrix} \ldots; & -x^2tq^{k+a-2} \\ q^{k+a}; & q^2 \end{matrix}\right], \end{aligned} \tag{2.7}$$

which is equivalent to the generating function

$$e_q(t){}_0\Phi_1\left[\begin{matrix} \ldots; & -xtq^{a+k-2} \\ q^{k+a}; & q^2 \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{t^n}{(q^{k+a})_n} {}_qL_n^{(a+k-1)}\left(xq^{n+a+k-1},1\right) \tag{2.8}$$

obtained by simplifying the left hand side of (2.7) and then replacing t by t/x.

If in (2.8) we replace t by $tT_{k,q,y}$ and operate on y^b, we get, by using [7, (3.21)], the operational relation

$$\begin{aligned} &e_q\left(tT_{k,q,y}\right)\left\{y^b{}_1\Phi_1\left[\begin{matrix} q^{k+b}; & -xytq^{a+k-2} \\ q^{k+a}; & q^2 \end{matrix}\right]\right\} \\ &= y^b \sum_{n=0}^{\infty} \frac{(q^{k+b})_n (yt)^n}{(q^{k+a})_n} {}_qL_n^{(a+k-1)}\left(xq^{n+a+k-1},1\right), \end{aligned}$$

which gives by virtue of [7, (3.18)] the generating function

$$\sum_{n=0}^{\infty} \frac{(q^{k+b})_n t^n}{(q^{k+a})_n} {}_qL_n^{(a+k-1)}\left(xq^{n+a+k-1}, 1\right) \tag{2.9}$$
$$= \frac{1}{(1-t)_{k+b}} {}_1\Phi_1 \begin{bmatrix} q^{k+b}; & -xtq^{a+k-2}/[1-tq^{k+b}] \\ q^{k+a}; & q^2 \end{bmatrix}.$$

For $b = \beta - k$, (2.9) becomes

$$\sum_{n=0}^{\infty} \frac{(q^{\beta})_n t^n}{(q^{k+a})_n} {}_qL_n^{(a+k-1)}\left(xq^{n+a+k-1}, 1\right) \tag{2.10}$$
$$= \frac{1}{(1-t)_{\beta}} {}_1\Phi_1 \begin{bmatrix} q^{\beta}; & -xtq^{a+k-2}/[1-tq^{\beta}] \\ q^{k+a}; & q^2 \end{bmatrix}.$$

We now put $f(x) = e_q(-x)$ in [7, (3.19)] and use [8, (4.9)] on the left hand side. We thus obtain the generating function

$$\sum_{n=0}^{\infty} t^n q^{\frac{1}{2}n(n+1)} {}_qL_n^{(a+k-1-n)}\left(xq^{a+k-1}, 1\right) = (1+tq)_{a+k-1} E_q(xtq^{k+a}). \tag{2.11}$$

Putting $k = 1$, replacing t by t/q and x by xq^{-a} in (2.11), we obtain

$$\sum_{n=0}^{\infty} t^n q^{\frac{1}{2}n(n+1)} {}_qL_n^{(a-n)}(x, 1) = (1+t)_a E_q(xt). \tag{2.12}$$

Multiplying (2.11) by $t^b e_q(xtq^{k+a})$ and operating on the variable t by $T^m_{k,q,t}$, we get

$$\sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} {}_qL_n^{(a+k-1-n)}\left(xq^{a+k-1}, 1\right) T^m_{k,q,t}\left\{t^{b+n} e_q(xtq^{k+a})\right\}$$
$$= T^m_{k,q,t}\left\{t^b {}_1\Phi_0[q^{1-a-k}; \ldots; -tq^{a+k}]\right\},$$

hence we obtain the following generalization of the generating function (2.11):

$$\sum_{n=0}^{\infty} t^n q^{\frac{1}{2}n(n+1)} {}_qL_n^{(a+k-1-n)}\left(xq^{a+k-1}, 1\right) {}_qL_m^{(b+n+k-1)} \tag{2.13}$$
$$\times \left(-xtq^{a+b+m+n+2k-1}, 1\right)$$
$$= \frac{(q^{k+b})_m}{(q)_m} E_q(xtq^{k+a}) {}_2\Phi_1\left[q^{1-a-k}; q^{k+b+m}; q^{k+b}; -tq^{k+a}\right].$$

Putting $k = 1$, replacing x by xq^{-a} and t by $-t/q$ in (2.13), we get the following generalization of (2.12):

$$\text{(2.14)} \qquad \sum_{n=0}^{\infty}(-t)^n q^{\frac{1}{2}n(n-1)} {}_qL_n^{(a-n)}(x,1)\,{}_qL_m^{(b+n)}\left(xtq^{b+m+n},1\right)$$
$$= \frac{(q^{1+b})_m}{(q)_m} E_q(-xt)\,{}_2\Phi_1\left[q^{-a};q^{1+b+m};q^{1+b};tq^a\right].$$

Also, for $k = 1$, (2.11) reduces to

$$\text{(2.15)} \qquad \sum_{n=0}^{\infty} t^n q^{\frac{1}{2}n(n+1)} {}_qL_n^{(a-n)}\left(xq^a\right) = (1+tq)_a E_q(xtq^{1+a}).$$

Now, multiplying (2.15) by $x^{1+a-k}e_q(-x)$, using [8, (4.9)] to express the q-Laguerre polynomial on the left hand side of (2.15) by its operational representation and then operating on both sides with $T_{k,q}^m$, we get by using [8, (4.11)].

$$\text{(2.16)} \qquad \sum_{n=0}^{\infty} \binom{m+n}{n}_q t^n q^{\frac{1}{2}n(n+1)} {}_qL_{m+n}^{(a-n)}(xq^{a+m},1)$$
$$= (1+tq)_a E_q(xtq^{1+a})\,{}_qL_m^{(a)}\left([1+tq^{1+a}]xq^{m+a},1\right).$$

Similarly, if we put $f(x) = E_q(x)$ in [7, (3.19)] and use [8, (4.10)] on the left hand side, we get the following generating function for ${}_qL_n^{(a)}(x)$:

$$\text{(2.17)} \qquad \sum_{n=0}^{\infty} \frac{t^n q^{\frac{1}{2}n(n+1)}}{(q)_n} {}_qL_n^{(a+k-n-1)}\left(xq^n\right) = (1+tq)_{a+k-1} e_q(-xtq^{k+a}).$$

If we multiply (2.17) by $x^{1+a-k}E_q(x)$ and operate with $T_{k,q}^m$, we get by replacing x by xq^{-m} and t by tq^{-1-a} that

$$\text{(2.18)} \qquad \sum_{n=0}^{\infty} \frac{t^n q^{\frac{1}{2}n(n+1)-n(1+a)}}{(q)_n} \binom{m+n}{n} {}_qL_{m+n}^{(a-n)}(xq^n)$$
$$= (1+tq^{-a})_a e_q(-x)\,{}_qL_m^{(a)}\left([1+t]x\right).$$

Since $T^n_{k,q,x} = T_{k,q,x}(T^{n-1}_{k,q,x})$, we have by virtue of [8, (4.9)] and [8, (4.10)] the following results:

$$T_{k,q}\Big\{T^{n-1}_{k,q}\{x^a e_q(-x)\}\Big\} = x^{a+n}(q)_n e_q(-x)_q L_n^{(a+k-1)}\big(xq^{n+a+k-1},1\big) \tag{3.1}$$

and

$$T_{k,q}\Big\{T^{n-1}_{k,q}\{x^a E_q(x)\}\Big\} = x^{a+n}(q)_n E_q(xq^n)_q L_n^{(a+k-1)}\big(xq^n\big). \tag{3.2}$$

In view of [8, (4.9)] and [8, (4.10)], results (3.1) and (3.2) give the following recurrence relations:

$$\begin{aligned}\Big\{[n+a+k-1] - \frac{xq^{n+a+k-1}}{(1-q)} + x(1+x)q^{n+a+k-1}D_q\Big\}_q L_{n-1}^{(a+k-1)} \\ \times \big(xq^{n+a+k-2},1\big) = [n]_q L_n^{(a+k-1)}\big(xq^{n+a+k-1},1\big)\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\Big\{[n+a+k-1] - \frac{xq^{n-1}}{(1-q)} + xq^{n+a+k-1}D_q\Big\}_q L_{n-1}^{(a+k-1)}\big(xq^{n-1}\big) \\ = [n]_q L_n^{(a+k-1)}\big(xq^n\big).\end{aligned} \tag{3.4}$$

Now, putting $k=1$ and replacing x by xq^{-n-a} in (3.3) and x by xq^{-n} in (3.4), one can obtain neat forms of (3.3) and (3.4).

Further, since we can write

$$T^n_{k,q}\{x^a e_q(-x)\} = T^n_{k,q}\{x^k \cdot x^m \cdot x^{a-m-k} e_q(-x)\}$$

and

$$T^n_{k,q}\{x^a E_q(x)\} = T^n_{k,q}\{x^k \cdot x^m \cdot x^{a-m-k} E_q(x)\}.$$

we have by making use of [7, (3.8)] with $u = x^m$ and $v = x^{a-m-k}e_q(-x)$ in the former case and $v = x^{a-m-k}E_q(x)$ in the latter case

$$\begin{aligned}&x^{a+n}(q)_n e_q(-x)_q L_n^{(a+k-1)}\big(xq^{n+a+k-1},1\big) \\ &= x^k \sum_{r=0}^{\infty} \binom{n}{r}_q q^{kr+r(r-n)}\{T^r_{k,q}x^m\}T^{n-r}_{k,q,xq^r}\{(xq^r)^{a-m-k}e_q(-xq^r)\}\end{aligned}$$

and

$$x^{a+n}(q)_n E_q(xq^n)\,{}_qL_{n}^{(a+k-1)}(xq^n)$$
$$= x^k \sum_{r=0}^{\infty} \binom{n}{r}_q q^{kr+r(r-n)} T_{k,q,xq^r}^{n-r}\left\{(xq^r)^{a-m-k} E_q(xq^r)\right\}\left\{T_{k,q}^{r} x^m\right\}.$$

Hence, we have

$$\text{(3.5)}\qquad {}_qL_n^{(a+k-1)}(xq^{n+a+k-1},1)$$
$$= \sum_{r=0}^{n} \frac{q^{r(a-m)}(q^{k+m})_r(1+x)_r}{(q)_r}\,{}_qL_{n-r}^{(a-m-1)}(xq^{n+a-m-1},1)$$

and

$$\text{(3.6)}\qquad {}_qL_n^{(a+k-1)}(xq^n) = \sum_{r=0}^{n} \frac{(q^{k+m})_r q^{r(a-m)}}{(q)}\,{}_qL_{n-r}^{(a-m-1)}(xq^n).$$

Putting $m = 0$ and $k = 1$ in (3.5), we get

$$\text{(3.7)}\qquad {}_qL_n^{(a)}(xq^{n+a},1) = \sum_{r=0}^{n} q^{ra}(1+x)_r\,{}_qL_{n-r}^{(a-1)}(xq^{n+a-1},1).$$

Replacing x by xq^{-a} in (3.7), we get

$$\text{(3.8)}\qquad {}_qL_n^{(a)}(xq^n,1) = \sum_{r=0}^{\infty}(q^\alpha + x)_r\,{}_qL_{n-r}^{(a-1)}(xq^{n-1},1).$$

Similarly, putting $k = 1$, $m = 0$ and replacing x by xq^{-n} in (3.6), we get

$$\text{(3.9)}\qquad {}_qL_n^{(a)}(x) = \sum_{r=0}^{\infty} q^{ra}\,{}_qL_{n-r}^{(a-1)}(x).$$

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Author's address: Department of Applied Mathematics Z. H. College of Engg. and Technology, Faculty of Engineering, A.M.U., Aligarh-202002, U.P., India.