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HOMOMORPHISMS BETWEEN A-PROJECTIVE ABELIAN GROUPS AND LEFT KASCH-RINGS

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Abstract. Glaz and Wickless introduced the class G of mixed abelian groups A which have finite torsion-free rank and satisfy the following three properties: i) A_p is finite for all primes p, ii) A is isomorphic to a pure subgroup of $\Pi_p A_p$, and iii) $\mathrm{Hom}(A, tA)$ is torsion. A ring R is a left Kasch ring if every proper right ideal of R has a non-zero left annihilator. We characterize the elements A of G such that $E(A)/tE(A)$ is a left Kasch ring, and discuss related results.

Keywords: mixed Abelian group, endomorphism ring, Kasch ring, A-solvable group

MSC 2000: 20K20

1. Introduction

One of the oldest problems in abelian group theory is to determine when a given sequence of abelian groups splits. The result which is perhaps most frequently cited in this context is Szele's Theorem [9, Proposition 27.1] describing when a subgroup of the form $\bigoplus_I \mathbb{Z}/p^n\mathbb{Z}$ of an abelian group is a direct summand. Szele's result was extended to valuated p-groups by one of the authors in [1]. The key in this extension was to study an indecomposable finite valuated p-group $E(A)$ as a module over its endomorphism ring. Since this ring is finite and local, the valuated version of Szele's result then follows from the fact that every module over a local Artinian ring is either projective or has infinite projective dimension. Another area of abelian group theory where local Artinian rings have played an important role is the investigation of quasi-decomposition of torsion-free abelian groups of finite rank. Indeed, one of the author showed in [2] that results in the spirit of Szele's Theorem also exist for torsion-free abelian groups of finite rank. For instance, an exact sequence $0 \to A^n \to A^m$ such that $n, m < \omega$ quasi-splits whenever $\mathbb{Q}E(A)$ has *finistic dimension* 0,

i. e. every finitely generated right $\mathbb{Q}E(A)$-module is either projective or has infinite projective dimension. This class of rings was discussed in [8] where Bass showed that a Noetherian ring R has finistic dimension 0 exactly if and only if every proper right ideal of R has a non-zero left annihilator, i. e. iff R is a *left Kasch-ring* in the sense of [12].

It is the goal of this paper to further investigate the relation between left Kasch-rings and splitting results for abelian groups similar to the previously mentioned. Whenever R is a Kasch-ring whose additive group is torsion-free, then R^+ is divisible since, for any non-zero integer n, nR is a left ideal whose right annihilator is zero. But this setting has been investigated in [2]. On the other hand, if A is a p-group with two non-isomorphic cyclic direct summands, then no Szele-style result can exist for A unless $A \cong \bigoplus_I \mathbb{Z}/p^n\mathbb{Z}$ for some $n < \omega$ since there always exists a monomorphism between these two cyclic summands which does not split. Therefore, the case of p-groups has been covered by Szele's original result. We may therefore assume that A is an honest mixed group.

The investigation of properties of mixed groups which are determined by the endomorphism ring has been particularly successful for the class $\mathcal{G}$ of mixed abelian groups which was introduced by Glaz and Wickless in [10]. In order to define $\mathcal{G}$, we first consider the class Γ of mixed groups G with the property that G is isomorphic to a pure subgroup of $\Pi_p\, G_p$ containing $\bigoplus_p G_p$. The symbol Γ_∞ denotes the groups in Γ which have finite torsion-free rank. Every $G \in \Gamma_\infty$ contains a finite subset X such that $F = \langle X \rangle$ is a free subgroup of G with G/F torsion. We view G as a pure subgroup of $\Pi_p\, G_p$, and write $X = \{x_i = (x_{ip}) \mid i = 1, \ldots, n\}$. Glaz and Wickless investigated the class $\mathcal{G}$ of groups in Γ_∞ for which G_p is finite for all p and satisfies $G_p = \langle x_{1p}, \ldots, x_{np} \rangle$ for all but finitely many p. Observe that every element of $\mathcal{G}$ is either an honest mixed group or finite. They showed in [10] that a group $G \in \Gamma_\infty$ such that G_p is finite for all p is in $\mathcal{G}$ if and only if $\mathrm{Hom}(G, tG)$ is a torsion group. In particular, $E(A)/tE(A)$ is a finite dimensional $\mathbb{Q}$-algebra for $A \in \mathcal{G}$. Goeters, Wickless, and the author continued the discussion of [10] in [6] by showing that the elements of $\mathcal{G}$ are the mixed self-small abelian groups which have finite torsion-free rank. Moreover, they showed that A is flat as an $E(A)$-module iff A/tA is a projective $E(A)/tE(A)$-module.

Section 2 investigates the groups $A \in \mathcal{G}$ for which $E(A)/tE(A)$ is a left Kasch-ring. We show that a group $A \in \mathcal{G}$ has this property if and only if, whenever $0 \to U \to A^n \to A^m \xrightarrow{\beta} G \to 0$ is an exact sequence with $n, m < \omega$ such that $S_A(U)$ is torsion, then U is finite and β quasi-splits (Theorem 2.2). The section concludes with a discussion of consequences of Theorem 2.2. Section 3 investigates the structure of groups G which arise as cokernels and kernels of maps $A^n \to A^m$ where $n, m < \omega$.

We show that each such G is isomorphic to a group of the form $T_A(M)$ where M is a finitely presented right $E(A)$-module (Theorem 3.1). We in particular investigate when such G's are a direct sum of a finite group and an A-projective group of finite A-rank. Here P is *A-projective (of finite A-rank)* if it is isomorphic to a direct summand of $\bigoplus_I A$ for some (finite) index-set I (Theorem 3.4).

2. Groups $A \in \mathcal{G}$ such that $E(A)/tE(A)$ is a Left Kasch-Ring

If A is an abelian group, then the symbol $H_A(G)$, where G is an abelian group, denotes the right $E(A)$-module $\mathrm{Hom}(A, G)$ whose $E(A)$-module structure is induced by composition of maps. If M is a right $E(A)$-module, then $T_A(M) = M \otimes_{E(A)} A$. Since H_A and T_A form an adjoint pair of functors, we can find natural maps $\theta_G\colon T_AH_A(G) \to G$ and $\varphi_M\colon M \to H_AT_A(M)$ for any abelian group G and right $E(A)$-module M. We say that G is A-solvable if θ_G is an isomorphism. For instance, if $A \in \mathcal{G}$, then A-projective groups are A-solvable, while φ_M is an isomorphism if M is projective [6]. We denote the image of θ_G by $S_A(G)$, and call a group G (*finitely*) *A-generated* if it is an image of an A-projective group of (finite) A-rank. Clearly, G is A-generated iff $S_A(G) = G$. We say that an A-generated group G is A-presented if $G \cong (\bigoplus_I A)/U$ for some index-set I and an A-generated subgroup U of $\bigoplus_I A$.

Theorem 2.1. *Let $A \in \mathcal{G}$, $p_1, \dots, p_n$ be primes of $\mathbb{Z}$, and C a fully invariant subgroup of A such that $A = B \oplus C$ where $B = A_{p_1} \oplus \dots \oplus A_{p_n}$.*

a) *If G is a direct summand of an A-presented group, then $G = S_B(G) \oplus S_C(G)$.*

b) *If $0 \to U \xrightarrow{\alpha} \bigoplus_I A \xrightarrow{\beta} G \to 0$ is an exact sequence such that $S_A(U) = U$, then the induced sequences $0 \to S_B(U) \xrightarrow{\alpha|_{S_B(U)}} \bigoplus_I B \xrightarrow{\beta|_{\oplus_I B}} S_B(G) \to 0$ and $0 \to S_C(U) \xrightarrow{\alpha|_{S_C(U)}} \bigoplus_I C \xrightarrow{\beta|_{\oplus_I C}} S_C(G) \to 0$ are exact.*

Proof. a) Let H be an A-presented group, and consider an exact sequence $0 \to U \xrightarrow{\alpha} \bigoplus_I A \xrightarrow{\beta} H \to 0$ with $S_A(U) = U$. For $g \in H_p$, we can find $x \in \bigoplus_I A$ such that $g = \beta(x)$. If $p^s g = 0$ for some $s < \omega$, then $p^s x = \alpha(u)$ for some $u \in U$. Since U is A-generated, U/tU is an epimorphic image of the divisible group A/tA. Hence, there are $u_1 \in U$ and $t \in tU$ with $u = p^s u_1 + t$. Then $p^s\bigl(x - \alpha(u_1)\bigr) = \alpha(t) \in \bigoplus_I tA$ yields $p^{s+r} m\bigl(x - \alpha(u_1)\bigr) = 0$ for some $r < \infty$ and integer m relatively prime to p. Since $m\bigl(x - \alpha(u_1)\bigr) \in \bigoplus_I A_p$, we have $p^{n_p} m\bigl(x - \alpha(u_1)\bigr) = 0$ where $n_p < \omega$ is chosen minimal with $p^{n_p} A_p = 0$. Then, $p^{n_p} mg = 0$ from which $p^{n_p} H_p = 0$ follows. In particular, $\mathbb{Z}(p^\infty) \nsubseteq H$.

We write $H = [\bigoplus_J \mathbb{Q}] \oplus L$ for some index-set J and a reduced group L. By [4, Theorem 2.2], $R_A(L) = 0$ where $R_A(L) = \cap \{\ker \varphi \mid \varphi \in \mathrm{Hom}(L, A)\}$, and we can find an index-set J_1 such that $L \subseteq B^{J_1} \oplus C^{J_1}$. If $S_B(L) \not\subseteq B^{J_1}$, then the projection of $B^{J_1} \oplus C^{J_1} \to C^{J_1}$ with kernel B^{J_1} would induce a non-zero map $\bigoplus_I B \to C^{J_1}$ which would contradict the fact that $\mathrm{Hom}(B, C) = 0$. A similar argument establishes $S_C(L) \subseteq C^{J_1}$. Moreover, since B is finite, $\bigoplus_J \mathbb{Q} \subseteq S_C(H)$. We obtain $S_B(H) = S_B(L)$ and $S_C(H) = S_C(L) \oplus [\bigoplus_J \mathbb{Q}]$ which yields $S_B(H) \cap S_C(H) = 0$. On the other hand, $A = B \oplus C$ yields $H = S_A(H) = S_B(H) + S_C(H)$. Finally, if $H = G \oplus W$, then $S_B(H) = S_B(G) \oplus S_B(W)$ and $S_C(H) = S_C(G) \oplus S_C(W)$. Therefore, $G = S_A(G) = S_B(G) \oplus S_C(G)$ as desired.

b) Since U is reduced, $\ker \theta_U$ is torsion-free divisible by [4, Theorem 2.2a]. Hence U is a direct summand of the A-presented group $T_A H_A(U)$, and $U = S_B(U) \oplus S_C(U)$. Since $\mathrm{Hom}(B, C) = 0 = \mathrm{Hom}(C, B)$, we have $\bigoplus_I B = S_B(\bigoplus_I A)$ and $\bigoplus_I C = S_C(\bigoplus_I A)$. Therefore, the maps in the sequences have the indicated domains and ranges and it remains to show that the restrictions of β are onto. For this, observe $G = S_B(G) \oplus S_C(G) \supseteq \beta(\bigoplus_I B) + \beta(\bigoplus_I C) \supseteq \beta(\bigoplus_I A) = G$. □

We now show that the investigation of splitting conditions similar to Szele's is closely related to the question for which $A \in \mathcal{G}$ the ring $E(A)/tE(A)$ is a left Kasch-ring. For reasons of simplicity, the symbol $\overline{M}$ denotes the $E(A)/tE(A)$-module M/tM.

Theorem 2.2. *The following conditions are equivalent for a group $A \in \mathcal{G}$:*

a) *$E(A)/tE(A)$ is a left Kasch-ring.*

b) *If $0 \to U \to A^n \xrightarrow{\alpha} A^m \xrightarrow{\beta} G \to 0$ is an exact sequence in which $S_A(U)$ is torsion, then U is finite and β quasi-splits.*

P r o o f. a) ⇒ b): Suppose that $E(A)/tE(A)$ is a left Kasch-ring, and consider an exact sequence as in b). It induces the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_A(U) & \longrightarrow & H_A(A^n) & \xrightarrow{H_A(\alpha)} & H_A(A^m) \\ & & \downarrow \pi_1 & & \downarrow \pi_2 & & \\ & & \overline{H_A(A^n)} & \xrightarrow{\overline{H_A(\alpha)}} & \overline{H_A(A^m)}. & & \end{array}$$

in which the vertical maps are the obvious projections. For $x \in \ker \overline{H_A(\alpha)}$, there is $y \in H_A(A^n)$ with $x = \pi_1(y)$. Then, $0 = \overline{H_A(\alpha)}\pi_1(y) = \pi_2 H_A(\alpha)(y)$ yields

$H_A(\alpha)(\ell_1 y) = 0$ for some non-zero $\ell_1 \in \mathbb{Z}$. Since $H_A(U) = H_A\big(S_A(U)\big)$ is a torsion group by [4, Lemma 2.1], we have $\ell_2\ell_1 y = 0$ for some non-zero $\ell_2 \in \mathbb{Z}$. Therefore, $\overline{H_A(\alpha)}$ is a monomorphism which splits since $E(A)/tE(A)$ is a left Kasch-ring. If $\delta\colon \overline{H_A(A^m)} \to \overline{H_A(A^n)}$ is a splitting-map for $\overline{H_A(\alpha)}$, then we can find a map $\gamma\colon H_A(A^m) \to H_A(A^n)$ with $\pi_1\gamma = \delta\pi_2$ since $H_A(A^m)$ is a projective $E(A)$-module. Furthermore, $\pi_1\gamma H_A(\alpha) = \delta\overline{H_A(\alpha)}\pi_1 = \pi_1$ yields that $\gamma H_A(\alpha) - 1_{H_A(A^n)}$ maps $H_A(A^n)$ into $tH_A(A^n)$. By [4, Lemma 2.1], tA^n is A-solvable and $tH_A(A^n) = H_A(tA^n)$. An application of the Adjoint-Functor-Theorem yields that $\operatorname{Hom}_{E(A)}\big(H_A(A^n), H_A(tA^n)\big)$ is isomorphic to the torsion group $\operatorname{Hom}(A^n, tA^n)$. Hence we can find a non-zero integer k such that $k\gamma H_A(\alpha) = k\,1_{H_A(A^n)}$. We obtain the diagram

$$\begin{array}{ccc} T_AH_A(A^n) & \underset{T_AH_A(\alpha)}{\overset{kT_A(\gamma)}{\rightleftarrows}} & T_AH_A(A^m) \\ \wr\downarrow \theta_{A^n} & & \wr\downarrow \theta_{A^m} \\ A^n & \xrightarrow{\alpha} & A^m. \end{array}$$

Therefore, we can find a mapping $\varepsilon\colon A^m \to A^n$ with $k\varepsilon\alpha = k\,1_{A^n}$.

Let $p_1, \ldots, p_s$ be the primes dividing k, and write $A = B \oplus C$ where $B = A_{p_1} \oplus \ldots \oplus A_{p_s}$ as in Theorem 2.1. Since $k\varepsilon\alpha = k1_{A^n}$, we have $kU = 0$. Therefore, U is finite since it is isomorphic to a subgroup of the finite group B^n. Furthermore, the sequence $0 \to C^n \xrightarrow{\alpha|_{C^n}} C^m \xrightarrow{\beta|_{C^m}} S_C(G) \to 0$ is exact by Theorem 2.1. Since $\varepsilon(C^m) \subseteq C^n$ and multiplication by k is an automorphism of C, we have that $\varepsilon|_{C^m}$ is a splitting map for $\alpha|_{C^n}$. But then $\beta|_{C^m}$ splits too. Since $G/\beta(C^m)$ is isomorphic to the finite group $\beta(B^m)$ by Theorem 2.1, $\beta\colon A^m \to G$ quasi-splits.

b) $\Rightarrow$ a): To show that $E(A)/tE(A)$ is a left Kasch-ring, it suffices to show by [8, Corollary 5.6] that, for every exact sequence $0 \to \overline{E(A)}^n \xrightarrow{\overline{\alpha}} \overline{E(A)}^m \to M \to 0$ of right $E(A)/tE(A)$-modules, M is projective. Let $\pi_1\colon E(A)^n \to \overline{E(A)}^n$ and $\pi_2\colon E(A)^m \to \overline{E(A)}^m$ be the canonical projections, and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & E(A)^n & \xrightarrow{\alpha} & E(A)^m & \xrightarrow{\beta} & X & \longrightarrow & 0 \\ & & & & \pi_1\downarrow & & \pi_2\downarrow & & & & \\ & & 0 & \longrightarrow & \overline{E(A)}^n & \xrightarrow{\overline{\alpha}} & \overline{E(A)}^m. & & & & \end{array}$$

A standard diagram chase shows that U is torsion. If V denotes the kernel of $T_A(\alpha)$, then we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_A(V) & \longrightarrow & H_A T_A\big(E(A)^n\big) & \xrightarrow{H_A T_A(\alpha)} & H_A T_A\big(E(A)^m\big) \\
 & & & & \wr\big\uparrow \varphi_{E(A)^n} & & \wr\big\uparrow \varphi_{E(A)^m} \\
0 & \longrightarrow & U & \longrightarrow & E(A)^n & \xrightarrow{\alpha} & E(A)^n
\end{array}$$

from which it follows that $H_A(V)$ is torsion. But this is only possible if $S_A(V)$ is torsion. By b), V is finite and $T_A(\beta)$ quasi-splits. We choose a map $\sigma\colon T_A(X) \to T_A\big(E(A)^m\big)$ such that $T_A(\beta)\sigma = k\,1_{T_A(X)}$ for some non-zero integer k. Without loss of generality, we may assume that $kV = 0$. Let $p_1, \ldots, p_s$ be the primes dividing k, and write $A = B \oplus C$ where $B = A_{p_1} \oplus \ldots \oplus A_{p_s}$. By Theorem 2.1, the sequence $0 \longrightarrow T_A\big(E(C)^n\big) \xrightarrow{T_A(\alpha|_{E(C)^n})} T_A\big(E(C)^m\big)$ is exact, and splits as before. Since $\overline{E(A)} = \overline{E(C)}$, no generality is lost if we assume that α is a monomorphism and $0 \to E(A)^n \overset{\alpha}{\to} E(A)^m \overset{\beta}{\to} X \to 0$ splits. Moreover, the sequence $0 \to tE(A)^n \to tE(A)^m \to tX \to 0$ is exact. We extend the first diagram using the kernels of the vertical maps. The $3{\times}3$-Lemma can be applied to this extended diagram and yields that the induced sequence $0 \to tX \to X \to M \to 0$ is exact. Since X has been shown to be a projective $E(A)$-module, $M \cong X/tX$ is a projective $E(A)/tE(A)$-module. $\square$

In particular, every essentially indecomposable group $A \in \mathcal{G}$ has the property that $E(A)/tE(A)$ is a left Kasch-ring since it is a local Artinian ring. Furthermore, every $A \in \mathcal{G}$ for which $E(A)/tE(A)$ is a quasi-Frobenius ring has this property.

Corollary 2.3. *Let $A \in \mathcal{G}$ have the property that $E(A)/tE(A)$ is a left Kasch-ring. Then every exact sequence $0 \to P \to A^m$ such that P is an A-projective group and in $\mathcal{G}$ splits if and only if A_p is homogeneous for each prime p.*

P r o o f. Clearly, a group $A \in \mathcal{G}$ with the stated splitting property has to have homogeneous p-components. To show the converse, observe that P is a direct summand of A^n for some $n < \omega$ by [4, Theorem 2.2]. The sequence $0 \to P \overset{\alpha}{\to} A^n$ quasi-splits, say $\sigma\alpha = k\,1_P$ for some $\sigma\colon A^n \to P$ and non-zero integer k. Write $A = B \oplus C$ where B and C are fully invariant subgroups of A such that B is finite and $k^s B = 0$ for some $s < \omega$. Observe that B and C are as in Theorem 2.1. As before, the sequence $0 \longrightarrow S_C(P) \xrightarrow{\alpha|S_C(P)} C^m$ splits. Moreover since the endomorphism ring of B is self-injective as a finite product of matrix-rings over self-injective rings, the sequence $0 \longrightarrow S_B(P) \xrightarrow{\alpha|S_B(P)} B^m$ splits too. By Theorem 2.1, $\alpha(P)$ is a direct summand of A^n. $\square$

In [2], it was shown that a torsion-free abelian group of finite rank whose quasi-endomorphism ring is a left Kasch-ring has the property that every exact sequence $0 \to P \xrightarrow{\alpha} G$ with $\alpha(P) \cap R_A(G) = 0$ quasi-splits. It is natural to ask if a similar extension of Corollary 2.3 exists in the setting of this paper. However, if $A \in \mathcal{G}$ is an honest mixed group, then A is isomorphic to a pure subgroup of $\Pi_p A_p$. Since $[\Pi_p A_p]/A$ is torsion-free divisible, this embedding does not quasi-split although $R_A(\Pi_p A_p) = 0$. Therefore, the splitting condition in Corollary 2.3 cannot be extended as in [2].

We conclude this section with an example that finitely A-generated subgroups of A^n need not quasi-summands if $E(A)/tE(A)$ is a Kasch-ring:

Example 2.4. There exists an essentially indecomposable group A which contains a finitely A-generated subgroup U which is not a quasi-summand of A.

Proof. Let $R = \left\{ \left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right) \mid x, y \in \mathbb{Q} \right\}$. It easy to see that R is a commutative local ring whose maximal ideal is $J = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ y & 0 \end{smallmatrix}\right) \mid y \in \mathbb{Q} \right\}$. Since there exists an exact sequence $0 \to J \to R \to J \to 0$ which does not split, J has infinite projective dimension. Moreover, a slight modification of the arguments used in the proof of [6, Theorem 4.1] yields a group $A \in \mathcal{G}$ of torsion-free rank 2 with $E(A)/tE(A) = R$ whose p-primary components are homogeneous. Since $\mathrm{Mat}_2(\mathbb{Q})$ is a projective R-module, A is flat as an $E(A)$-module. Furthermore, it easy to see that A/tA is faithful as an $E(A)/tE(A)$-module. Hence, A is a faithfully flat left $E(A)$-module by [6]. We choose a cyclic ideal I of $E(A)$ with $[I + tE(A)]/tE(A) = J$. Consider the induced exact sequence $0 \to IA \to A$. If it were to quasi-split, then it would induce the exact sequence $0 \to H_A(IA) \to H_A(A)$ which also quasi-splits. Since A is faithfully flat as an $E(A)$-module, we have that I is a quasi-summand of $E(A)$. But then $\overline{I}$ were a direct summand of $\overline{E(A)}$ from which it would follow that $J \cong \overline{I}$ is projective, a contradiction. □

3. Tensor-Products of Finitely Presented Modules

In [4], we investigated A-solvable groups in $\mathcal{G}$. We showed that these are precisely the finitely A-generated A-solvable groups, i. e. images of A-projective groups of finite A-rank. In this discussion we considered almost A-balanced sequences $0 \to B \to C \xrightarrow{\beta} G \to 0$ of abelian groups, i. e. sequences for which $H_A(G)/\operatorname{Im} H_A(\beta)$ is a torsion group. We showed that an almost A-balanced exact sequence is A-balanced, i. e. $H_A(\beta)$ is onto, if $A \in \mathcal{G}$ has homogeneous p-components. Particular attention was given to $\mathcal{G}_A$*-presented groups*, i. e. those $G \in \mathcal{G}$ admitting an almost A-balanced sequence $0 \to U \to \oplus_n A \to G \to 0$ in which U is A-generated and in $\mathcal{G}$. Our first result investigates what happens if we remove the assumption that the sequence is almost A-balanced.

Since a reduced A-generated group G is finitely A-generated if and only if $G \in \mathcal{G}$ [4, Theorem 2.2], a group G admits a sequence of the desired form exactly if it is the cokernel of a map $A^n \to A^m$ for some $m, n < \omega$.

Theorem 3.1. *Let $A \in \mathcal{G}$. The following conditions are equivalent for a group G:*

a) *G admits an exact sequence $0 \to U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \to 0$ in which U is a finitely A-generated group.*

b) *$G \cong T_A(M)$ for some finitely presented right $E(A)$-module M.*

Proof. a) $\Rightarrow$ b): By [3, Lemma 2.1], the submodule $M = \operatorname{Im} H_A(\beta)$ of $H_A(G)$ satisfies $G \cong T_A(M)$. Moreover, since $U \in \mathcal{G}$, there is an exact sequence $A^m \xrightarrow{\sigma} U \to 0$ for some $m < \omega$. While this sequence need not be almost A-balanced in general, we can find a finitely generated submodule V of $H_A(U)$ such that $H_A(U)/V$ is torsion as an abelian group. This is possible since $H_A(U)$ has finite torsion-free rank as an abelian group. Let $\varphi_1, \ldots, \varphi_\ell$ be generators of V, and define a map $\lambda\colon A^m \oplus A^\ell \to U$ by $\lambda\big(x, (a_1, \ldots, a_\ell)\big) = \sigma(x) + \Sigma_{i=1}^{\ell} \varphi_i(a_i)$ for all $x \in A^m$ and $a_1, \ldots, a_\ell \in A$. Clearly, λ is onto. Since $\varphi_1, \ldots, \varphi_\ell \in \operatorname{Im} H_A(\lambda)$, the sequence $A^m \oplus A^\ell \xrightarrow{\lambda} U \to 0$ is almost A-balanced. Therefore, we may assume that σ induces an almost A-balanced sequence.

The finitely generated submodule $N = \operatorname{Im} H_A(\sigma)$ of $H_A(A^n)$ satisfies the condition $T_A\big(H_A(U)/N\big) = 0$. To see this, it is enough to show $T_A\big((H_A(U)/N)_p\big) = 0$ for all primes p of $\mathbb{Z}$ since $H_A(U)/N$ is torsion. We write $A = A_p \oplus A^p$ where A^p is a fully invariant subgroup of A for which multiplication by p is an automorphism. Since $E(A) = E(A_p) \times E(A^p)$, we can write $N = N_p \oplus N^p$ where N_p is an $E(A_p)$-module and N^p is an $E(A^p)$-module. Furthermore, since $U \in \mathcal{G}$, we can find find a similar decomposition for U, say $U = U_p \oplus U^p$. Then, $H_A(U_p)$ is an $E(A_p)$-module, while $H_A(U^p)$ is an $E(A^p)$-module. Therefore $N_p \subseteq H_A(A_p)$ and $N^p \subseteq H_A(U^p)$, from which we obtain that $H_A(U)/N = H_A(U_p)/N_p \oplus H_A(U^p)/N^p$. Since $(H_A(U)/N)_p$ is bounded as an abelian group, and $H_A(U^p)/N^p$ is p-divisible, we have $H_A(U_p)/N_p \cong (H_A(U)/N)_p$. Moreover, if σ_p denotes the restriction of σ to A_p^m, then $N_p = \operatorname{Im} H_A(\sigma_p)$. Since $\ker \sigma_p$ is A-generated as a subgroup of A_p^m, the evaluation map $\theta\colon T_A(N_p) \to U_p$ is an isomorphism by [3, Lemma 2.1]. It fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_A(N_p) & \xrightarrow{T_A(\iota)} & T_A H_A(U_p) & \longrightarrow & T_A\big(H_A(U_p)/N_p\big) & \longrightarrow & 0 \\ & & \theta \big\downarrow \wr & & \theta_{U_p} \big\downarrow & & & & \\ & & U_p & \xrightarrow{1_{U_p}} & U_p & & & & \end{array}$$

whose top-row is induced by the inclusion map $\iota\colon N_p \to H_A(U_p)$ and is exact by [4, Lemma 2.1]. By the same result, U_p is an A-solvable group, and $T_A(\iota)$ is an isomorphism. Therefore, $T_A\big((H_A(U)/N)_p\big) = 0$. Now observe that the exact sequence $0 \to H_A(U)/N \to H_A(A^n)/N \to M \to 0$ induces $0 = T_A\big(H_A(U)/N\big) \to T_A\big(H_A(A^n)/N\big) \to T_A(M) \to 0$. Hence G is isomorphic to $T_A\big(H_A(A^n)/N\big)$, and $H_A(A^n)/N$ is finitely presented.

b) $\Rightarrow$ a): We consider a projective resolution $0 \to U \to E(A)^n \xrightarrow{\beta} M \to 0$ in which U is finitely generated. It induces the exact sequence $T_A\big(E(A)^n\big) \xrightarrow{T_A(\beta)} T_A(M) \to 0$ in which $\ker T_A(\beta)$ is A-generated as an image of $T_A(U)$. Since U is a finitely generated $E(A)$-module, $T_A(U)$ is finitely A-generated, and the same holds for $\ker T_A(\beta)$. By [3, Theorem 2.2], $\ker T_A(\beta) \in \mathcal{G}$. □

Clearly, the arguments used in the proof of b) implies a) apply to show that any group $G \cong T_A(M)$ with M finitely presented arises as a cokernel of a map $A^n \to A^m$ regardless of what the actual structure of A is.

Corollary 3.2. *Let A in $\mathcal{G}$ have homogeneous p-components. The following conditions are equivalent for a group G:*

a) *G is $\mathcal{G}_A$-presented.*

b) *G is an A-solvable group for which $H_A(G)$ is finitely presented.*

Proof. a) $\Rightarrow$ b): By [4], G is A-solvable and admits an A-balanced exact sequence $0 \to U \to A^n \xrightarrow{\beta} G \to 0$ in which $U \in \mathcal{G}$ is A-generated. Using the notation of the proof of Theorem 3.1, we have that $M = H_A(G)$. Moreover, N is a submodule of $H_A(U)$ for which $H_A(U)/N$ is torsion and satisfies $T_A\big(H_A(U)/N\big) = 0$. Since A has homogeneous p-components, $H_A(U) = N$ as in [4]. This shows that $H_A(G)$ is finitely presented.

b) $\Rightarrow$ a): We can find an exact sequence $0 \to W \to E(A)^m \xrightarrow{\beta} H_A(G) \to 0$ for some $m < \omega$ in which W is finitely generated. Tensoring with T_A induces an exact sequence as in Theorem 3.1a. It remains to show that this sequence is A-balanced. However, this is an immediate consequence of the commutative diagram

$$\begin{array}{ccccc} H_A T_A(E(A)^m) & \xrightarrow{H_A T_A(\beta)} & H_A T_A H_A(G) & & \\ \varphi_{E(A)^m} \uparrow \wr & & \varphi_{H_A(G)} \uparrow & & \\ E(A)^m & \xrightarrow{\beta} & H_A(G) & \longrightarrow & 0 \end{array}$$

in which the last vertical map is an isomorphism since G is A-solvable. □

We now turn to the question how the group-structure of a group G of the form $G \cong T_A(M)$ such that M is finitely presented is related to the module structure of

M. We begin our discussion with a result which is stated for the sake of an easier reference.

Lemma 3.3. *Let $A \in \mathcal{G}$.*

a) *If $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} W \to 0$ is an exact sequence of right $E(A)$-modules, then the induced sequence $0 \to \overline{N} \xrightarrow{\overline{\alpha}} \overline{M} \xrightarrow{\overline{\beta}} \overline{W} \to 0$ of $E(A)/tE(A)$-modules is exact.*

b) *If M is a right $E(A)$-module, then*

$$\operatorname{proj.dim.}_{E(A)/tE(A)} \overline{M} \leqslant \operatorname{proj.dim.}_{E(A)} M$$

for all right $E(A)$-modules M.

Proof. a) To see that the induced sequence is exact, we suppose that $\overline{\beta}(\overline{x}) = 0$ for some $x \in M$. Then, $\beta(x) \in tW$, say $\beta(\ell x) = 0$ for some non-zero integer ℓ. Then, $\ell x = \alpha(u)$ for some $u \in N$. Since N/tN is divisible as an abelian group, we can find $v \in N$ and $t \in tN$ such that $u = \ell v + t$. Then, $\ell(u - \alpha(v)) \in tM$ from which $x = \alpha(v) + t_1$ for some $t_1 \in tM$ follows. Hence $\overline{x} = \overline{\alpha}(\overline{v})$. b) There is nothing to show if M has infinite projective dimension. If M is projective, then so is $\overline{M}$. Thus assume that $0 < n = \operatorname{proj.dim.}_{E(A)} M < \infty$. Consider an exact sequence $0 \to U \to \bigoplus_I E(A) \to M \to 0$ for some index-set I which induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{\alpha} & \bigoplus_I E(A) & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \\
0 & \longrightarrow & \overline{U} & \xrightarrow{\overline{\alpha}} & \overline{\bigoplus_I E(A)} & \xrightarrow{\overline{\beta}} & \overline{M} & \longrightarrow & 0.
\end{array}$$

By induction, $\operatorname{proj.dim.}_{E(A)/tE(A)} \overline{U} \leqslant \operatorname{proj.dim.}_{E(A)} U = n-1$, which implies that $\operatorname{proj.dim.}_{E(A)/tE(A)} \overline{M} \leqslant n = \operatorname{proj.dim.}_{E(A)} M$. □

Theorem 3.4. *Let A be in $\mathcal{G}$. The following conditions are equivalent for a group G:*

a) *$G = T \oplus P$ with T a finite A-generated group and P A-projective of finite A-rank.*

b) *$G \cong T_A(M)$ for some finitely presented $E(A)$-module such that M/tM is a projective $E(A)/tE(A)$-module.*

P r o o f. a) $\Rightarrow$ b): Observe that G is A-solvable by [4, Lemma 2.1] and $H_A(P)$ is a finitely generated projective right $E(A)$-module such that $\overline{H_A(G)} = \overline{H_A(P)}$ is projective.

b) $\Rightarrow$ a): We consider an exact sequence $0 \to U \xrightarrow{\alpha} E(A)^n \xrightarrow{\beta} M \to 0$ in which U is finitely generated. It induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{\ \alpha\ } & E(A)^n & \xrightarrow{\ \beta\ } & M & \longrightarrow & 0 \\
 & & \pi_1 \big\downarrow & & \pi_2 \big\downarrow & & \pi_3 \big\downarrow & & \\
0 & \longrightarrow & \overline{U} & \xrightarrow{\ \overline{\alpha}\ } & \overline{E(A)^n} & \xrightarrow{\ \overline{\beta}\ } & \overline{M} & \longrightarrow & 0
\end{array}$$

in which the vertical maps are the obvious projections, and the bottom-row is exact as in the proof of Lemma 3.3a. Since $\overline{M}$ is projective, we can find a splitting map $\delta\colon \overline{M} \to \overline{E(A)^n}$ for $\overline{\beta}$. Since M is finitely presented, it is projective with respect to the sequence $0 \to tE(A)^n \to E(A)^n \to \overline{E(A)^n} \to 0$ by [5], and we can find a map $\gamma\colon M \to E(A)^n$ with $\pi_2\gamma = \delta\pi_3$. We have that $\pi_3\beta\gamma = \overline{\beta}\pi_2\gamma = \pi_3$. Thus, $\beta\gamma - \mathrm{id}_M \in \mathrm{Hom}_{E(A)}(M, tM)$. Consider the exact sequence $0 \to \mathrm{Hom}_{E(A)}(M, tM) \xrightarrow{\mathrm{Hom}(\beta, tM)} \mathrm{Hom}_{E(A)}\big(E(A)^n, tM\big)$. Since the last module in this sequence is torsion, $k(\beta\delta - \mathrm{id}_M) = 0$ for some non-zero integer k.

An application of the functor T_A gives the exact sequence $T_A\big(E(A)^m\big) \xrightarrow{T_A(\beta)} T_A(M) \to 0$ in which $T_A(\beta)\, T_A(k\delta) = k\, 1_{T_A(M)}$. If $p_1, \ldots, p_m$ are the primes dividing k, then $A = B \oplus C$ where $B = A_{p_1} \oplus \ldots \oplus A_{p_m}$ and multiplication by k is an automorphism of C. As in the proof of Theorem 2.1, the map $T_A(\beta)\colon\ T_C\big(E(A)^m\big) \to T_C(M)$ splits. Thus, $T_C(M)$ is A-projective. On the other hand, $T_B(M)$ is a finite group with $G \cong T_A(M) = T_B(M) \oplus T_C(M)$. $\square$

Corollary 3.5. *Let $A \in \mathcal{G}$ have homogeneous p-components. If $E(A)/tE(A)$ is a left Kasch-ring, and G is a $\mathcal{G}_A$-presented group such that $H_A(G)$ has finite projective dimension, then G is A-projective.*

P r o o f. By Corollary 3.2, $H_A(G)$ is a finitely presented module of finite projective dimension with $G \cong T_A H_A(G)$. Because of Lemma 3.3b, the $E(A)/tE(A)$-module $\overline{H_A(G)}$ has finite projective dimension. Since $E(A)/tE(A)$ is a left Kasch-ring, the latter module is projective. By Theorem 3.4, $G = P \oplus T$ for some A-projective group P and some finite T. $\square$

Example 3.6. There exists a group A in $\mathcal{G}$ such that $E(A)/tE(A)$ is right hereditary, but not a left Kasch-ring.

P r o o f. Let R be the ring of upper triangular matrices over $\mathbb{Q}$. Then R is a right semi-hereditary ring. In particular, $N(R)$, the nilradical of R, is a projective ideal of R such that $R/N(R)$ is not projective. By [6], we can find a group $A \in \mathcal{G}$ with $pA_p = 0$ for all primes p such that $R = E(A)/tE(A)$. Observe that $E(A)$ is a right semi-hereditary ring by [6, Theorem 5.2]. We choose a cyclic right ideal I of $E(A)$ such that $[I + tE(A)]/tE(A) = N(R)$. Since I is projective, $T_A(I)$ is A-projective, and hence reduced. Therefore, the sequence $0 \to T_A(I) \to T_A\big(E(A)\big)$ is exact since $\operatorname{Tor}^1_{E(A)}\big(E(A)/I, A\big)$ is divisible by [4]. If the last sequence were to quasi-split, then I were isomorphic to a quasi-summand of $E(A)$ from which it would follow that $R/N(R)$ is projective. Therefore, $E(A)/tE(A)$ is not a left Kasch-ring. □

We conclude this paper with a short discussion of the structure of the kernels of maps $A^n \to A^m$. In particular, we are interested in the question when such a kernel is A-generated and in $\mathcal{G}$. Clearly, the fact that the kernel of each such map is A-generated immediately restricts our discussion to groups that are flat as modules over their endomorphism ring.

Proposition 3.7. *Consider the following conditions for a group $A \in \mathcal{G}$:*

a) *A is an $E(A)$-flat group whose endomorphism ring is right coherent.*

b) *If $\varphi\colon A^n \to A^m$ for some $m < \omega$, then* $\ker \varphi$ *is A-generated and in $\mathcal{G}$.*

Then, a) *implies* b), *and the converse holds if A has homogeneous p-components.*

Proof. a) ⇒ b): Consider an exact sequence $0 \to U \to A^n \xrightarrow{\beta} A^m$ for some $n, m < \omega$. It induces $0 \to H_A(U) \to H_A(A^n) \xrightarrow{H_A(\beta)} H_A(A^m)$. Since $\operatorname{Im} H_A(\beta)$ is a finitely generated submodule of $H_A(A^m)$ and $E(A)$ is right coherent, $H_A(U)$ is finitely generated. Thus, we can find an epimorphism $E(A)^\ell \to H_A(U) \to 0$ for some $\ell < \omega$, which induces an epimorphism $T_A\big(E(A)^\ell\big) \to T_A(U) \to 0$. Since A is flat as an $E(A)$-module, U is A-solvable. Therefore, U is finitely A-generated, and so $U \in \mathcal{G}$ by [4, Theorem 2.2].

b) ⇒ a): Since A is flat as an $E(A)$-module by Ulmer's Theorem [13], every exact sequence $0 \to U \to E(A)^n \to E(A)^m$ of right $E(A)$-modules induces an exact sequence $0 \to T_A(U) \to T_A\big(E(A)^n\big) \to T_A\big(E(A)^m\big)$ from which it follows that $T_A(U)$ is finitely A-generated. Furthermore, an application of the functor H_A yields the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_A T_A(U) & \longrightarrow & H_A T_A\big(E(A)^n\big) & \longrightarrow & H_A T_A\big(E(A)^m\big) \\
 & & \big\downarrow \varphi_U & & \wr\big\downarrow \varphi_{E(A)^n} & & \wr\big\downarrow \varphi_{E(A)^m} \\
0 & \longrightarrow & U & \longrightarrow & E(A)^n & \longrightarrow & E(A)^m
\end{array}$$

which gives that φ_U is an isomorphism. Consequently, $T_A(U)$ is A-solvable, and there exists an A-balanced exact sequence $A^\ell \to T_A(U) \to 0$ by [4, Corollary 3.5] since

A has homogeneous p-components. But this gives that $U \cong H_A T_A(U)$ is finitely generated. □

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