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CHARACTERIZATIONS OF ABSOLUTE $F_{\sigma\delta}$-SETS

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Abstract. We give several internal characterizations for the metrizable absolute $F_{\sigma\delta}$-spaces. The characterizing conditions involve the existence of compatible bicomplete quasi-metrics, of complete sequences of σ-discrete closed covers and of compact σ-discrete closed networks.

Keywords: metric space, $F_{\sigma\delta}$-set, bicomplete quasi-metric, complete sequence of covers, compact family of sets, cotopology

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1. Introduction

It is a basic problem of classical descriptive set theory to find simple internal conditions which are necessary and sufficient for a metrizable space to belong to one of the absolute Borel classes F_α or G_α, for $\alpha \in \omega_1$. This problem has been solved only for a few small values of α. A. H. Stone remarks in [22] that there does not even exist satisfactory internal characterizations for absolute F_2-sets, that is, for absolute $F_{\sigma\delta}$-sets (the characterization of *separable* absolute $F_{\sigma\delta}$-sets given by W. Sierpiński in [21] is already quite complicated). It is the purpose of this note to provide simple internal characterizations of metrizable absolute $F_{\sigma\delta}$-spaces. These characterizations involve the existence of "complete" sequences of covers, of "compact" networks and of "bicomplete" quasi-metrics.

The conditions used in this paper to characterize metrizable absolute $F_{\sigma\delta}$-spaces are variations of conditions that have appeared before in characterizations of metriz-

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able absolute G_δ-spaces. The best-known of such characterizations is E. Čech's theorem [5] according to which the metrizable absolute G_δ-spaces coincide with the completely metrizable spaces. We provide a corresponding characterization for metrizable absolute $F_{\sigma\delta}$-spaces by replacing "complete metrizability" with "bicomplete quasi-metrizability." Other well-known characterizations of metrizable absolute G_δ's involve the existence of a "complete" sequence of open covers (Z. Frolík [12] and A. V. Arhangel'skij [3]) and of a "compact" closed quasi-base (J. M. Aarts, J. de Groot and R. H. McDowell [1]); we show that these characterizations turn into characterizations of metrizable absolute $F_{\sigma\delta}$'s when the open covers in the complete sequence are replaced by σ-discrete closed covers and the compact closed quasi-base is replaced by a compact σ-discrete closed network.

In [20], S. Romaguera and S. Salbany pose the problem of characterizing those quasi-metrizable spaces that admit a bicomplete quasi-metric. So far only a few results have been obtained in this area. For instance it is known that a quasi-metrizable space admits only bicomplete quasi-metrics if and only if it is hereditarily compact and sober [19] and that every quasi-metrizable and every sober space admits a bicomplete quasi-uniformity [10]. In this note we show that, for metrizable spaces, the question proposed by Romaguera and Salbany has the simple and elegant solution indicated above: a metrizable space has a compatible bicomplete quasi-metric if, and only if, the space is an absolute $F_{\sigma\delta}$-set.

Let us first recall the necessary terminology and introduce the appropriate notation.

A metrizable space X is an *absolute $F_{\sigma\delta}$-set* (or an *absolute $F_{\sigma\delta}$-space*) provided that X is an $F_{\sigma\delta}$-subset in every metrizable space in which X is embedded.

Let X be a nonempty set. A function d from $X \times X$ into the nonnegative real numbers is called a *quasi-metric* of X if

(i) $d(x,y) = 0 \Longleftrightarrow x = y$ for all $x, y \in X$, and

(ii) $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x, y, z \in X$.

For a quasi-metric d of X, let $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$. Then d^{-1} is also a quasi-metric on X.

Let d be a quasi-metric of X. For all $x \in X$ and $n \in \omega$, set $U_n^d(x) = \{y \in X\colon\ d(x,y) < 2^{-n}\}$. The family $\{U_n^d(x)\colon\ n \in \omega, x \in X\}$ is a base for a topology τ_d on X. The quasi-metric d is called a *strong* quasi-metric if $\tau_d \subseteq \tau_{d^{-1}}$.

Let d be a quasi-metric of the topological space X. If the topology τ_d is coarser than the topology of X, then we say that d is an *admissible* quasi-metric of X, and if the topology τ_d coincides with the topology of X, then we say that d is a *compatible* quasi-metric of X. Note that a metric d of X is admissible if, and only if, d is a continuous function on $X \times X$.

For a quasi-metric d of X, the function $d^* = \max\{d, d^{-1}\}$ is a metric of X. The quasi-metric d is called *bicomplete* provided that the metric d^* is complete.

A binary relation V on a topological space X is called a *neighbornet* [15] provided that $V(x) = \{y \in X\colon (x,y) \in V\}$ is a neighborhood at x whenever $x \in X$. A neighbornet V on X is called *unsymmetric* [15, p. 88] if for all $a, b \in X$, $a \in V(b)$ and $b \in V(a)$ imply that $V(a) = V(b)$.

A sequence $(\mathcal{G}_n)_{n\in\omega}$ of covers of a topological space X is called *complete* provided that any filter $\mathcal{F}$ on X such that $\mathcal{F} \cap \mathcal{G}_n \neq \emptyset$ whenever $n \in \omega$ has a cluster point in X (compare [12] and [3]).

A family of sets will be called *compact* provided that every subfamily with the finite intersection property has nonempty intersection.

Recall that a *network* of a topological space X is a family $\mathcal{N}$ of subsets of X such that every open subset of X is the union of some subfamily of $\mathcal{N}$. Note that a network of X is a base for the topology of X if, and only if, the network consists of open sets.

If $\mathcal{H}$ is a collection of subsets of a set X and x is a point of X, then $(\mathcal{H})_x$ will denote the collection $\{H \in \mathcal{H}\colon x \in H\}$.

Our topological terminology follows that of [9]. Basic facts concerning Borel sets can also be found in [9]. For terminology and basic facts on quasi-uniformities we refer the reader to [11].

2. The results

The following theorem contains the promised characterizations of metrizable absolute $F_{\sigma\delta}$-sets.

Theorem 1. *The following conditions are equivalent for a metrizable space:*
(a) *The space is an absolute $F_{\sigma\delta}$-set.*
(b) *The space has a compatible bicomplete quasi-metric.*
(c) *The space has a compact σ-discrete network consisting of closed sets.*
(d) *The space possesses a complete sequence of σ-discrete closed covers.*

Proof. (a) $\Rightarrow$ (b): Since every metrizable absolute $F_{\sigma\delta}$-space is an $F_{\sigma\delta}$-subset of its completion, it suffices to show that every $F_{\sigma\delta}$-subspace of a completely metrizable space has a compatible bicomplete quasi-metric. Let (Y, d) be a complete metric space. We show first that every F_σ-subspace Z of Y has a compatible bicomplete quasi-metric; by [20, Theorem 3.7], it suffices to show that the subspace Z has a compatible quasi-metric ϱ such that the metric topology τ_{ϱ^*} is completely metrizable. Let $Z = \bigcup_{n=0}^{\infty} F_n$, where the sets F_n are closed in Y. We may assume that $F_0 = \emptyset$

and $F_n \subset F_{n+1}$ for every n. For each $x \in Z$, let n_x be the minimal (positive) integer n such that $x \in F_n$. For all $x, y \in Z$, set $\varrho(x,y) = d(x,y) + 1$ if $n_y < n_x$ and $\varrho(x,y) = d(x,y)$ otherwise. Clearly ϱ is a compatible quasi-metric on Z. Note that, for each $n \in \omega$, the set $F_{n+1} \setminus F_n$ is a G_δ-set in (Y,d) and a clopen set in (Z, ϱ^*), and the topologies τ_d and τ_{ϱ^*} coincide on this set. Hence the metric space (Z, ϱ^*) is the discrete sum of the countably many completely metrizable spaces $F_{n+1} \setminus F_n$ $(n \in \omega)$ and thus the topology τ_{ϱ^*} is completely metrizable [9, Theorem 3.9.6].

Suppose now that $Z = \bigcap_{n=0}^{\infty} Z_n$ and each subspace Z_n of (Y,d) admits a bicomplete quasi-metric ϱ_n. We can assume that each ϱ_n is bounded by 1. Then the topological product $\prod_{n=0}^{\infty} Z_n$ admits a bicomplete quasi-metric ϱ, where

$$\varrho((x_n)_{n\in\omega}, (z_n)_{n\in\omega}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \varrho_n(x_n, z_n)\,.$$

Since Z is homeomorphic to the τ_ϱ-closed subspace $\{(x, x, \ldots, x, \ldots)\colon\ x \in Z\}$ of this product, we see that Z admits a bicomplete quasi-metric.

It follows by the foregoing that every $F_{\sigma\delta}$-subset of Y admits a bicomplete quasi-metric.

(b) $\Rightarrow$ (c): Let X be a metrizable space. Suppose that ϱ is a compatible quasi-metric on X that is bicomplete. Let $n \in \omega$. By [15, Theorem 4.4], there exists an unsymmetric neighbornet S_n of X such that $S_n \subseteq (U^{\varrho}_{n+1})^2 \subseteq U^{\varrho}_n$. The relation $S_n \cap S_n^{-1}$ is an equivalence relation on X. By [15, Theorem 4.8], the partition $\{(S_n \cap S_n^{-1})(x)\colon\ x \in X\}$ has a refinement $\mathcal{G}_n = \bigcup_{k\in\omega} \mathcal{D}_{nk}$ such that each collection $\mathcal{D}_{nk}$ is closed and discrete in X. Furthermore, by the proof of [15, Theorem 4.8], we can assume that each class C belonging to the partition contains at most one element $D_k(C)$ of $\mathcal{D}_{nk}$ and that the sequence $(D_k(C))$ is increasing with k. Then the σ-discrete closed network $\bigcup_{n\in\omega} \mathcal{G}_n$ of X is compact:

Consider $\mathcal{A} \subseteq \bigcup_{n\in\omega} \mathcal{G}_n$ having the finite intersection property. Assume that $\bigcap \mathcal{A} = \emptyset$. Because of the properties of the collections $\mathcal{D}_{nk}$ this can only happen if $\mathcal{A} \cap \mathcal{G}_n \neq \emptyset$ for infinitely many $n \in \omega$. But then $\mathcal{A}$ is a subbase of a closed Cauchy filter on the complete metric space (X, ϱ^*) and thus $\bigcap \mathcal{A} \neq \emptyset$.

(c) $\Rightarrow$ (d): Suppose that the metrizable space X has a compact network $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ where each collection $\mathcal{F}_n$ is closed and discrete. For all $x \in X$ and $n \in \omega$, let $F_{x,n} \in \mathcal{F}$ be such that x belongs to $F_{x,n}$, and either $F_{x,n}$ belongs to $\mathcal{F}_n$ or $F_{x,n}$ is disjoint from the union of $\mathcal{F}_n$. Then the covers $\{F_{x,n}\colon\ x \in X\}$ are closed and σ-discrete and they form a complete sequence: Let $\mathcal{P}$ be a filter on X that contains sets $F_{x_n,n}$ for $n \in \omega$. Then by compactness, there exists x belonging to all those sets. Now for every $n \in \omega$, if x belongs to a member F of $\mathcal{F}_n$, then we must have $F_{x_n,n} = F$. It follows that if x belongs to $F \in \mathcal{F}$, then F is in $\{F_{x_n,n} \colon n \in \omega\}$. Hence $\mathcal{P}$ contains a network at x, and x is thus a cluster point of $\mathcal{P}$.

In order to establish the implication (d) $\Rightarrow$ (a) we shall need the following lemma:

Lemma 1. *Let Y be a metrizable space and X a subspace of Y. Furthermore let $\mathcal{L}$ be a locally finite closed family of X. Then there exists a G_δ-subspace A of Y such that $X \subseteq A$, $\mathcal{L}$ is locally finite in A and*

$$\overline{K}^A \cap \overline{L}^A = \overline{K \cap L}^A$$

for all $K \in \mathcal{L}$ and $L \in \mathcal{L}$.

Proof. For each $x \in X$, let O_x be an open neighborhood of x in Y such that O_x meets only finitely many sets of $\mathcal{L}$. Set $O = \bigcup\limits_{x \in X} O_x$. Then O is open, $X \subseteq O$ and $\mathcal{L}$ is locally finite in O. It follows that also the family $\{\overline{L}^O \colon L \in \mathcal{L}\}$, and hence the family $\{\overline{L}^O \cap \overline{K}^O \colon L, K \in \mathcal{L}\}$ is locally finite in O. Let d be a compatible metric for O. For every $n \in \omega$ and for all $L, K \in \mathcal{L}$, the set $S_n(L, K) = \{s \in \overline{L}^O \cap \overline{K}^O \colon d(s, L \cap K) \geqslant 2^{-n}\}$ is a closed subset of $\overline{L}^O \cap \overline{K}^O$. (We have set $d(x, \emptyset) = \infty$ for all $x \in O$.) As a consequence, for every $n \in \omega$, the family $\mathcal{S}_n = \{S_n(L, K) \colon L, K \in \mathcal{L}\}$ is locally finite and closed in O and hence the set $S_n = \bigcup \mathcal{S}_n$ is closed in O; note that $S_n \cap X = \emptyset$, since $\mathcal{L}$ is a closed family in X. Set $A = O \setminus \bigcup\limits_{n \in \omega} S_n$. Then A is a G_δ-set in Y containing X, the collection $\mathcal{L}$ is locally finite in A and $\overline{K}^A \cap \overline{L}^A = \overline{K \cap L}^A$ whenever $K, L \in \mathcal{L}$. $\square$

We are now ready to continue the proof of Theorem 1.

(d) $\Rightarrow$ (a): Let Y be a metrizable space containing X as a subspace. Moreover let $(\mathcal{G}_n)_{n \in \omega}$ be a complete sequence of covers of X such that $\mathcal{G}_n = \bigcup\limits_{k \in \omega} \mathcal{G}_{nk}$ whenever $n \in \omega$ and such that any collection $\mathcal{G}_{nk}$ is closed and discrete in X. Considering the locally finite closed collections $\mathcal{H}_s$ of X obtained by taking all intersections of finitely many elements in $\bigcup\limits_{k,n=0}^{s} \mathcal{G}_{nk}$ where $s \in \omega$, we see in the light of Lemma 1 that there is a G_δ-set A in Y containing X such that $\overline{K}^A \cap \overline{L}^A = \overline{K \cap L}^A$ whenever $K, L \in \bigcup\limits_{s=0}^{\infty} \mathcal{H}_s$ and such that each collection $\mathcal{H}_s$ is locally finite in A. Clearly we can suppose that $A \subseteq \operatorname{cl}_Y X$, since Y is perfect. Observe that A is an $F_{\sigma\delta}$-set in Y.

Set $P = \bigcap\limits_{n \in \omega} \big(\bigcup\limits_{k \in \omega} \operatorname{cl}_Y (\bigcup \mathcal{G}_{nk}) \big) \cap A$. Obviously P is an $F_{\sigma\delta}$-set in Y. Furthermore $X \subseteq P$. Consider $x \in P$. Let $\mathcal{F}$ be the filter generated by the family $\mathcal{H} = \{F \in$

$\mathcal{G}_{nk}\colon\ x \in \overline{F}^A; n, k \in \omega\}$ on X. Note that $\mathcal{F}$ is well defined, because, by the definition of A, the family $\mathcal{H}$ has the finite intersection property. Furthermore x is a cluster point of the filterbase $\mathcal{F}$ on A. Let $\mathcal{N}$ be the trace of the neighborhood filter at x in Y on X. Since $\mathcal{F}$ contains a member of each cover $\mathcal{G}_n$, the filter $\sup\{\mathcal{F}, \mathcal{N}\}$ on X has a cluster point, say y, in X. Because Y is a Hausdorff space, it follows that $y = x$. We conclude that $P = X$. Hence we have shown that X is an $F_{\sigma\delta}$-set in Y. □

Remarks. (i) Note that if ϱ is a compatible bicomplete quasi-metric on X and δ is a compatible metric on X, then $\max\{\varrho, \delta\}$ is a compatible bicomplete strong quasi-metric on X. It follows that we can replace condition (b) in Theorem 1 by the following condition:

(b′) The space has a compatible bicomplete strong quasi-metric.

(ii) The basic idea in the last step of the preceding proof is due to Z. Frolík (compare [13, proof of Theorem 6]).

(iii) It is easy to see that the above theorem remains true if we replace "σ-discrete" by "σ-locally finite" in conditions (c) and (d).

(iv) The above theorem makes it possible to point out relatively simple examples of metrizable spaces which do not have compatible bicomplete quasi-metrics. For example, every non-Borel subset of the reals (e. g. a "Bernstein set") fails to have such a quasi-metric. The next two examples are more concrete.

The infinite power $\mathbb{Q}^\omega$ cannot be represented as the union of countably many completely metrizable subspaces (see e. g. [8] or [17]). As a consequence, $\mathbb{Q}^\omega$ is not a $G_{\delta\sigma}$-subset in the completely metrizable space $\mathbb{R}^\omega$; therefore the complement of $\mathbb{Q}^\omega$ in $\mathbb{R}^\omega$ is not an $F_{\sigma\delta}$-set. It follows that the space $\mathbb{R}^\omega \setminus \mathbb{Q}^\omega$ has no compatible bicomplete quasi-metric.

By results given in [7], the function spaces $C_p(X)$, where X is a countable nondiscrete metrizable space and the set $C(X)$ of all continuous real-valued functions on X is equipped with the topology of pointwise convergence, serve as examples of absolute $F_{\sigma\delta}$-sets which are not absolute $G_{\delta\sigma}$-sets. It follows, for instance, that the set c_0 of all null-sequences is an $F_{\sigma\delta}$-set, but not a $G_{\delta\sigma}$-set, in the set ℓ^∞ of all bounded sequences (of real numbers), when ℓ^∞ has been equipped with the topology of pointwise convergence. As a consequence, no bicomplete quasi-metric is compatible with the topology of pointwise convergence in the set $\ell^\infty \setminus c_0$.

The characterization of completely metrizable spaces as the absolute G_δ-subsets of metrizable spaces suggests the existence of a direct proof of the implication (b) ⇒ (a) in Theorem 1 that is based on the idea of extending quasi-metrics. Since these ideas seem to be of independent interest, we would like to include such an argument here. To state some necessary auxiliary results, we need the concept of a *quasi-pseudometric*. A quasi-pseudometric of a set X is an "unsymmetric pseudometric" of X, that is, a function from $X \times X$ to the nonnegative reals which satisfies the triangle inequality and vanishes on the diagonal. The concept of "admissibility" for quasi-pseudometrics is defined similarly as for quasi-metrics.

Lemma 2. *Let Y be a metrizable space, let $X \subseteq Y$ and let $\varrho \leqslant 2$ be an admissible quasi-pseudometric of X. Then there exist an $F_{\sigma\delta}$-set A in Y containing X and a compatible quasi-metric d of A such that $\varrho \leqslant d$ on $X \times X$.*

Proof. Since the fine quasi-uniformity of the metrizable space X coincides with its point-finite covering quasi-uniformity [15, p. 101], we can find, for every $n \in \omega$, a point-finite open family $\mathcal{O}_n$ in X such that for all $x, z \in X$, if $z \in \bigcap(\mathcal{O}_n)_x$, then $\varrho(x, z) < 2^{-n}$. We may assume that $n > k$ implies that $\mathcal{O}_k \subseteq \mathcal{O}_n$. For every $O \in \bigcup_{n\in\omega} \mathcal{O}_n$, let O^* be open in Y such that $O^* \cap X = O$. For every $n \in \omega$ let $\mathcal{O}_n^* = \{O^*\colon O \in \mathcal{O}_n\} \cup \{Y\}$. Set $A = \{y \in Y\colon (\mathcal{O}_n^*)_y$ is finite for all $n \in \omega\}$. Note that A is an $F_{\sigma\delta}$-set in Y and $X \subseteq A$. Define a function d' on $A \times A$ by setting $d'(x, y) = \inf\{2^{-(n-1)}\colon y \in \bigcap(\mathcal{O}_n^*)_x\}$. Note that d' is an admissible quasi-pseudometric on A. Let $x, y \in X$. We show that $d'(x, y) \geqslant \varrho(x, y)$. The inequality is obvious if $\varrho(x, y) = 0$. If $\varrho(x, y) > 0$, then choose $n \in \omega$ so that $2^{-n} < \varrho(x, y) \leqslant 2^{-(n-1)}$; note that then $y \notin \bigcap(\mathcal{O}_n)_x$ and thus $d'(x, y) > 2^{-(n-1)} \geqslant \varrho(x, y)$. To obtain the required compatible quasi-metric d, let δ be a compatible metric on A and set $d = d' + \delta$. □

The following auxiliary result, with "(continuous) pseudometric" in room of "(admissible) quasi-pseudometric," was proved (implicitly) by R. H. Bing in [4]; J. Deák observed that Bing's argument works also for "unsymmetric distance-functions."

Lemma 3. [6] *Let Y be a topological space, let $X \subseteq Y$ and let ϱ be a quasi-pseudometric defined on X. If there exists an admissible quasi-pseudometric d of Y such that $\varrho \leqslant d$ on $X \times X$, then ϱ can be extended to an admissible quasi-pseudometric $\overline{\varrho}$ of Y such that $\overline{\varrho} \leqslant d$.*

It is a consequence of the "symmetric" version of the above lemma that any continuous pseudometric defined on a subset of a metrizable space can be extended to a continuous pseudometric defined on a G_δ-subset of the space. This result does not generalize from the case of continuous pseudometrics to the case of admissible quasi-pseudometrics; a simple example is furnished by the compatible quasi-metric d of $\mathbb{Q}$ obtained from an enumeration $\mathbb{Q} = \{q_n\colon n \in \mathbb{N}\}$ as follows: set $d(q_n, q_k) = |q_n - q_k|$ if $n \leqslant k$ and set $d(q_n, q_k) = 1$ if $n > k$; a category argument together with [15, Theorems 4.7 and 4.8] shows that the quasi-metric d does not extend to an admissible quasi-pseudometric over any G_δ-subset of $\mathbb{R}$ containing $\mathbb{Q}$. If we are willing to relax the requirement that the extension should be made to a G_δ-set, then we have the following analogue of the result on extending continuous pseudometrics.

Proposition 1. *Every bounded admissible quasi-pseudometric defined on a subset of a metrizable space Y can be extended to an admissible quasi-pseudometric of an $F_{\sigma\delta}$-subset of Y.*

P r o o f. The assertion is a direct consequence of Lemmas 2 and 3. □

We are now ready to sketch the direct argument (b) $\Rightarrow$ (a): Suppose that X is a subspace of the metric space (Y,d) and let ϱ be a compatible bicomplete quasi-metric on X; by the first remark following Theorem 1, we may assume that ϱ is a strong quasi-metric and that $\varrho \geqslant d$ on $X \times X$. Without loss of generality we can suppose that $d, \varrho \leqslant 1$. Extend ϱ according to Proposition 1 to an admissible quasi-pseudometric δ of an $F_{\sigma\delta}$-set A in Y that contains X. Since $\varrho \geqslant d$ on $X \times X$, we can choose δ so that $\delta \geqslant d$ on $A \times A$; note that this makes δ a strong compatible quasi-metric of A. Because δ^* and ϱ^* agree on $X \times X$ and because the metric space (X, ϱ^*) is complete, X is a G_δ-set in the metric space (A, δ^*). Since the compatible quasi-metric δ on A is strong, each τ_{δ^*}-open set in A is clearly an F_σ-set in (A,δ). Hence X is an $F_{\sigma\delta}$-set in (A,δ); since the topologies τ_δ and τ_d agree on A and since A is an $F_{\sigma\delta}$-set in (Y,d), it follows that X is an $F_{\sigma\delta}$-set in (Y,d).

We close this paper with a result which gives several characterizations of *separable* absolute $F_{\sigma\delta}$-sets. Two of the conditions in the following theorem are just modifications of the corresponding conditions which appeared in Theorem 1, but we also have a new condition for separable absolute $F_{\sigma\delta}$-sets in terms of cotopologies.

The following notions are discussed in [2]: Let (X,τ) be a topological space. A topology π on X is called a *cotopology* of τ—and the space (X,π) is called a *cospace* of (X,τ)—if

(i) π is weaker than τ, and

(ii) for each point x and each closed neighborhood V of x in (X,τ) there exists a neighborhood U of x in (X,τ) such that U is contained in V and U is closed in (X,π).

Cotopologies are related to the concepts discussed earlier in this paper in several ways. For example, if ϱ is a strong quasi-metric on a set Z, then it is a consequence of [18, Theorem 4] that (X,τ_ϱ) is a cospace of $(X,\tau_{\varrho^{-1}})$ and both spaces are of the same weight. The following lemma indicates a connection between cotopologies and σ-discrete networks.

Lemma 4. *Let $\mathcal{F}$ be a σ-discrete network for a T_1-space (Y,π) consisting of closed sets. Denote by τ the topology of Y generated by the family $\mathcal{F}$. Then the topology*

τ is metrizable, and the space (Y, π) is a cospace of the space (Y, τ). If the network $\mathcal{F}$ is compact, then the topology τ is completely metrizable.

P r o o f. It is easy to see that (Y, π) is a cospace of (Y, τ), and it follows from [16, Lemma 2.1] that τ is a metrizable topology. Assume that $\mathcal{F}$ is a compact family. Then $\mathcal{F}$ is a compact network of the space (Y, τ) consisting of clopen sets. Similarly as in the proof of the implication (c) $\Rightarrow$ (d) in Theorem 1, we construct from $\mathcal{F}$ a complete sequence $(\{F_{y,n} \colon y \in Y\})$ of covers for the space (Y, τ); we note that, in the case at hand, the covers in the complete sequence consist of clopen sets. Since the metrizable space (Y, τ) has a complete sequence of open covers, the space is completely metrizable (by the result of Frolík [12] and Arhangel'skij [3]). $\square$

We are now ready to characterize the separable metrizable absolute $F_{\sigma\delta}$-sets.

Theorem 2. *The following conditions are equivalent for a metrizable space:*
(a) *The space is a separable absolute $F_{\sigma\delta}$-set.*
(b) *The space is a cospace of a Polish space.*
(c) *The space has a compact countable network consisting of closed sets.*
(d) *The space possesses a complete sequence of countable closed covers.*

P r o o f. The equivalence of conditions (a) and (c) follows from Theorem 1. It is a consequence of Lemma 4 that (c) $\Rightarrow$ (b). To see that (b) $\Rightarrow$ (d), let (X, τ) be a topological space and let d be a complete separable metric on X such that the space (X, τ) is a cospace of (X, τ_d). For all $x \in X$ and $n \in \omega$, let $F_{x,n}$ be a τ-closed τ_d-neighborhood of x of d-diameter at most 2^{-n}. For every $n \in \omega$, let $\mathcal{F}_n$ be a countable subcover of the cover $\{F_{x,n} \colon x \in X\}$ of X. Since the metric d is complete and $\tau \subset \tau_d$, the sequence $\big(\mathcal{F}_n\big)$ of covers is complete in the space (X, τ).

(d) $\Rightarrow$ (a): Assume that (d) holds for a metrizable space X. Then X is an absolute $F_{\sigma\delta}$-set by Theorem 1. Moreover, [13, Theorem 7] shows that X is an $F_{\sigma\delta}$-subset of the compact space βX; as a consequence, X is a Lindelöf space. $\square$

Remark. We have not seen any parts of Theorem 2 stated explicitly in the literature; however, one part of the theorem can be easily derived from older results: it is quite easy to prove the equivalence of conditions (a) and (d) above with the help of [13, Theorem 7] and [14, Corollary to Theorem 2].

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