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ON A BOUND ON ALGEBRAIC CONNECTIVITY: THE CASE OF EQUALITY

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Abstract. In a recent paper the authors proposed a lower bound on $1-\lambda_i$, where λ_i, $\lambda_i \neq 1$, is an eigenvalue of a transition matrix T of an ergodic Markov chain. The bound, which involved the group inverse of $I-T$, was derived from a more general bound, due to Bauer, Deutsch, and Stoer, on the eigenvalues of a stochastic matrix other than its constant row sum. Here we adapt the bound to give a lower bound on the algebraic connectivity of an undirected graph, but principally consider the case of equality in the bound when the graph is a weighted tree. It is shown that the bound is sharp only for certain Type I trees. Our proof involves characterizing the case of equality in an upper estimate for certain inner products due to A. Paz.

1. Introduction

Let $\mathcal{G}$ be a weighted undirected graph on n vertices and let L be its Laplacian so that L is a positive semidefinite M-matrix. Let $0=\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ be the eigenvalues of L. Fiedler [6] has shown that $\nu := \lambda_2 > 0$ if and only if $\mathcal{G}$ is connected and, since increasing the number of edges in $\mathcal{G}$ cannot decrease the value ν, he has termed ν **the algebraic connectivity of $\mathcal{G}$.**

Now let $L^{\#}$ be the group generalized inverse[4] of L. For the case in which $\mathcal{G}$ is an unweighted graph, that is, $\mathcal{G}$ is a weighted graph whose edges have the uniform weight of 1, an **upper bound** on ν in terms of the dominant diagonal entry of $L^{\#}$

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[4] For background material on generalized inverses of matrices see Ben-Israel and Greville [2] and Campbell and Meyer [4].

was given in Kirkland, Neumann, and Shader [8]. Moreover the case of equality in the bound was characterized and, for the case in which $\mathcal{G}$ is a tree, a graph-theoretic interpretation for the entries of $L^{\#}$ was provided. This graph-theoretic interpretation has since been extended to the case in which $\mathcal{G}$ is a weighted tree in Kirkland, Neumann, and Shader [9]. In Kirkland, Neumann, and Shader [11] a certain **lower bound** was given, in terms of the group inverse of a singular M-matrix associated with the transition matrix of an ergodic Markov chain, for the nonzero eigenvalues of that M-matrix. This lower bound, which is an immediate consequence of a spectral bound due to Bauer, Deutsch, and Stoer [1] (see Theorem BDS below) can be easily adapted to yield a lower bound for ν. The main purpose of this paper is to consider the equality case of this lower bound for weighted graphs $\mathcal{G}$ that are trees.

The spectral bound due to Bauer, Deutsch, and Stoer is as follows:

Theorem BDS (Bauer, Eckart Deutsch, and Stoer [1], see also Seneta [14, p. 63, Theorem 2.10]). *Suppose that $B = (b_{i,j})$ is an $n \times n$ matrix with constant row sums b and assume that μ is an eigenvalue of B other than b. Then*

$$|\mu| \leqslant \frac{1}{2} \max_{1 \leqslant i,j \leqslant n} \sum_{s=1}^{n} |b_{i,s} - b_{j,s}|. \tag{1.1}$$

Let $e^{(n)}$ be the n-dimensional vector of all 1's. As $L^{\#}e^{(n)} = 0$ because L and $L^{\#}$ have identical nullspaces and as the nonzero eigenvalues of $L^{\#}$ are the reciprocals of the nonzero eigenvalues of L (see, for example, [2] and [4]), the following is an immediate consequence of Theorem BDS:

Observation 1.1. *Let $\mathcal{G}$ be a weighted connected graph on n vertices with Laplacian matrix L and algebraic connectivity ν. Set*

$$\mathcal{Z}(\mathcal{G}) := \frac{1}{2} \max_{1 \leqslant i,j \leqslant n} \sum_{s=1}^{n} |L^{\#}_{i,s} - L^{\#}_{j,s}|. \tag{1.2}$$

Then

$$\frac{1}{\mathcal{Z}(\mathcal{G})} \leqslant \nu. \tag{1.3}$$

Let us now describe the paper in more detail. In Theorem 2.3 we shall show that for weighted trees, equality in (1.3) holds if and only if $\mathcal{G}$ is a Type I tree of a certain kind. (Recall that a Type I tree is one in which an eigenvector of L corresponding to

ν has a zero entry, see, e.g., Fiedler [7].) Viewing an unweighted graph as a weighted graph with all weights on the edges equal to 1, we shall show as a corollary to this theorem that for an unweighted graph $\mathcal{G}$ equality in (1.3) holds if and only if $\mathcal{G}$ is a star. In another corollary we shall show precisely which weighted trees admit a re-weighting for which equality now holds in (1.3). In the course of the proof of Theorem BDS use is made of the following estimate which follows from Lemma 2.4 in Seneta [14], but which Seneta attributes to Paz in a somewhat different form:

Lemma PS (Paz [13, Chp. IIa], Seneta [14, p. 63]). *Let $z = (z_1, \dots, z_n)$ be an arbitrary row vector of complex numbers. Then for any real vector $\delta \neq 0$ with $\delta^T e^{(n)} = 0$,*

$$|z^T \delta| \leqslant \frac{1}{2} \max_{1 \leqslant i,j \leqslant n} |z_i - z_j| \, \|\delta\|_1. \tag{1.4}$$

Our proof of the main theorem requires knowledge of the case of equality in Lemma PS. We therefore open Section 2 by characterizing equality in this lemma.

2. Main results

We begin by characterizing the case of equality in the inequality (1.4).

Theorem 2.1. *Let $\delta \in \mathbb{R}^n$ be a vector such that $\delta^T e^{(n)} = 0$ and let $z \in \mathbb{C}^n$. Then equality holds in* (1.4), *viz.*

$$|z^T \delta| = \frac{1}{2} \max_{1 \leqslant i,j \leqslant n} |z_i - z_j| \, \|\delta\|_1$$

if and only if z and δ can be reordered simultaneously such that

$$\delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_m \\ -\delta_{m+1} \\ \vdots \\ -\delta_{m+k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} a \\ \vdots \\ a \\ b \\ \vdots \\ b \\ c_1 \\ \vdots \\ c_{n-k-m} \end{bmatrix}, \tag{2.1}$$

and where

$$\max_{1\leqslant i,j\leqslant n}|z_i-z_j|=|a-b| \text{ and } \delta_i>0,\ i=1,\dots,k+m. \tag{2.2}$$

P r o o f. The sufficiency of the conditions (2.1) and (2.2) is straightforward upon using the condition that $\delta^T e^{(n)}=0$. For the necessity, suppose that equality holds in (1.4). We shall induct on the number of nonzero entries in δ, upon noting first that the result holds trivially if δ has just two nonzero entries. We begin by assuming, without loss of generality, that $\delta_1\geqslant\delta_{m+1}$. Then

$$\delta=\delta_{m+1}\begin{bmatrix}1\\0\\\vdots\\0\\-1\\0\\0\\0\\\vdots\\\vdots\\0\end{bmatrix}+\begin{bmatrix}\delta_1-\delta_{m+1}\\\delta_2\\\vdots\\\delta_m\\0\\-\delta_{m+2}\\\vdots\\-\delta_{m+k}\\0\\\vdots\\0\end{bmatrix}=:\ \delta_{m+1}(e_1^{(n)}-e_{m+1}^{(n)})+\hat{\delta}.$$

Note that $\|\delta\|_1=2\delta_{m+1}+\|\hat{\delta}\|_1, \hat{\delta}^Te^{(n)}=0$, and that $z^t\delta=\delta_{m+1}(z_1-z_{m+1})+z^T\hat{\delta}$. Now,

$$\begin{aligned}\frac{1}{2}(2\delta_{m+1}+\|\hat{\delta}\|_1)\max_{1\leqslant i,j\leqslant n}|z_i-z_j|&=|z^T\delta|\\&\leqslant\delta_{m+1}|z_1-z_{m+1}|+|z^T\hat{\delta}|\\&\leqslant\frac{1}{2}\max_{1\leqslant i,j\leqslant n}|z_i-z_j|(2\delta_{m+1}+\|\hat{\delta}\|_1),\end{aligned}$$

from which we can conclude the following:

(i) $|z^T\hat{\delta}|=\frac{1}{2}\max_{1\leqslant i,j\leqslant n}|z_i-z_j|\ \|\hat{\delta}\|_1$.

(ii) $|z_1-z_{m+1}|=\max_{1\leqslant i,j\leqslant n}|z_i-z_j|$.

(iii) The argument of z_1-z_{m+1} is the same as that of $z^T\hat{\delta}$.

From the induction step on $\hat{\delta}$ we find that z must have the form

$$z = \begin{bmatrix} u \\ a \\ \vdots \\ a \\ v \\ b \\ \vdots \\ b \\ c_1 \\ \vdots \\ c_{n-k-m} \end{bmatrix},$$

where $\max\limits_{1\leqslant i,j\leqslant n} |z_i - z_j| = |a-b|$. Moreover, if $\delta_1 > \delta_{m+1}$, then $u = a$. We further claim that the argument of $z^T\hat{\delta}$ must now be the same as that of $a-b$. Note that

$$\begin{aligned} z^T\hat{\delta} &= u(\delta_1 - \delta_{m+1}) + \sum_{i=2}^{m} a\delta_i - \sum_{i=m+2}^{m+k-1} b\delta_i \\ &\quad + b\Bigg(\delta_{m+1} - \delta_1 - \sum_{i=2}^{m}\delta_i + \sum_{i=m+2}^{m+k-1}\delta_i\Bigg) \\ &= (u-b)(\delta_1 - \delta_{m+1}) + \sum_{i=2}^{m}(a-b)\delta_i. \end{aligned}$$

This shows that $z^T\hat{\delta}$ is a positive multiple of $a-b$. Consequently, $|z_1 - z_{m+1}| = |a-b|$ and $\arg(z_1 - z_{m+1}) = \arg(z^T\hat{\delta})$, so that $z_1 - z_{m+1} = a - b$.

If $z_1 > z_{m+1}$, then we can conclude that $z_1 = a$ and $z_{m+1} = b$. On the other hand, if $\delta_1 = \delta_{m+1}$, then we repeat the argument with either δ_1 and δ_{m+2} or δ_2 and δ_{m+1} to deduce that $z_1 = a$ and $z_{m+1} = b$. Our proof is done. $\square$

To prove our main result we further need the following fact:

Theorem 2.2. *Let $\mathcal{G}$ be a weighted tree on n vertices with Laplacian matrix L. Then the maximum of the expression*

$$\big\| e_i^{(n)T} L^{\#} - e_j^{(n)T} L^{\#} \big\|_1 \tag{2.3}$$

can only be attained at a pair of pendant vertices.

P r o o f. Without loss of generality, we can assume (2.3) is maximized for some $1 \leqslant i \leqslant n-1$ and $j = n$. In [9] it is shown that $L^{\#}$ is given by

$$L^{\#} = \begin{bmatrix} (L^{\#})_{1,1} & (L^{\#})_{1,2} \\ (L^{\#})_{2,1} & (L^{\#})_{2,2} \end{bmatrix}, \tag{2.4}$$

where

$$(L^{\#})_{1,1} = M - \frac{1}{n}MJ - \frac{1}{n}JM + \frac{e^{(n-1)^T}Me^{(n-1)}}{n^2}J, \tag{2.5}$$

$$(L^{\#})_{1,2} = -\frac{1}{n}Me^{(n-1)} + \frac{e^{(n-1)T}Me^{(n-1)}}{n^2}e^{(n-1)}, \tag{2.6}$$

$$(L^{\#})_{2,1} = (L^{\#})_{1,2}^T, \tag{2.7}$$

and

$$(L^{\#})_{2,2} = \frac{e^{(n-1)^T}Me^{(n-1)}}{n^2}, \tag{2.8}$$

where M is the $(n-1) \times (n-1)$ matrix whose (i,j)-th entry is given by

$$\sum_{\varepsilon \in \mathcal{P}_{i,j}} \frac{1}{w(\varepsilon)}, \tag{2.9}$$

with $\mathcal{P}_{i,j}$ denoting the set of edges of $\mathcal{G}$ which are on both the path from vertex i to vertex n and the path from vertex j to vertex n, and where $w(\varepsilon)$ is the weight of edge $\varepsilon \in \mathcal{G}$.

Fix $1 \leqslant i \leqslant n-1$ and note that the last entry of the n-vector

$$e_i^{(n)T}L^{\#} - e_n^{(n)T}L^{\#} = \Big[e_i^{(n-1)T}M - \frac{1}{n}\big(e_i^{(n-1)T}Me^{(n-1)}\big)e^{(n-1)T} \;\Big|\; -\frac{1}{n}\big(e_i^{(n-1)T}Me^{(n-1)}\big)\Big] \tag{2.10}$$

is negative; a similar argument applied to $e_n^{(n)T}L^{\#} - e_i^{(n)T}L^{\#}$ shows that the i-th entry of $e_i^{(n)T}L^{\#} - e_n^{(n)T}L^{\#}$ is positive.

Suppose now that i is not a pendant vertex of $\mathcal{G}$. Then there is a vertex i_0 which is adjacent to i such that the path from i_0 to n includes i. Let θ be the weight of the edge between i and i_0 and let S be the set of vertices in $\mathcal{G}$ whose path to n includes the vertex i_0. Let the cardinality of S be σ and let v_S^T be the row vector with a $1/\theta$ in position k if and only if $k \in S$ and 0 otherwise. It follows that

$$e_{i_0}^{(n-1)T}M = e_i^{(n-1)T}M + v_S^T$$

and, if $k \in S$, then

$$e_i^{(n-1)T} M e_k^{(n-1)} = e_i^{(n-1)T} M e_i^{(n-1)}.$$

Note that in position k, v_S^T is positive only if $k \in S$, in which case the k-th entry of $e_i^T M - \frac{1}{n}(e_i^{(n-1)T} M e^{(n-1)}) e^{(n-1)T}$ agrees with the i-th entry of the vector, which is positive. Hence,

$$\begin{aligned}\Big\| e_i^{(n-1)T} M - \frac{1}{n}(e_i^{(n-1)T} M e^{(n-1)}) e^{(n-1)T} + v_S^T \Big\|_1 \\ = \Big\| e_i^{(n-1)T} M - \frac{1}{n}(e_i^{(n-1)T} M e^{(n-1)}) e^{(n-1)T} \Big\|_1 + \frac{\sigma}{\theta},\end{aligned}$$

Consequently,

$$\begin{aligned}\big\| e_{i_0}^{(n)T} L^{\#} - e_n^{(n)T} L^{\#} \big\|_1 &\geqslant \frac{1}{n}\Big(e_i^{(n-1)T} M e^{(n-1)} + \frac{\sigma}{\theta} \Big) \\ &+ \Big\| e_i^{(n-1)T} M - \frac{1}{n}(e_i^{(n-1)T} M e^{(n-1)}) e^{(n-1)T} \Big\|_1 \\ &+ \frac{\sigma}{\theta} - \frac{\sigma}{n\theta} \big\| e^{(n-1)} \big\|_1 \\ &= \big\| e_i^{(n)T} L^{\#} - e_n^{(n)T} L^{\#} \big\|_1 + \frac{2\sigma}{n\theta} > \big\| e_i^{(n)T} L^{\#} - e_n^{(n)T} L^{\#} \big\|_1.\end{aligned}$$

As a result we see that if i is not a pendant vertex, then $\big\| e_i^{(n)T} L^{\#} - e_n^{(n)T} L^{\#} \big\|_1$ is not maximal and, arguing similarly on n, this shows that n too must be pendant. □

In order to state our main result we require the following result due to Fiedler [7].[5] Let $\mathcal{G}$ be a weighted tree on n vertices with Laplacian matrix L. Let ν be the algebraic connectivity and let y be any corresponding eigenvector. Then exactly one of the following situations holds:

i) Some entry of y is zero. Then there is a unique vertex k such that $y_k = 0$ and k is adjacent to a vertex m with $y_m \neq 0$. Further, the entries of y are either increasing, decreasing or identically 0 along any path in $\mathcal{G}$ which starts at k.

ii) No entry of y is zero. Then there is a unique pair of vertices i and j such that i is adjacent to j and $y_i > 0$ and $y_j < 0$. Futher, the entries of y are increasing along any path which starts at i and does not contain j, while the entries of y are decreasing along any path which starts at j and does not contain i.

A weighted tree is of Type I if (i) holds and it is of Type II if (ii) holds. In the former case, we call the unique vertex k the *characteristic vertex* and, in the latter case, the unique pair of vertices i and j are called the *characteristic vertices.*

[5] We remark that strictly speaking that paper only deals with unweighted trees, but the results extends to the weighted case.

The identification of the characteristic vertices is independent of the choice of the eigenvector y as shown by Merris [12].

At a vertex u of $\mathcal{G}$, a *branch at* u is a connected component which is obtained by deleting vertex u and all edges incident to u. Let u be a vertex of $\mathcal{G}$ and let B denote a branch of the tree at u. Associated with B is a positive matrix formed from the vertices of B as follows: if v_i and v_j are on B, then the entry in row i and column j of the matrix is $\sum\limits_{\varepsilon\in\mathcal{P}_{v_i,v_j}} \frac{1}{w(\varepsilon)}$, where $\mathcal{P}_{v_i,v_j}$ is the set of edges which are on both the path from v_i to u and on the path from v_j to u. For each branch of $\mathcal{G}$ at u we compute the Perron value of the associated positive matrix; if a branch yields the maximal Perron value over all branches at u, then the branch is called a *Perron branch at* u. In Kirkland, Neumann, and Shader [10] it is shown that $\mathcal{G}$ is a Type I tree if and only if there are at least two Perron branches at the characteristic vertex; in that case ν is equal to the reciprocal of the maximal Perron value over all branches at the characteristic vertex.

We are now ready to state our main theorem:

Theorem 2.3. *Let $\mathcal{G}$ be a weighted tree with Laplacian matrix L. Then equality holds in* (1.3) *if and only if $\mathcal{G}$ is a Type I tree whose characteristic vertex u has the property that two of its Perron branches consist of single edges adjacent to pendant vertices, v_1 and v_2, and*

$$\max_{1\leqslant i,j\leqslant n} \big\|e_i^{(n)T}L^{\#} - e_j^{(n)T}L^{\#}\big\|_1 = \frac{2}{w(u,v_1)} \ \Big(= \frac{2}{w(u,v_2)}\Big). \tag{2.11}$$

P r o o f. We begin with the "necessity" part of the proof. For a vector $z\in\mathbb{C}^n$ let $f(z) := \max\limits_{1\leqslant i,j\leqslant n} |z_i - z_j|$. Next, for any $1\leqslant i,j\leqslant n$, let $\delta_{i,j}^T = e_i^T L^{\#} - e_j^T L^{\#}$, and suppose that y is an eigenvector of L corresponding to ν. We then have that

$$\begin{aligned}\frac{1}{\nu}f(y) &= \frac{1}{\nu}\max_{1\leqslant i,j\leqslant n}|y_i - y_j| := \frac{1}{\nu}|y_{i_0} - y_{j_0}|\\ &= |\delta_{i_0,j_0}^T y| = \frac{1}{2}f(y)\|\delta_{i_0,j_0}\|_1\\ &= \frac{1}{2}f(y)\max_{1\leqslant i,j\leqslant n}\|\delta_{i,j}\|_1.\end{aligned}$$

In particular, $\|\delta_{i_0,j_0}\|_1 = \max\limits_{1\leqslant i,j\leqslant n}\|\delta_{i,j}\|_1$, so that, by Theorem 2.2, i_0 and j_0 are pendant vertices of $\mathcal{G}$. Furthermore, $|\delta_{i_0,j_0}^T y| = \frac{1}{2}\max\limits_{1\leqslant i,j\leqslant n}|y_i - y_j|\ \|\delta_{i_0,j_0}\|_1$ so that Theorem 2.1 applies.

Without loss of generality, let us take $i_0 = 1$ and $j_0 = n$. Then, by Theorem 2.1, the vector y is constant on the indices where $e_1^{(n)T}L^{\#} - e_n^{(n)T}L^{\#}$ is positive and constant on the indices where $e_1^{(n)T}L^\# - e_n^{(n)T}L^\#$ is negative. Moreover, if the k_1-th and k_2-th entries of $e_1^{(n)T}L^\# - e_n^{(n)T}L^\#$ are positive and negative, respectively, then $|y_{k_1} - y_{k_2}| = \max_{1\leqslant i,j\leqslant n} |y_i - y_j|$.

It is not difficult to show that the entries of $e_n^{(n)T}L^\#$ are strictly decreasing along any path which starts at n, while the entries of $e_1^{(n)T}L^\#$ are strictly increasing along any path which ends at 1. Hence, the entries of $e_1^{(n)T}L^\# - e_n^{(n)T}L^\#$ are strictly increasing as we move along the path which starts at n and ends at 1. In particular, there is at most one vertex on that path for which the corresponding entry in $e_1^{(n)T}L^\# - e_n^{(n)T}L^\#$ is 0.

As we observed in the proof of Theorem 2.2, following equation (2.10), the first entry of the vector $e_1^{(n)T}L^\# - e_n^{(n)T}L^\#$ is positive and its last entry is negative, so that $|y_1 - y_n| = \max_{1\leqslant i,j\leqslant n} |y_i - y_j|$. Since the entries in y sum to 0, without loss of generality we can assume that $y_1 = \max_{1\leqslant i\leqslant n} y_i > 0$ and $y_n = \min_{1\leqslant i\leqslant n} y_i < 0$.

Suppose now that vertex k is on the path joining vertex 1 with vertex n. Then from the above, we see that there are three possibilities:

(a) $e_1^{(n)T}L^\# e_k^{(n)} - e_n^{(n)T}L^\# e_k^{(n)} > 0$ and $y_k = y_1 > 0$.
(b) $e_1^{(n)T}L^\# e_k^{(n)} - e_n^{(n)T}L^\# e_k^{(n)} < 0$ and $y_k = y_n < 0$.
(c) $e_1^{(n)T}L^\# e_k^{(n)} - e_n^{(n)T}L^\# e_k^{(n)} = 0$ and $|y_1 - y_k|, |y_n - y_k| \leqslant y_1 - y_n$.

Also note that there is at most one index k for which (c) holds.

Recall now that if $\mathcal{G}$ is a Type I tree, then along any path starting from the characteristic vertex, the entries in y are either strictly increasing and positive, or strictly decreasing and negative, or all zero. Since $y_1 > 0 > y_n$, we conclude that if $\mathcal{G}$ is of Type I, then the characteristic vertex must be on the path from vertex 1 to vertex n. Similarly, if $\mathcal{G}$ is a Type II tree, we find that both characteristic vertices must be on that path.

Suppose first that $\mathcal{G}$ is a Type I tree whose characteristic vertex is k. Then, $y_k = 0$ and so, necessarily, (c) holds. The entries of y are monotonically increasing along a path from k to 1. However, since the entries of $e_1^{(n)T}L^\# - e_n^{(n)T}L^\#$ are also monotonically increasing on that path, y_j must equal y_1 if vertex j is on the path and $j \neq k$. We conclude that the path must have length 1, that is, k is adjacent to the pendant vertex 1. Similarly, we obtain that k is adjacent to n. Thus we see that the characteristic vertex k has two Perron branches which consist of single edges adjacent to the pendant vertices 1 and n. Now from (2.4)–(2.9), we find that

$$e_1^{(n)T}L^\# - e_n^{(n)T}L^\# = \left[\frac{1}{w(1,k)}\ 0\ \ldots\ 0\ -\frac{1}{w(1,k)}\right]$$

and hence

$$\max_{1\leqslant i,j\leqslant n} \left\|e_i^{(n)T}L^\# - e_j^{(n)T}L^\#\right\|_1 = \frac{2}{w(1,k)}.$$

Suppose now that $\mathcal{G}$ is a Type II tree with characteristic vertices i and j so that the entries of y are increasing along the path from i to 1 and decreasing along the path from j to n. But then there is at most one vertex on the path from 1 to n which (c) can hold and so either $y_i = y_1$ or $y_j = y_n$, both of which yield contradictions. Consequently, if $\mathcal{G}$ is a Type II tree, equality in (1.3) can not hold. This completes the proof of the necessity part.

For the "sufficiency" part of the theorem assume that $\mathcal{G}$ is a Type I tree whose characteristic vertex u has the property that two of its Perron branches consist of single edges adjacent to pendant vertices, v_1 and v_2 and (2.11) holds. Then $\nu = 1/w(u,v_1) = \frac{1}{2}\max\limits_{1\leqslant i,j\leqslant n} \left\|e_1^{(n)T}L^{\#} - e_n^{(n)T}L^{\#}\right\|_1$. This concludes the proof of the theorem. □

We now have the following corollary:

Corollary 2.4. *Let $\mathcal{G}$ be an unweighted tree with Laplacian matrix L and algebraic connectivity ν. Then equality holds in* (1.3) *if and only if $\mathcal{G}$ is a star.*

Proof. The only unweighted tree whose characteristic vertex is adjacent to a pendant vertex is a star, so if (1.3) holds, then $\mathcal{G}$ is a star. Conversely, it is not difficult to show that if $\mathcal{G}$ is a star then $\nu = 1$ and

$$\max_{1\leqslant i,j\leqslant n} \left\|e_i^{(n)T}L^{\#} - e_j^{(n)T}L^{\#}\right\|_1 = 2.$$

□

We conclude the paper with a corollary concerning when a tree can be weighted to yield equality in (1.3).

Corollary 2.5. *Let $\mathcal{G}$ be a tree on n vertices having two pendant vertices which are adjacent to the same vertex. Then there is a weighting of $\mathcal{G}$ yielding equality in* (1.3).

Proof. Without loss of generality we can re-label the vertices of $\mathcal{G}$ so that vertices n and $n-1$ are pendant and adjacent to vertex $n-2$. Now weight the edges from n and $n-1$ to $n-2$ with $\varepsilon > 0$ and weight every other edge with 1. We shall use the results of Theorem 2.3 to show that for a choice of ε sufficiently small, equality holds in (1.3).

Consider the branches of $\mathcal{G}$ at vertex $n-2$. Associated with the branches containing vertices n and $n-1$ we have a Perron value of $1/\varepsilon$, while every other branch at $n-2$ yields a Perron value which is independent of ε. As a result, when ε is sufficiently small, we find that $\mathcal{G}$ is a Type I tree with characteristic vertex $n-2$ and that the branches at $n-2$ containing n and $n-1$ are Perron branches.

Now let L be the Laplacian matrix of the (newly) weighted tree. It remains to show that

$$\max_{1\leqslant i,j\leqslant n} \left\| e_i^{(n)T} L^{\#} - e_j^{(n)T} L^{\#} \right\|_1 = \frac{2}{\varepsilon}.$$

As in the proof of Theorem 2.3 we find that

$$e_{n-1}^{(n)T} L^{\#} - e_n^{(n)T} L^{\#} = \begin{bmatrix} 0 & \dots 0 & \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \end{bmatrix}$$

and so

$$\left\| e_{n-1}^{(n)T} L^{\#} - e_n^{(n)T} L^{\#} \right\|_1 = \frac{2}{\varepsilon}.$$

Now suppose that $1 \leqslant i \leqslant n-2$. Letting $M \in \mathbb{R}^{(n-1),(n-1)}$ be the matrix whose i,j-th entry is given by (2.9), we find that for some $(n-1)$-vector x whose entries are positive and independent of $\varepsilon > 0$,

$$e_i^{(n-1)T} M = x^T + \frac{1}{\varepsilon} e^{(n-1)T}.$$

Hence,

$$e_i^{(n)T} L^{\#} - e_n^{(n)T} L^{\#} = \Big[\frac{1}{n\varepsilon} e^{(n)T} + x^T - \frac{x^T e^{(n)}}{n} e^{(n)} \ \Big| \ -\frac{n-1}{n\varepsilon} - \frac{x^T e^{(n)}}{n} \Big].$$

It follows that for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \left\| e_i^{(n)T} L^{\#} - e_n^{(n)T} L^{\#} \right\|_1 &= \left\| \frac{1}{n\varepsilon} e^{(n)T} + x^T - \frac{x^T e^{(n)}}{n} e^{(n)} \right\|_1 + \frac{n-1}{n\varepsilon} + \frac{x^T e^{(n)}}{n} \\ &\leqslant 2\frac{n-1}{n\varepsilon} + \frac{x^T e^{(n)}(n+2)}{n} < \frac{2}{\varepsilon}. \end{aligned}$$

An analogous argument also shows that

$$\left\| e_i^{(n)T} L^{\#} - e_{n-1}^{(n)T} L^{\#} \right\|_1 < \frac{2}{\varepsilon}$$

when $\varepsilon > 0$ is sufficiently small. Finally, suppose that $1 \leqslant i,j \leqslant n-2$. Then, again, from (2.9) and the equations leading to it, we have that

$$\begin{aligned} \left\| e_i^{(n)T} L^{\#} - e_j^{(n)T} L^{\#} \right\|_1 = \Big[&\big(e_i^{(n-1)T} - e_j^{(n-1)T} \big) M \Big(I - \frac{1}{n} J \Big) \ \Big| \\ &- \frac{1}{n} \big(e_i^{(n-1)T} - e_j^{(n-1)T} \big) M e^{(n-1)} \Big]. \end{aligned}$$

Since there are positive $(n-1)$-vectors y and z, independent of $\varepsilon > 0$, such that $e_i^{(n-1)T} M = y^T + (1/\varepsilon) e^{(n-1)}$ and $e_j^{(n-1)T} M = z^T + (1/\varepsilon) e^{(n-1)}$, we find that $\left\| e_i^{(i)T} L^{\#} - e_j^{(j)T} L^{\#} \right\|_1$ is independent of ε, so that

$$\left\| e_i^{(n)T} L^{\#} - e_j^{(n)T} L^{\#} \right\|_1 \leqslant \frac{2}{\varepsilon}$$

when $\varepsilon > 0$ is sufficiently small. Thus for such ε we have that

$$\max_{1\leqslant i,j\leqslant n} \left\| e_i^{(n)T} L^{\#} - e_j^{(n)T} L^{\#} \right\|_1 = \frac{2}{\varepsilon}$$

and so, by Theorem 2.3, equality holds in (1.3). This completes the proof. □

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