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ON OPERATORS WITH THE SAME LOCAL SPECTRA

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Abstract. Let $B(X)$ be the algebra of all bounded linear operators in a complex Banach space X . We consider operators $T_1, T_2 \in B(X)$ satisfying the relation $\sigma_{T_1}(x) = \sigma_{T_2}(x)$ for any vector $x \in X$, where $\sigma_T(x)$ denotes the local spectrum of $T \in B(X)$ at the point $x \in X$. We say then that T_1 and T_2 have the same local spectra. We prove that then, under some conditions, $T_1 - T_2$ is a quasinilpotent operator, that is $\|(T_1 - T_2)^n\|^{1/n} \rightarrow 0$ as $n \rightarrow \infty$. Without these conditions, we describe the operators with the same local spectra only in some particular cases.

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1. Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X . For any $T \in B(X)$, denote by $\sigma(T)$ the spectrum of T , and by $\sigma_T(x)$ the local spectrum of T at a point $x \in X$. It is known (see [4] or [5]) that

- (a) $\sigma_T(x) \subseteq \sigma(T) \quad (x \in X)$;
- (b) $\sigma(T) = \bigcup \{\sigma_T(x) : x \in X\}$;
- (c) $\sigma_T(x)$ is a compact set in \mathbb{C} , for any $x \in X$;
- (d) $\sigma_T(x) = \emptyset$ if and only if $x = 0$.

An operator $N \in B(X)$ is called quasinilpotent if $\|N^n\|^{1/n} \rightarrow 0$ as $n \rightarrow \infty$, i.e. if $\sigma(N) = \{0\}$. N is obviously quasinilpotent if and only if $\sigma_N(x) = \{0\}$ for any vector $x \in X \setminus \{0\}$.

For arbitrary operators $T_1, T_2 \in B(X)$, we can consider the relation

$$(1) \quad \sigma_{T_1}(x) = \sigma_{T_2}(x) \quad \text{whenever } x \in X.$$

It is obvious that in the relation considered we can suppose that $x \neq 0$.

If T_1, T_2 satisfy the relation (1) (for any vector $x \in X$), we say that they have the same local spectra (*SLS* in short).

If T_1 and T_2 have *SLS*, then obviously $\sigma(T_1) = \sigma(T_2)$. In the general case, the point spectra $\sigma_p(T_1)$ and $\sigma_p(T_2)$ can be different. As an example, take any nilpotent operator N_1 in the complex Hilbert space $X = L^2(0, 1)$, and the Volterra integral operator N_2 in the same space (N_2 is a quasinilpotent but not nilpotent operator with empty point spectrum). Then $\sigma(N_1) = \sigma(N_2) = \{0\}$, and immediately $\sigma_{N_1}(x) = \sigma_{N_2}(x) = \{0\}$ for any vector $x \neq 0$. But, on the other hand, $\sigma_p(N_1) = \{0\}$ and $\sigma_p(N_2) = \emptyset$.

2. The next theorem completely describes the relationship between operators with *SLS* if they are commuting and decomposable in the sense of Foias [3].

Theorem 1. *Let $T_1, T_2 \in B(X)$ be commuting decomposable operators. Then they have *SLS* if and only if $T_1 - T_2$ is a quasinilpotent operator.*

P r o o f. Assume that $T_1 - T_2$ is a quasinilpotent operator. Then, by a result of [1] (see also [2, Ch. 4]), we have

$$\sigma_{T_1}(x) = \sigma_{T_2+N}(x) = \sigma_{T_2}(x)$$

for any vector $x \in X$.

Next assume that T_1 and T_2 have *SLS*. Then with notation from [3], we have for any closed set $F \subseteq C$:

$$\begin{aligned} X_{T_1}(F \cap \sigma(T_1)) &= \{x: \sigma_{T_1}(x) \subseteq F\}, \\ X_{T_2}(F \cap \sigma(T_2)) &= \{x: \sigma_{T_2}(x) \subseteq F\}. \end{aligned}$$

By relation (1), it is obvious that

$$X_{T_1}(F \cap \sigma(T_1)) = X_{T_2}(F \cap \sigma(T_2))$$

for any closed set $F \subseteq C$, so that all conditions from Theorem 3.2 from [3] are satisfied. Hence $T_1 - T_2$ is a quasinilpotent operator, Q.E.D. \square

The above theorem completely describes commuting operators with *SLS* in the class of all normal operators in a Hilbert space, or in the class of all compact operators in a Banach space. Indeed, as is well-known, all these operators are decomposable.

Corollary 1. *Let T_1, T_2 be commuting operators in a finite-dimensional space X . Then they have *SLS* if and only if $T_1 - T_2$ is a nilpotent operator.*

P r o o f. As is known, any operator in a finite-dimensional space is decomposable, and the assertion follows immediately by Theorem 1. As is also known, nilpotent and quasinilpotent operators in a finite-dimensional space always coincide. \square

The next examples show that operators with *SLS*, in the general case, need not be commuting nor decomposable.

Example 1. Take $X = C^2$, and let $\{e_1, e_2\}$ be the standard basis in C^2 . Define

$$\begin{aligned} N_1(x) &= N_1(\xi_1, \xi_2) = (\xi_1 + \xi_2, -\xi_1 - \xi_2), \\ N_2(x) &= N_2(\xi_1, \xi_2) = (\xi_1 - \xi_2, \xi_1 - \xi_2). \end{aligned}$$

Then $N_1^2 = N_2^2 = 0$, thus N_1 and N_2 are nilpotent, and therefore they have *SLS*. Nevertheless, they are obviously noncommuting.

Example 2. Let T_1 be the right shift operator in the space ℓ^2 , defined by

$$T_1(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$$

for any $x = (\xi_1, \xi_2, \dots) \in \ell^2$.

Take $T_2 = T_1^2$, i.e.

$$T_2(\xi_1, \xi_2, \dots) = (0, 0, \xi_1, \xi_2, \dots).$$

It is known (see [4], [5] or [6]) that

$$\sigma_{T_1}(x) = \sigma(T_1) = \{\lambda: |\lambda| \leq 1\}$$

for any vector $x \in \ell^2 \setminus \{0\}$.

It can be also verified that

$$\sigma_{T_2}(x) = \sigma(T_2) = \{\lambda: |\lambda| \leq 1\}$$

for any vector $x \in \ell^2 \setminus \{0\}$.

Hence, the condition (1) is obviously fulfilled, and T_1 and $T_2 = T_1^2$ obviously commute. On the other hand, if $D = T_1 - T_2 = T_1 - T_1^2$, then

$$\begin{aligned} \|D^n e_1\|^2 &= \|(1 - T_1)^n e_{n+1}\|^2 = \left\| \sum_{k=0}^n \binom{n}{k} (-1)^k e_{n+k+1} \right\|^2 \\ &= \sum_{k=0}^n \binom{n}{k}^2 \geq \sum_{k=0}^n \binom{n}{k} = 2^n, \end{aligned}$$

so that $\|D^n e_1\|^{1/n} \geq \sqrt{2}$ for all $n \geq 0$. Hence, D is not a quasinilpotent operator. Together with Theorem 1 this shows that not both operators T_1 and T_2 are decomposable. Moreover, since the analytic functions of decomposable operators are also decomposable, we have that both operators T_1 and T_2 are nondecomposable.

We also note that the operator T_1 from this example has appeared several times in literature as a very useful example for different aims (see [4], [5], [6]).

Describing the general operators which have *SLS* remains an open question in this paper. We have succeeded only in some particular classes of operators. In the next section we shall analyze general (not necessarily commuting) operators in a finite-dimensional space X which have *SLS*.

3. Let T be an arbitrary linear operator in a finite-dimensional space X . Denote $\sigma(T) = \sigma_p(T) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$, where $\lambda_i \neq \lambda_j$ for $i \neq j$, and let $\mathcal{L}_T(\lambda_1), \mathcal{L}_T(\lambda_2), \dots, \mathcal{L}_T(\lambda_r)$ be the corresponding root spaces of T . Then we have

$$(2) \quad X = \mathcal{L}_T(\lambda_1) \dot{+} \mathcal{L}_T(\lambda_2) \dot{+} \dots \dot{+} \mathcal{L}_T(\lambda_r).$$

Denote by $E_T(\lambda_i)$ ($i = 1, 2, \dots, r$) the projection from X to $\mathcal{L}_T(\lambda_i)$ according to the decomposition (2). Since X is finite-dimensional, all these projections $E_T(\lambda_i)$ ($i = 1, 2, \dots, r$) are bounded.

Using the definition of the local spectrum, it is not difficult to see the following.

Proposition 1. *Let T be an arbitrary operator in a finite-dimensional space X and assume that $x \in X \setminus \{0\}$. Then a complex value $\lambda \in \sigma_T(x)$ if and only if $\lambda \in \sigma(T)$ and $E_T(\lambda)x \neq 0$.*

In particular, $\sigma_T(x) = \{\lambda_i\}$ if and only if $x \in \mathcal{L}_T(\lambda_i) \setminus \{0\}$.

Proposition 2. *Let X be a finite-dimensional space. Then operators T_1, T_2 have *SLS* if and only if they have the same spectra and the same corresponding root spaces.*

Remark. As an immediate consequence, we get that then T_1 and T_2 also have the same algebraic multiplicities of their eigenvalues.

P r o o f. Let T_1 and T_2 have *SLS*. Then we obviously have $\sigma(T_1) = \sigma(T_2)$. Denote their common spectrum by $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$ where $\lambda_i \neq \lambda_j$ for $i \neq j$, and let $\mathcal{L}_{T_1}(\lambda_i), \mathcal{L}_{T_2}(\lambda_i)$ ($i = 1, 2, \dots, r$) be the corresponding root spaces of T_1 and T_2 , respectively. If $x \in \mathcal{L}_{T_1}(\lambda_i) \setminus \{0\}$ then $\sigma_{T_1}(x) = \{\lambda_i\} = \sigma_{T_2}(x)$, whence $x \in \mathcal{L}_{T_2}(\lambda_i)$ by Proposition 1. Hence $\mathcal{L}_{T_1}(\lambda_i) \subseteq \mathcal{L}_{T_2}(\lambda_i)$. Similarly $\mathcal{L}_{T_2}(\lambda_i) \subseteq \mathcal{L}_{T_1}(\lambda_i)$, and hence $\mathcal{L}_{T_1}(\lambda_i) = \mathcal{L}_{T_2}(\lambda_i)$ for any $i = 1, 2, \dots, r$.

Conversely, assume that T_1 and T_2 have the same spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$ and the same corresponding root spaces. Take any $x \in X \setminus \{0\}$. Then $x = x_1 + x_2 + \dots + x_r$, where $x_i \in \mathcal{L}_{T_1}(\lambda_i) = \mathcal{L}_{T_2}(\lambda_i)$ for any $i = 1, 2, \dots, r$. Denoting $S = \{i \in \{1, 2, \dots, r\} : x_i \neq 0\}$ and $F = \{\lambda_i : i \in S\}$, we have that $\sigma_{T_1}(x) = \sigma_{T_2}(x) = F$ by Proposition 1.

This completes the proof. □

Using the Jordan form of operators in a finite-dimensional space, one can easily obtain

Corollary 1. *Let X be a finite-dimensional space. Then operators T_1, T_2 have SLS if and only if T_1 is similar to an operator S_1 and T_2 is similar to an operator S_2 such that $S_1 - S_2 = N_1 - N_2$, where N_1, N_2 are nilpotent operators commuting respectively with T_1, T_2 .*

4. Next, consider two similar operators $T, S \in B(X)$ in a Banach space X ; thus $S = KTK^{-1}$, where $K, K^{-1} \in B(X)$. We know that T and S always have the same spectra. But the following example shows that similar operators need not have the same local spectra.

Example 3. Let $X = C^2$ with the standard basis $\{e_1, e_2\}$. Define operators T and S by

$$T(\xi_1, \xi_2) = (\xi_1, 0), \quad S(\xi_1, \xi_2) = (0, \xi_2).$$

Operators T and S are similar with respect to the operator $K(\xi_1, \xi_2) = (\xi_2, \xi_1)$. Next we have that $\sigma(T) = \sigma(S) = \{0; 1\}$, but $\sigma_T(e_1) = \{1\}$, $\sigma_S(e_1) = \{0\}$. Hence T and S have not the same local spectra.

The next example shows that some similar operators can have the same local spectra.

Example 4. Let X be an arbitrary Banach space, and assume that $T = \lambda I + N$, where $\lambda \in \mathbb{C}$ and N is a quasinilpotent operator. If $S = KTK^{-1}$, where $K, K^{-1} \in B(X)$, we find that $\sigma(S) = \sigma(T) = \{\lambda\}$. For any $x \in X \setminus \{0\}$, we easily get $\sigma_T(x) = \sigma_S(x) = \{\lambda\}$, so that T and S have SLS.

We also see that such operators T have, in a sense, extremely large local spectra, for $\sigma_T(x) = \sigma(T)$ for every $x \in X \setminus \{0\}$.

These examples motivate us to introduce the following definition.

Let $\mathcal{M}(X)$ be the class of all operators $T \in B(X)$ in a Banach space X such that T and every operator S similar to T have SLS.

Let $\mathcal{S}(x)$ be the class of all operators $T \in B(X)$ such that

$$(3) \quad \sigma_T(x) = \sigma(T)$$

for every vector $x \in X \setminus \{0\}$.

Obviously, both classes $\mathcal{M}(X)$ and $\mathcal{S}(X)$ contain all operators of the form $\lambda I + N$, where $\lambda \in C$ and N is an arbitrary quasinilpotent operator.

Proposition 3. *Classes $\mathcal{M}(X)$ and $\mathcal{S}(X)$ coincide.*

Proof. Assume that $T \in \mathcal{S}(X)$. By the definition of the local spectrum it is not difficult to see that

$$\sigma_{KT K^{-1}}(x) = \sigma_T(K^{-1}x)$$

whenever $K, K^{-1} \in B(X)$. Hence, obviously,

$$\sigma_{KT K^{-1}}(x) = \sigma(T) = \sigma_T(x)$$

for every vector $x \in X \setminus \{0\}$. Therefore $T \in \mathcal{M}(X)$.

Conversely, assume that $T \in \mathcal{M}(X)$. Then we have

$$(4) \quad \sigma_T(K^{-1}x) = \sigma_T(x)$$

whenever $x \in X$ and $K, K^{-1} \in B(X)$.

If $x, y \neq 0$ are arbitrary but fixed vectors in X , we are now proving that there is an invertible operator $K \in B(X)$ such that $K^{-1}x = y$, i.e. $Ky = x$. If $y = x$, take $K = I$. If $y = \alpha x$ ($\alpha \neq 0$), take $K = \alpha^{-1}I$. If x, y are linearly independent, denote by $E = \text{sp}\{x, y\}$ the linear span over x and y . Then E is closed, and as is well-known, there is a closed subspace F in X such that $X = E \dot{+} F$. Define $Kx = y$, $Ky = x$, $Ku = u$ ($u \in F$), and further by linearity. Then $K = K^{-1}$ is bounded on closed invariant subspaces E and F , and the projection from X onto E along F is also bounded. Hence K is bounded on the entire space X .

Now consider relation (4).

By the previous remark we get

$$(5) \quad \sigma_T(x) = \sigma_T(y) \quad (x, y \neq 0).$$

Since $\sigma(T) = \bigcup\{\sigma_T(x): x \in X\}$, we immediately have relation (3) for every $x \in X \setminus \{0\}$. Therefore $T \in \mathcal{S}(X)$. \square

The class of operators $\mathcal{S}(X)$ seems to be important and interesting. However, we shall not develop the questions concerning this class in this paper. We only note that the operators T_1, T_2 from Example 2 both belong to this class.

The class $\mathcal{S}(X)$ can be completely described at least in a finite-dimensional space.

Proposition 4. *If X is a finite-dimensional space, then*

$$\mathcal{S}(X) = \{\lambda I + N: \lambda \in C, N \text{ is a nilpotent operator}\}.$$

P r o o f. Example 4 proves a half of this assertion.

Next, assume that $T \in \mathcal{S}(X)$. By Proposition 1, it is easy to see that then T has exactly one point in its spectrum, thus $T = \lambda I + N$ for some $\lambda \in C$ and a nilpotent operator N .

This completes the proof. \square

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