

Józef Dudek

Small idempotent clones. I

*Czechoslovak Mathematical Journal*, Vol. 48 (1998), No. 1, 105–118

Persistent URL: <http://dml.cz/dmlcz/127403>

## Terms of use:

© Institute of Mathematics AS CR, 1998

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

## SMALL IDEMPOTENT CLONES I

JÓZEF DUDEK, Wrocław

(Received July 18, 1995)

*Abstract.* G. Grätzer and A. Kisielewicz devoted one section of their survey paper concerning  $p_n$ -sequences and free spectra of algebras to the topic “Small idempotent clones” (see Section 6 of [18]). Many authors, e.g., [8], [14, 15], [22], [25] and [29, 30] were interested in  $p_n$ -sequences of idempotent algebras with small rates of growth. In this paper we continue this topic and characterize all idempotent groupoids  $(G, \cdot)$  with  $p_2(G, \cdot) \leq 2$  (see Section 7). Such groupoids appear in many papers see, e.g. [1], [4], [21], [26, 27], [25], [28, 30, 31, 32] and [34].

## 1. INTRODUCTION

Recall that by  $p_n = p_n(\mathcal{A})$  we denote the number of all essentially  $n$ -ary polynomials (term functions) of a given algebra  $\mathcal{A}$ . Note that  $p_0(\mathcal{A})$  denotes the number of unary constant polynomials over  $\mathcal{A}$ . The notation and notions used in this paper are standard and we refer to [17] and [16, 18].

In [28] J. Płonka characterized all groupoids  $(G, \cdot)$  (algebras) with  $p_n(G, \cdot) = n$  for all  $n$  (see also [17]).

P.P. Pálffy described all groupoids  $(G, \cdot)$  whose clones are minimal and  $p_2(G, \cdot) \leq 2$  (see [26]).

In [19] G. Grätzer and R. Padmanabhan described all groupoids  $(G, \cdot)$  representing the sequence  $(0, 1, 1, 3, 5)$ . They proved that any groupoid  $(G, \cdot)$  representing this sequence is a nontrivial affine space over  $GF(3)$ .

Recall that a sequence  $a = (a_0, \dots)$  (finite or infinite) of cardinals is representable if there exists an algebra  $\mathcal{A}_0$  such that  $p(\mathcal{A}_0) = a$ , i.e.,  $p_i(\mathcal{A}_0) = a_i$  for  $i = 0, 1, \dots$

In [24] A. Mischke and H. Werner among other described by means of identities the variety of groupoids  $(G, \cdot)$  which are polynomially equivalent to affine spaces over  $GF(4)$ . Obviously for such groupoids  $(G, \cdot)$  we have  $p_2(G, \cdot) \leq 2$ . Recall that two

algebras  $(A, F_1), (A, F_2)$  are polynomially equivalent if  $\mathbf{A}(F_1) = \mathbf{A}(F_2)$ , where  $\mathbf{A}(F)$  denotes the set of all polynomials over  $(A, F)$ .

In [12] we proved that a groupoid  $(G, \cdot)$  represents the sequence  $(0, 1, 2, 7)$  if and only if  $(G, \cdot)$  is a nontrivial affine space over  $GF(4)$ .

For more details see also [3],[7] and [8]. Note that in [28] R. Park considered groupoids  $(G, \cdot)$  with  $p_2(G, \cdot) = 1$  which are not finitely based.

The simplest nontrivial groupoids  $(G, \cdot)$  with  $p_2(G, \cdot) \leq 2$  are obviously the following ones:  $T_1 = (\{0, 1\}, e_1)$ ,  $T_2 = (\{0, 1\}, e_2)$ ,  $S_0 = (\{0, 1\}, \vee)$ , where  $e_1$  is the first projection,  $e_2$  is the second projection and also  $A_0 = (\{0, 1, 2\}, 2x +_3 2y)$ . We have  $p_2(T_1) = p_2(T_2) = 0$  and  $p_2(S_0) = p_2(A_0) = 1$ . It is easy to see that  $T_1 \times T_2$  is a groupoid satisfying  $x^2 = x$ ,  $(xy)z = x(yz) = xz$ . Groupoids which satisfy these identities are called diagonal semigroups or rectangular bands. Note that  $p(T_1 \times T_2) = (0, 1, 2, 0, 0, \dots)$ .

For a given groupoid  $(G, \cdot)$  we put  $xy^1 = xy$  and  $xy^{n+1} = (xy^n)y$  and dually we define  ${}^n yx$  ( $n = 1, \dots$ ). We also write  $x_1x_2x_3$  instead of  $(x_1x_2)x_3$  and in general  $x_1x_2 \dots x_{n-1}x_n$  stands for  $(x_1x_2 \dots x_{n-1})x_n$  ( $n \geq 3$ ).

A commutative idempotent groupoid  $(G, \cdot)$  satisfying  $xy^2 = x$  is called a Steiner quasigroup and if  $xy^2 = xy$ , then  $(G, \cdot)$  is called a near-semilattice. Further, a groupoid  $(G, \cdot)$  is distributive if  $(G, \cdot)$  satisfies  $(xy)z = (xz)(yz)$  and  $z(xy) = (zx)(zy)$  for  $x, y, z \in G$ , and  $(G, \cdot)$  is medial if  $(G, \cdot)$  satisfies the medial law, i.e.,  $(xy)(uv) = (xu)(yv)$  for all  $x, y, u, v \in G$ .

We recall the groupoids introduced in [9], namely for a nonnegative integer  $n$  by  $N_n$  we mean a groupoid  $(\{-1, 0, 1, \dots, n-1, n\}, \cdot)$ , where the fundamental operation  $\cdot$  is defined as follows:

$$xy = \begin{cases} x & \text{if } x = y, \\ 1 + \max(x, y) & \text{if } x \neq y \text{ and } x, y \leq n-1, \\ n & \text{otherwise.} \end{cases}$$

By  $\mathbf{N}_n$  we denote the variety of all commutative idempotent groupoids  $(G, \cdot)$  satisfying  $xy^2 = yx^2$  and  $xy^n = xy^{n+1}$  for a fixed  $n$ .

Recall also that a groupoid  $(G, \cdot)$  is totally commutative if every essentially binary polynomial  $f$  over  $(G, \cdot)$  is commutative, i.e.,  $f(x, y) = f(y, x)$  for all  $x, y \in G$ .

Following [31], we say that an identity is regular if the sets of variables on both sides coincide.

## 2. GENERAL REMARKS

We start with a result from [9].

**Theorem 2.1.** *If  $(G, \cdot)$  is an idempotent groupoid, then  $p_2(G, \cdot) = 1$  if and only if  $(G, \cdot)$  is either a nontrivial Steiner quasigroup or a nontrivial near-semilattice.*

Concerning commutative idempotent groupoids we have

**Theorem 2.2.** *Let  $(G, \cdot)$  be a commutative idempotent groupoid. Then*

(i) *If  $p_2(G, \cdot) = 2$ , then  $(G, \cdot)$  contains isomorphically as a subgroupoid the groupoid  $N_2$  (see [9]).*

(ii)  *$p_2(G, \cdot) = 2$  if and only if  $(G, \cdot)$  satisfies  $xy^2 = yx^2$ ,  $xy^2 = xy^3$  (i.e.,  $(G, \cdot) \in \mathbf{N}_2$ ) and  $(G, \cdot)$  is not a near-semilattice (see [9]).*

(iii) *If  $(G, \cdot)$  satisfies  $xy^2 = yx^2$ , then  $(G, \cdot)$  is totally commutative (see Theorem 4 of [6]) and if  $\text{card } G > 1$ , then such groupoids  $(G, \cdot)$  satisfy only regular identities.*

We also have (easy to prove)

**Theorem 2.3.** *For every  $n$ ,  $N_n \in \mathbf{N}_n$  and every nontrivial member from the variety  $\mathbf{N}_n$  satisfies only regular identities.*

Recall also the results from [10] and [11].

**Theorem 2.4.** *If  $(G, \cdot)$  is a commutative idempotent groupoid, then  $(G, \cdot)$  is either a semilattice (here  $p(G, \cdot) = (0, 1, 1, 1, \dots)$  provided  $\text{card } G > 1$ ) or  $(G, \cdot)$  is an affine space over  $GF(3)$  (with  $p(G, \cdot) = (0, 1, 1, 3, \dots, \frac{2^n - (-1)^n}{3}, \dots)$  or else  $p_n(G, \cdot) \geq 3^{n-1}$  for all  $n \geq 4$ ).*

Moreover, a commutative groupoid  $(G, \cdot)$  represents the sequence  $(0, 1, 3, \dots, 3^{n-1}, \dots)$  if and only if  $(G, \cdot)$  is a nontrivial Płonka sum of affine spaces over  $GF(3)$  which are not all singletons (for details see [10] and [31]).

**Theorem 2.5.** *There are no distributive, commutative and idempotent groupoids  $(G, \cdot)$  with  $p_2(G, \cdot) = 2$ .*

Note that all distributive noncommutative idempotent groupoids  $(G, \cdot)$  with  $p_2(G, \cdot) \leq 2$  are described in [5].

According to the formula of the description of the set of all  $n$ -ary polynomials over a given algebra  $\mathcal{A} = (A, F)$  we have  $\mathbf{A}^{(n)}(\mathcal{A}) = \bigcup_{k=0}^{\infty} \mathbf{A}_k^{(n)}(\mathcal{A})$ , where  $\mathbf{A}_0^{(n)} = \mathbf{A}_0^{(n)}(\mathcal{A}) = \{e_1^{(n)}, \dots, e_n^{(n)}\}$ ,  $e_i^{(n)}(x_1, \dots, x_n) = x_i$ ,  $i = 1, \dots, n$ ,  $\mathbf{A}_{k+1}^{(n)} = \mathbf{A}_{k+1}^{(n)}(\mathcal{A}) = \mathbf{A}_k^{(n)}(\mathcal{A}) \cup \{f(f_1, \dots, f_m) : f_1, \dots, f_m \in \mathbf{A}_k^{(n)}(\mathcal{A}), f \in F\}$  (for details see [23]). For an

idempotent groupoid  $(G, \cdot)$  with  $e_1^{(2)}(x, y) = x$ ,  $e_2^{(2)}(x, y) = y$  we have  $\mathbf{A}_2^{(2)}(G, \cdot) = \{x, y, xy, yx, xy^2, yx^2, {}^2yx, {}^2xy, (xy)x, (yx)y, x(yx), y(xy), (xy)(yx), (yx)(xy)\}$ . So, dealing with idempotent groupoids  $(G, \cdot)$  satisfying  $p_2(G, \cdot) = 2$  we have to consider the set  $\mathbf{A}_2^{(2)}(G, \cdot)$ .

According to Theorems 2.1 and 2.2 we mainly deal with noncommutative idempotent groupoids  $(G, \cdot)$  satisfying  $p_2(G, \cdot) = 2$ . (Keep in mind that our main aim is to describe such groupoids by means of identities). The main role in our considerations play by the polynomial  $xy^2$ .

### 3. GROUPOIDS WITH $XY^2 = X$ .

We start with

**Lemma 3.1.** *Let  $(G, \cdot)$  be an idempotent groupoid satisfying  $xy^2 = x$ . Then we have*

- (i)  $(G, \cdot)$  satisfies  $xy = x$  iff  $(G, \cdot)$  satisfies  $(xy)x = x$ .
- (ii)  $(G, \cdot)$  is a Steiner quasigroup iff  $(G, \cdot)$  satisfies  $(xy)x = y$  (or dually  $x(yx) = y$ ).
- (iii)  $(G, \cdot)$  satisfies  $(xy)x = yx$  iff  $\text{card } G = 1$ .
- (iv)  $(G, \cdot)$  is a left-sided quasigroup, i.e., the equation  $xa = b$  has a unique solution for  $a, b \in G$ .
- (v) If  $p_2(G, \cdot) = 2$ , then  $(G, \cdot)$  is a noncommutative groupoid satisfying  $(xy)x = xy$ .
- (vi) If  $(G, \cdot)$  satisfies  $(xy)x = xy$ , then  $(G, \cdot)$  satisfies  $x(xy) = x$ .

*Proof.* We prove only (v). If  $(G, \cdot)$  is commutative, then  $(G, \cdot)$  is a Steiner quasigroup and therefore  $p_2(G, \cdot) \leq 1$ , which contradicts the assumption  $p_2(G, \cdot) = 2$ . Take the polynomial  $(xy)x$ . Using the assumption  $p_2(G, \cdot) = 2$  and the conditions (i)–(iii) we get  $(xy)x = xy$ , completing the proof of the lemma.  $\square$

Similarly we get

**Lemma 3.2.** *Let  $(G, \cdot)$  be an idempotent groupoid satisfying  $xy^2 = x$ . Then we have*

- (i)  $(G, \cdot)$  satisfies  $x(yx) = x$  iff  $(G, \cdot)$  satisfies  $xy = x$ .
- (ii)  $(G, \cdot)$  satisfies  $x(yx) = y$  iff  $(G, \cdot)$  is a Steiner quasigroup.
- (iii)  $(G, \cdot)$  satisfies  $x(yx) = yx$  iff  $\text{card } G = 1$ .
- (iv) If  $p_2(G, \cdot) = 2$ , then  $(G, \cdot)$  satisfies  $x(yx) = xy$  and  $(xy)(yx) = x$ .

Note that there exist idempotent groupoids  $(G, \cdot)$  satisfying  $xy^2 = x$  in which the polynomial  $(xy)x = x(yx)$  is commutative. Take e.g. an abelian group  $(G, +)$  of exponent 5. Then  $(G, \cdot)$  where  $xy = 4x + 2y$  is the required groupoid. Such groupoids are obviously polynomially equivalent to affine spaces over  $GF(5)$ .

For a given groupoid  $(G, \cdot)$  with the fundamental polynomial  $xy$  we consider the groupoid  $(G, \circ)$  where  $x \circ y = yx$  for all  $x, y \in G$ ; it is called the dual groupoid of  $(G, \cdot)$ . If  $K$  is a class of groupoids, then by  $K^d$  we denote the class of all dual groupoids  $(G, \cdot)$  from  $K$ . Note that a groupoid  $(G, \cdot)$  is called proper if the fundamental operation  $xy$  is essentially binary.

We have

**Lemma 3.3.** *Let  $(G, \cdot)$  be an idempotent groupoid satisfying  $xy^2 = x$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs to the variety  $K_1: xy = (xy)x = x(yx)$  and  $(G, \cdot)$  is proper.*

(The dual class  $K_1^d$  of the class  $K_1$  is defined by  $x = y(yx)$ ,  $yx = x(yx) = (xy)x$ ).

*Proof.* If  $p_2(G, \cdot) = 2$ , then the proof follows from the last two lemmas. If  $(G, \cdot) \in K_1$  and  $(G, \cdot)$  is proper, then obviously  $(G, \cdot)$  is noncommutative and  $xy$  is essentially binary. Using the identities of the groupoid  $(G, \cdot)$  one can compute the elements of the set  $\mathbf{A}^{(2)}(G, \cdot)$ . We simply show that  $\mathbf{A}^{(2)}(G, \cdot) = \{x, y, xy, yx\}$  and hence  $p_2(G, \cdot) = 2$ , completing the proof.  $\square$

Note that there exist proper (noncommutative) idempotent groupoids  $(G, \cdot)$  belonging to the variety  $K_1$ . For example, let  $(G, +)$  be an abelian group of exponent 4. Take  $(G, \cdot)$  with  $xy = 3x + 2y$ . Then  $(G, \cdot) \in K_1$ .

#### 4. GROUPOIDS WITH $XY^2 = Y$

Concerning  $(xy)x$  we have

**Lemma 4.1.** *Let  $(G, \cdot)$  be an idempotent groupoid satisfying  $xy^2 = y$ . Then  $(G, \cdot)$  is one-element iff one of the following conditions holds: either  $(G, \cdot)$  is commutative or  $(xy)x = y$  or  $(xy)x = xy$ .*

*Proof.* If e.g.  $(xy)x = xy$ , then using  $xy^2 = y$  we get  $xy = (xy)x = ((xy)x)x = x$ , which gives  $x = xy = (xy)y = y$  as required, completing the proof.  $\square$

According to this lemma we have the following possibilities for the polynomial  $(xy)x$ :  $(xy)x = x$  or  $(xy)x = yx$ .

**Lemma 4.2.** *If  $(G, \cdot)$  is a proper idempotent groupoid satisfying  $xy^2 = y$  and  $(xy)x = x$ , then  $(G, \cdot)$  satisfies  $x(xy) = y(xy) = xy$  and  $(xy)(yx) \notin \{y, xy\}$ .*

*Proof.* Putting  $xy$  for  $x$  in  $(xy)x = x$  we get  $xy = ((xy)y)(xy) = y(xy)$ . If e.g.  $(xy)(yx) = y$ , then  $y = ((xy)y)(y(xy)) = y(y(xy)) = y(xy) = xy$ , which is impossible.  $\square$

From this lemma we get

**Lemma 4.3.** *Let  $(G, \cdot)$  be a proper idempotent groupoid satisfying  $y = xy^2 = (yx)y$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs to one of the following varieties:*

$$K_2: xy = x(xy) = y(xy) \text{ and } (xy)(yx) = x,$$

$$K_3: xy = x(xy) = y(xy) = (yx)(xy).$$

Recall that the dual classes of  $K_2$  and  $K_3$  are defined by

$$K_2^d: y = y(yx) = y(xy), yx = (yx)x = (yx)y \text{ and } (xy)(yx) = x,$$

$$K_3^d: y = y(yx) = y(xy), yx = (yx)x = (yx)y = (yx)(xy).$$

Further we consider idempotent groupoids  $(G, \cdot)$  with  $p_2(G, \cdot) = 2$  satisfying  $xy^2 = y$  and  $(xy)x = yx$ . We have

**Lemma 4.4.** *Let  $(G, \cdot)$  be an idempotent groupoid satisfying  $xy^2 = y$  and  $(xy)x = yx$ . Then we have*

$$(i) \text{ card } G = 1 \text{ iff } x(xy) \in \{x, yx\} \text{ or } x(yx) \in \{y, xy\}.$$

$$(ii) (G, \cdot) \text{ satisfies } xy = y \text{ iff } x(yx) = x.$$

*Proof.* (i) If e.g.  $x(xy) = yx$ , then  $x = (yx)x = (x(xy))x = (xy)x = yx$ . The identities  $xy = y$  and  $x(xy) = yx$  give  $\text{card } G = 1$ .

$$(ii) \text{ If } x(yx) = x, \text{ then } x = x(yx) = (x(yx))(yx) = yx, \text{ completing the proof. } \square$$

As a corollary of this lemma we get

**Lemma 4.5.** *Let  $(G, \cdot)$  be a proper idempotent groupoid satisfying  $xy^2 = y$  and  $(xy)x = yx$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs to one of the following varieties:*

$$K_1^d: x(xy) = y \text{ and } x(yx) = yx,$$

$$K_4: x(xy) = xy \text{ and } x(yx) = yx.$$

Moreover, the varieties  $K_1^d$  (and also  $K_1$ ),  $K_4$  have proper models (see e.g. p. 394 [17]).

From Lemmas 4.3 and 4.5 we get

**Lemma 4.6.** *Let  $(G, \cdot)$  be a proper idempotent groupoid satisfying  $xy^2 = y$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs to one of the varieties*

$$K_2: xy = x(xy) = y(xy), x = (xy)x = (xy)(yx),$$

$$K_3: xy = x(xy) = y(xy) = (yx)(xy) \text{ and } (xy)x = x,$$

$$K_4: xy = x(xy), (xy)x = x(yx) = yx,$$

and to the dual of  $K_1$ .

We continue our characterization and deal with idempotent groupoids  $(G, \cdot)$  satisfying  $p_2(G, \cdot) = 2$  and either  $xy^2 = xy$  or  $xy^2 = yx$ . It is obvious that such groupoids are proper and noncommutative.

## 5. GROUPOIDS WITH $XY^2 = XY$

First we consider the polynomial  $(xy)x$ . We have

**Lemma 5.1.** *Let  $(G, \cdot)$  be a proper idempotent groupoid satisfying  $xy^2 = xy$  and  $(xy)x = x$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs to the variety*

$$K_5: x = x(yx) = (xy)(yx).$$

*Proof.* First observe that the identity  $(xy)x = x$  implies  $x(xy) = xy$ . Assume now that  $(G, \cdot)$  satisfies  $xy^2 = xy$ ,  $(xy)x = x$  and  $p_2(G, \cdot) = 2$ . Take  $x(yx)$ . Obviously  $x(yx) \neq y$ . If  $x(yx) = xy$ , then putting  $yx$  for  $x$  in this identity we get  $y = (yx)y = (yx)(y(yx)) = yx$ , a contradiction. If  $(G, \cdot)$  satisfies  $x(yx) = yx$ , then  $yx = (yx)x = (x(yx))x = x$ , which is impossible. Thus we have proved that  $(G, \cdot)$  satisfies  $x(yx) = x$ . In this case we consider the polynomial  $(xy)(yx)$ . If  $(xy)(yx) = y$ , then using  $yx = y(yx)$  we get  $yx = y(yx) = ((xy)(yx))(yx) = (xy)(yx) = y$ , a contradiction. If  $(xy)(yx) = xy$ , then putting  $xy$  for  $y$  we get  $xy = x(xy) = (x(xy))((xy)x) = (xy)x = x$ , again a contradiction. Analogously we get  $(xy)(yx) \neq yx$  and therefore  $(G, \cdot)$  satisfies  $(xy)(yx) = x$  as required.

If  $(G, \cdot)$  is a proper member of  $K_5$ , then the standard method shows that  $p_2(G, \cdot) = 2$ , completing the proof of the lemma.  $\square$

Note that  $K_5 = K_5^d$  and any near-rectangular band, i.e. an idempotent groupoid  $(G, \cdot)$  satisfying  $(xy)z = xz$  and  $x(yx) = x$ , is a member of  $K_5$  (for details see [2]).

Now we deal with groupoids  $(G, \cdot)$  satisfying either  $(xy)x = xy$  or  $(xy)x = yx$ .

It is easy to prove

**Lemma 5.2.** *Let  $(G, \cdot)$  be an idempotent groupoid satisfying  $xy = xy^2 = (xy)x$ . Then we have*

- (i)  $(G, \cdot)$  satisfies  $x(yx) = y$  iff  $\text{card } G = 1$ .
- (ii) If  $p_2(G, \cdot) = 2$ , then  $(G, \cdot)$  satisfies either  $x(yx) = x$  or  $x(yx) = xy$  or  $x(yx) = yx$ .

Further we have

**Lemma 5.3.** *Let  $(G, \cdot)$  be a proper idempotent groupoid satisfying  $xy = xy^2 = (xy)x$  and  $x(yx) = x$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs either to  $K_2^d$  or to  $K_3^d$ .*



**P r o o f.** Recall that  $K_2^d$  and  $K_3^d$  are defined by

$$K_2^d: xy^2 = xy = (xy)x, x(yx) = (xy)(yx) = x, \text{ and}$$

$$K_3^d: xy^2 = xy = (xy)x = (xy)(yx), x(yx) = x$$

(see Lemma 4.6). Assume that  $p_2(G, \cdot) = 2$ . Consider the polynomial  $(xy)(yx)$ . If  $(xy)(yx) = y$ , then  $y = (xy^2)(y(xy)) = xy$ , a contradiction. Analogously we have  $(xy)(yx) \neq yx$ . Thus we see that  $(G, \cdot)$  satisfies either  $(xy)(yx) = x$  or  $(xy)(yx) = xy$  and therefore  $(G, \cdot) \in K_2^d \cup K_3^d$ . It is routine to prove the converse, completing the proof.  $\square$

Concerning Lemma 5.2 we have

**Lemma 5.4.** *Let  $(G, \cdot)$  be a proper noncommutative idempotent groupoid satisfying  $xy = xy^2 = (xy)x = x(yx)$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs either to  $K_4^d$  or to the variety  $K_6$ :  $x(xy) = xy$ .*

**P r o o f.** Assume that  $p_2(G, \cdot) = 2$ . Take  $x(xy)$ . If  $x(xy) = y$ , then  $y = x(xy) = x(x(yx)) = yx$ , which is impossible. If  $x(xy) = yx$ , then putting  $xy$  for  $x$  we get  $yx = y(xy) = (xy)(xy^2) = xy$  which proves that  $(G, \cdot)$  is commutative, a contradiction. Since  $p_2(G, \cdot) = 2$  we infer that  $(G, \cdot)$  satisfies either  $x(xy) = x$  or  $x(xy) = xy$ . If the first case occurs, then  $(G, \cdot) \in K_4^d$ , i.e.  $(G, \cdot)$  satisfies  $xy = xy^2 = (xy)x = x(yx) = (xy)(yx)$  and  $x(xy) = x$ . If the second case occurs, then  $(G, \cdot)$  satisfies  $xy = xy^2 = (xy)x = x(yx) = x(xy) = (xy)(yx)$ , i.e.  $(G, \cdot) \in K_6$ , completing the proof.  $\square$

According to Lemma 5.2 it remains to consider yet in this case the identity  $x(yx) = yx$ . We have

**Lemma 5.5.** *There is no idempotent groupoid with  $p_2(G, \cdot) = 2$  satisfying  $xy = xy^2 = (xy)x = y(xy)$ .*

**P r o o f.** First we prove that  $x(xy) = xy$ . Indeed, we have  $x(xy) = x((xy)x) = (xy)x = xy$ . Take now the polynomial  $(xy)(yx)$ . If  $(xy)(yx) = x$ , then  $x = (x(xy))((xy)x) = (xy)(xy) = xy$ , a contradiction. Analogously we get  $(xy)(yx) \neq y$ . If  $(xy)(yx) = yx$ , then  $xy = (yx)(xy) = ((xy)(yx))(xy) = (xy)(yx) = yx$  and hence  $(G, \cdot)$  is a near-semilattice, which is impossible. Analogously, using  $xy = y(xy)$  we infer that  $(xy)(yx) \neq yx$ , which completes the proof of the lemma.  $\square$

According to Lemmas 5.1 and 5.2 to complete the case  $xy^2 = xy$  it remains to consider yet the identity  $(xy)x = yx$ . Note that if  $(G, \cdot)$  satisfies  $(xy)x = y$ , then  $(G, \cdot)$  is a quasigroup and hence  $(G, \cdot)$  is cancellative. In this case  $xy^2 = xy$  gives  $x = y$ . Further we have.

**Lemma 5.6.** *Let  $(G, \cdot)$  be a noncommutative idempotent groupoid satisfying  $xy^2 = xy$  and  $(xy)x = yx$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs to the dual of  $K_6$ , i.e.,  $(G, \cdot)$  satisfies  $K_6^d$ :  $xy = xy^2 = (yx)y = y(xy) = x(xy) = (yx)(xy)$ .*

**Proof.** Putting  $xy$  for  $x$  in  $(xy)x = yx$  we get  $y(xy) = (xy^2)(xy) = xy$  and we have  $(yx)(xy) = ((xy)x)(xy) = x(xy)$ . Consider now the polynomial  $x(xy)$ . We see that  $x(xy) \neq y$ . If  $x(xy) = x$ , then using  $(xy)x = yx$  we get  $(xy)(yx) = (xy)((xy)x) = xy$  and hence  $xy = (xy)(yx) = y(yx) = y$ , a contradiction. Let  $x(xy) = yx$ . Then  $(yx)(xy) = (x(xy))(xy) = x(xy) = yx$  and  $xy = (xy)(yx) = (xy)((yx)(xy)) = (yx)(xy) = yx$ , which is impossible. Thus we have proved that  $(G, \cdot)$  satisfies  $x(xy) = xy$  and consequently  $(G, \cdot) \in K_6^d$ ; the fact that if  $(G, \cdot) \in K_6^d$ , then  $p_2(G, \cdot) = 2$  is obvious, which completes the proof of the lemma.  $\square$

To complete our characterization of idempotent groupoids  $(G, \cdot)$  with  $p_2(G, \cdot) = 2$  it remains to consider the last case, namely the identity  $xy^2 = yx$ .

## 6. GROUPOIDS WITH $XY^2 = YX$

We begin with

**Lemma 6.1.** *If  $(G, \cdot)$  is a groupoid satisfying  $xy^2 = yx$ , then  $(G, \cdot)$  satisfies  $(xy)x = x(yx)$ .*

**Proof.** Using  $xy^2 = yx$  we have  $(yx)y = ((xy)y)y = y(xy)$  as required.  $\square$

**Lemma 6.2.** *Let  $(G, \cdot)$  be a proper idempotent groupoid satisfying  $xy^2 = yx$ . Then  $p_2(G, \cdot) = 2$  iff  $(G, \cdot)$  belongs to  $K_7$ :  $(xy)x = y$ .*

Moreover, if  $(G, \cdot) \in K_7$  and  $a, b \in G$   $a \neq b$ , then the subgroupoid  $G(a, b)$  generated by the set  $\{a, b\}$  is a four-element affine space over  $GF(4)$ .

**Proof.** Let  $p_2(G, \cdot) = 2$ . Using the preceding lemma we get  $(xy)x = x(yx)$ . Take the polynomial  $(xy)x$ . If  $(xy)x = x$ , then  $x = x(yx) = (x(yx))(yx) = (yx)x = xy$ , a contradiction. If  $(xy)x = yx$ , then  $xy = (yx)x = ((xy)x)x = x(xy)$ . Using  $xy = x(xy)$  and  $(xy)x = yx$  we get  $(yx)(xy) = ((xy)x)(xy) = x(xy) = xy$ . The identity  $(xy)(yx) = yx$  gives  $xy = ((yx)(xy))(xy) = (xy)(yx) = yx$  and therefore  $(G, \cdot)$  is a near-semilattice, a contradiction. If  $(xy)x = xy$ , then  $(xy)(yx) = ((xy)x)(yx) = (x(yx))(yx) = (yx)x = xy$ . Using  $xy = (xy)(yx)$  we get  $xy = (xy)(yx) = ((xy)(yx))(yx) = (yx)(xy) = yx$ , a contradiction. Thus we have proved that  $(G, \cdot)$  satisfies  $(xy)x = y$ .

Let now  $(G, \cdot) \in K_7$  i.e.,  $(G, \cdot)$  satisfies  $xy^2 = yx$ ,  $(xy)x = y$  and also  $x(xy) = yx$ ,  $x(yx) = y$ ,  $(xy)(yx) = y$ ,  $xy^3 = x$  and  ${}^3yx = x$ . Let  $G(a, b)$  be the subgroupoid

generated by the set  $\{a, b\}$  where  $a \neq b$  and  $a, b \in G$ . One can check that  $G(a, b) = \{a, b, ab, ba\}$  and the groupoid  $G(a, b)$  is an affine space over  $GF(4)$ , which completes the proof of the lemma.  $\square$

## 7. A CHARACTERIZATION THEOREM

As a corollary of Lemmas 3.3, 4.6, 5.3, 5.4, 5.5, 5.6, 6.2 and Theorems 2.1, 2.2 we get

**Theorem.** *Let  $(G, \cdot)$  be an idempotent groupoid. Then  $p_2(G, \cdot) \leq 2$  if and only if  $(G, \cdot)$  belongs to one of the following varieties:*

$$K_1: xy^2 = x, xy = (xy)x = x(yx), x(xy) = (xy)(yx) = x;$$

$$K_2: xy^2 = y, (xy)(yx) = (xy)x = x, xy = x(xy) = y(xy);$$

$$K_3: xy^2 = y, (xy)x = x, xy = x(xy) = y(xy) = (yx)(xy);$$

$$K_4: xy^2 = y, xy = (yx)y = y(xy) = x(xy) = (yx)(xy);$$

$$K_5: xy^2 = xy, (xy)x = x(yx) = (xy)(yx) = x;$$

$$K_6: xy^2 = xy = (xy)x = x(yx) = x(xy) = (xy)(yx);$$

$$K_7: xy^2 = yx, (xy)x = x(yx) = y, x(xy) = yx, (xy)(yx) = y;$$

$$K_8: xy^2 = x, xy = yx \text{—the variety of Steiner quasigroups};$$

$$K_9: xy = yx, xy^2 = yx^2, xy^2 = xy^3 \text{—the variety } \mathbf{N}_2$$

and to the varieties  $K_i^d$  ( $i = 1, \dots, 9$ ).

Note that in the above varieties some identities can be omitted (they are inserted only for the sake of symmetry). Proper models were given earlier from each variety  $K_i$  ( $i = 1, \dots, 9$ ), except for  $i = 2$  and  $i = 3$ . The next two sections are devoted to these varieties. Note also that the variety  $K_9$  plays an important role in Grätzer's problem of the minimal extension of sequences (see [16]).

**On the variety  $K_2$ .** In this section we shall deal with the variety  $K_2$ , i.e. the variety of all idempotent groupoids  $(G, \cdot)$  defined by  $xy^2 = y$ ,  $(xy)x = (xy)(yx) = x$  and  $xy = y(xy)$ . Note that  $xy = x(xy)$  is a consequence of  $(xy)x = x$ .

We define the four-element groupoid  $k_2 = (\{1, 2, 3, 4\}, \cdot)$ , where  $1 \cdot 2 = 3$ ,  $2 \cdot 3 = 4$ ,  $3 \cdot 4 = 1$ ,  $4 \cdot 3 = 2$  and otherwise the fundamental operation  $xy$  is the second projection. It is not hard to prove that  $k_2$  is a proper groupoid belonging to the variety  $K_2$ . We have

**Lemma 8.1.** *If  $(G, \cdot)$  is a proper groupoid from the variety  $K_2$ , then  $(G, \cdot)$  contains isomorphically the groupoid  $k_2$  as a subgroupoid.*

**P r o o f.** Since  $(G, \cdot)$  is proper we infer that there exist elements  $a, b \in G$  such that  $ab \neq b$  and  $a \neq b$ . Take the subgroupoid  $G(a, b)$  generated by the set  $\{a, b\}$ . We have  $G(a, b) = \{a, b, ab, ba\}$ . Now we show that  $\text{card } G(a, b) = 4$ . If  $a = ab$ , then  $a = ab = (ab)b = b$ , which is impossible. If  $a = ba$ , then  $ab = (ba)b = b$ . If  $b = ba$ , then  $b = ba = (ba)a = a$ . If again  $ab = ba$ , then  $a = (ab)(ba) = (ba)(ab) = b$ . The required isomorphism between  $G(a, b)$  and the groupoid  $k_2$  is given by the mapping  $a \rightarrow 1, b \rightarrow 2, ab \rightarrow 3$  and  $ba \rightarrow 4$ , which completes the proof of the lemma.  $\square$

Further we try to estimate the number  $p_n$  of proper groupoids  $(G, \cdot)$  from  $K_2$ . In this connection we recall some definitions.

For a given function  $f = f(x_1, \dots, x_n)$  on a set  $A$  we denote by  $G(f)$  the symmetry group of the function  $f$ , i.e., the subgroup of the group  $S_n$  of all permutations  $\sigma \in S_n$  such that  $f(x_1, \dots, x_n) = f(x_{\sigma_1}, \dots, x_{\sigma_n})$  for all  $x_1, \dots, x_n \in A$  (for more details see [20]). It is clear that if  $f = f(x_1, \dots, x_n)$  is essentially  $n$ -ary, then  $f^\sigma$  where  $f^\sigma(x_1, \dots, x_n) = f(x_{\sigma_1}, \dots, x_{\sigma_n})$  is also essentially  $n$ -ary. So, from an essentially  $n$ -ary function  $f$  on a set  $A$  we can get (permuting its variables)  $\frac{n!}{\text{card } G(f)}$  different essentially  $n$ -ary functions on  $A$ .

**Lemma 8.2.** *If  $(G, \cdot)$  is a proper groupoid from  $K_2$ , then the polynomials  $(xy)z$ ,  $x(yz)$  are essentially ternary, their symmetry groups are one-element and  $(xy)z \neq \sigma x(\sigma y \sigma z)$  for every permutation  $\sigma$ .*

**P r o o f.** Let  $(G, \cdot) \in K_2$ . Using the identities  $xy^2 = y, (xy)x = (xy)(yx) = x$  and  $xy = x(xy) = y(xy)$  one can prove that both  $(xy)z$  and  $x(yz)$  are essentially ternary. Since  $(G, \cdot)$  is not a semilattice we infer that  $(xy)z \neq (yz)x$  and  $x(yz) \neq y(zx)$  (see e.g. Theorem 8 [7]). If  $(xy)z = (yx)z$ , then  $xy = (xy)(xy) = (yx)(xy) = y$ , a contradiction.

If  $(xy)z = (zy)x$ , then  $y = (xy)y = yx$ , a contradiction. If  $(xy)z = (xz)y$ , then  $xy = (xy)x = x$ , which is again impossible. Thus we have proved that  $\text{card } G((xy)z) = 1$ . Consider now the polynomial  $x(yz)$ . If  $x(yz) = x(zy)$ , then  $yx = x(yx) = x(xy) = xy$ , a contradiction. If  $x(yz) = z(yx)$ , then putting  $y = z$  we get  $xy = yx$ . If  $x(yz) = y(xz)$ , then  $x = (xy)(yx) = y((xy)x) = yx$  and this proves that  $\text{card } G(x(yz)) = 1$ . Let now  $(xy)z \in \{x(yz), y(zx), z(xy), y(xz), z(yx), x(zy)\}$ . We see that in any case if  $(xy)z = \sigma x(\sigma y \sigma z)$  holds, then putting  $y = z$  we get a contradiction, which completes the proof.  $\square$

Analogously as above we prove

**Lemma 8.3.** *If  $(G, \cdot)$  is a proper groupoid from  $K_2$ , then the polynomial  $f(x, y, z) = ((xy)z)x$  is essentially ternary, its symmetry group is one-element,  $f \notin \{(xy)z, (yz)x, (zx)y, (yx)z, (zy)x\}$  and  $f \notin \{x(yz), y(zx), z(xy), y(xz), z(yx), x(zy)\}$ .*

As a corollary from Lemmas 8.1 and 8.2 we get

**Proposition 8.4.** *If  $(G, \cdot)$  is a proper groupoid from the variety  $K_2$ , then  $p_n(G, \cdot) \geq p_n(k_2)$  for all  $n$  and  $p_3(G, \cdot) \geq 12$ .*

## 9. ON THE VARIETY $K_3$

Recall that  $K_3$  is a variety of idempotent groupoids  $(G, \cdot)$  defined by the following identities:

$$xy^2 = y, (xy)x = x, xy = x(xy) = y(xy) = (yx)(xy).$$

We start with an easy

**Lemma 9.1.** *The groupoid  $k_3 = (\{0, 1, 2\}, \cdot)$ , where  $0 \cdot 1 = 1 \cdot 0 = 2$  and otherwise the operation  $xy$  is the second projection, belongs to the variety  $K_3$ . Moreover, this groupoid satisfies the identity  $x(yz) = y(xz)$ .*

**Lemma 9.2.** *If  $(G, \cdot)$  is a proper groupoid from  $K_3$ , then the polynomials  $(xy)z, x(yz)$  are essentially ternary and the symmetry group of  $(xy)z$  is one-element. If the symmetry group of  $x(yz)$  is nontrivial, then  $(G, \cdot)$  satisfies  $x(yz) = y(xz)$  and the transposition  $(x, y)$  is the only admissible nontrivial permutation of  $x(yz)$ .*

*Proof.* Analogously as in Lemma 8.2 we prove that  $(xy)z, x(yz)$  are essentially ternary. Now we examine the symmetry groups of these polynomials. According to Theorem 8 of [7] we infer that  $(xy)z \neq (yz)x$  and  $x(yz) \neq y(zx)$ . If  $(xy)z = (yx)z$ , then  $yz = ((xy)y)z = (y(xy))z = (xy)z$ , which contradicts the fact that  $(xy)z$  is essentially ternary. If  $(xy)z = (zy)x$ , then  $y = (xy)y = yx$ , a contradiction. Similarly  $(xy)z \neq (xz)y$ . Thus we get  $\text{card } G((xy)z) = 1$ . We also have  $x(yz) \neq x(zy)$  and  $x(yz) \neq z(yx)$ . If e.g.  $x(yz) = x(zy)$ , then  $xy = (yx)(xy) = yx$  and hence  $x = (xy)x = x(xy) = xy$ , a contradiction which completes the proof.  $\square$

We have (similarly as in Lemma 8.3)

**Lemma 9.3.** *If  $(G, \cdot)$  is a proper groupoid from  $K_3$ , then the polynomial  $f(x, y, z) = ((xy)z)x$  is essentially ternary, its symmetry group is trivial and the groupoid  $k_3$  satisfies the identity  $((xy)z)x = yzx$ .*

*Proof.* Since  $f(x, y, y) = yx$  we infer that  $f$  depends on  $x$ . If  $f$  does not depend on  $y$ , then  $(G, \cdot)$  satisfies  $x = (xz)x = zx$ , a contradiction. If  $f$  does not depend on  $z$ , then  $x = ((xy)x)x = ((xy)y)x = yx$ , again a contradiction. If  $((xy)z)x = ((yx)z)y$  or  $((xy)z)x = ((yx)z)y$ , then putting  $x = z$  we get  $x = xy$ , a contradiction. If

$((xy)z)x = ((xz)y)x$ , then putting  $z = xy$  we obtain  $x = (xy)x = ((x(xy))y)x = ((xy)y)x = yx$ , again a contradiction. If  $((xy)z)x = ((zy)x)z$ , then  $y = z$  gives  $yx = y$ . Thus we have proved that  $\text{card } G((xy)z) = 1$ . Further observe that if  $(G, \cdot) \in K_3$ , then  $(G, \cdot)$  satisfies  $xyzx = yzxx$  under any identification of variables and now it is not difficult to check that the groupoid  $k_2$  satisfies this identity, completing the proof of the lemma.  $\square$

As a corollary from Lemmas 9.1 and 9.2 we get

**Proposition 9.4.** *If  $(G, \cdot)$  is a proper groupoid from the variety  $K_3$ , then  $p_3(G, \cdot) \geq 9$ .*

### References

- [1] *J. Berman*: Free spectra of 3-element algebras. Universal Algebra and Lattice Theory (Puebla, 1982). Lecture Notes in Math., 1004. Springer-Verlag, Berlin, New York, 1983, pp. 10–53.
- [2] *S. Crvenkovič and J. Dudek*: Rectangular groupoids. Czechoslovak Math. J. 35 (1985), 405–414.
- [3] *B. Csákány*: On affine spaces over prime fields. Acta Sci. Math. (Szeged) 37 (1975), 33–36.
- [4] *B. Csákány*: All minimal clones on the three-element set. Acta Cybernet. 6 (1983), 227–238.
- [5] *J. Dudek*: Some remarks on distributive groupoids. Czechoslovak Math. J. 31 (1981), 58–64.
- [6] *J. Dudek*: On binary polynomials in idempotent commutative groupoids. Fund. Math. 120 (1984), 187–191.
- [7] *J. Dudek*: Varieties of idempotent commutative groupoids. Fund. Math. 120 (1984), 193–204.
- [8] *J. Dudek*: Polynomial characterization of some idempotent algebras. Acta Sci. Math. 50 (1986), 39–49.
- [9] *J. Dudek*: On the minimal extension of sequences. Algebra Universalis 23 (1986), 308–312.
- [10] *J. Dudek*: Polynomials in idempotent commutative groupoids. Dissertationes Math. 286 (1989), 1–55.
- [11] *J. Dudek*:  $p_n$ -sequences. The minimal extension of sequences. Abstract. Presented at the Conference on Logic and Algebra dedicated to Roberto Magari, on his 60<sup>th</sup> Birthday. Pontignano (Siena) 26–30 April 1994.
- [12] *J. Dudek and J. Tomasik*: Affine spaces over  $\text{GF}(4)$ . To appear.
- [13] *B. Ganter and H. Werner*: Equational classes of Steiner systems. Algebra Universalis 5 (1975), 125–140.
- [14] *J. A. Gerhard*: The lattice of quational classes of idempotent semigroups. J. Algebra 15, 195–224.
- [15] *J. A. Gerhard*: The number of polynomials of idempotent semigroups. J. Algebra 18, 366–376.
- [16] *G. Grätzer*: Composition of functions. Proceedings of the conference on universal algebra (Kingston, 1969). Queen's Univ., Kingston, Ont., 1970, pp. 1–106.

- [17] *G. Grätzer*: Universal Algebra. Second edition. Springer-Verlag, New York-Heidelberg-Berlin, 1979.
- [18] *G. Grätzer and A. Kisielewicz*: A survey of some open problems on  $p_n$ -sequences and free spectra of algebras and varieties. Universal Algebra and Quasigroup Theory (A. Romanowska and J. D. H. Smith, eds.). Helderman Verlag (Berlin), 1992, pp. 57–88.
- [19] *G. Grätzer and R. Padmanabhan*: On commutative idempotent and nonassociative groupoids. Proc. Amer. Math. 28 (1971), 75–78.
- [20] *G. Grätzer and J. Płonka*: On the number of polynomials of an idempotent algebra I. Pacific J. Math. 22 (1970), 697–709.
- [21] *H. Kaiser*: On a problem in the theory of primal algebras. Algebra Universalis 5 (1974), 307–311.
- [22] *A. Kisielewicz*: On idempotent algebra with  $p_n = 2n$ . Algebra Universalis 23 (1981), 313–323.
- [23] *E. Marczewski*: Independence and homomorphisms in abstract algebras. Fund. Math. 50 (1961), 45–61.
- [24] *A. Mitschke and H. Werner*: On groupoids representable by vector spaces over finite fields. Arch. Math. 24 (1973), 14–20.
- [25] *R. Padmanabhan*: Characterization of a class of groupoids. Algebra Universalis 1 (1972), 374–382.
- [26] *P. P. Pálffy*: Minimal clones. Preprint No. 27/1984, Budapest, May 1984. Math. Inst. of the Hungarian Academy of Sciences, H-1053 Budapest, Reáltanoda u. 13–15. Hungary.
- [27] *P. P. Pálffy*: The arity of minimal clones. Acta Sci. Math. 50 (1986), 331–333.
- [28] *R. E. Park*: A four-element algebra whose identities are not finitely based. Algebra Universalis 11 91980, 255–260.
- [29] *J. Płonka*: On algebras with  $n$  distinct  $n$ -ary operations. Algebra Universalis 1 (1971), 73–79.
- [30] *J. Płonka*: On algebras with at most  $n$  distinct  $n$ -ary operations. Algebra Universalis 1 (1971), 80–85.
- [31] *J. Płonka*: On equational classes of abstract algebras defined by regular equations. Fund. Math. 64 (1969), 241–247.
- [32] *J. Płonka*: On  $k$ -cyclic groupoids. Math. Japonica 30 (1985), no. 3, 371–382.
- [33] *J. Płonka*: Subdirectly irreducible groupoids in some varieties. CMUC 24/4 (1983), 631–645.
- [34] *S. K. Stein*: Homogeneous quasigroup. Pacific J. Math. 14 (1964), 1091–1102.

*Author's address*: Mathematical Institute, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.