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Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 1, 135–144

Persistent URL: <http://dml.cz/dmlcz/127405>

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SOME CARDINAL CHARACTERISTICS OF ORDERED SETS

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(Received August 17, 1995)

Abstract. For ordered (= partially ordered) sets we introduce certain cardinal characteristics of them (some of those are known). We show that these characteristics—with one exception—coincide.

0. PRELIMINARIES

An *ordered set* is a pair $(G, <)$ where G is a set and $<$ is an irreflexive and transitive binary relation on G . We shall write briefly G instead of $(G, <)$. Such a set will be always assumed to be nonempty. The symbol $x \prec y$ means that y is a *cover* of x , i.e. $x < y$ and $x < z < y$ holds for no $z \in G$. If $x \leq y$ or $y \leq x$ then the elements x, y are *comparable*; otherwise they are *incomparable*, notation $x \parallel y$. A *chain* is an ordered set any two elements of which are comparable; an *antichain* is an ordered set any two distinct elements of which are incomparable. By the symbol $\mathbf{2}$ we denote the two-element chain, i.e. $\mathbf{2} = (\{0, 1\}; 0 < 1)$.

An *ideal* in an ordered set G is such a subset $A \subseteq G$ that the following holds: $y \in A$, $x \in G$, $x \leq y \Rightarrow x \in A$. The empty set \emptyset will be also assumed to be an ideal in G . If $x \in G$, then $(x] = \{t \in G; t \leq x\}$ is an ideal in G , called the *principal ideal* generated by the element x . If G, H are ordered sets then the *cardinal power* G^H ([1]) is the set of all order preserving mappings $f: H \rightarrow G$ ordered by $f \leq g \iff f(x) \leq g(x)$ for all $x \in H$. Especially, if H is an antichain, then G^H is the set of all mappings $f: H \rightarrow G$ ordered by this rule. The symbol $\max G$ ($\min G$) denotes the *greatest* (*least*) element of G , if this element exists.

1. 2-PSEUDODIMENSION

Let G be an ordered set. The *dimension* of G ([3]) can be defined in the following manner:

$\dim G = \min\{\text{card } T; \text{ there exists a system } (L_t; t \in T) \text{ of chains and a system } (f_t; t \in T) \text{ where } f_t: G \rightarrow L_t \text{ is injective and order preserving for any } t \in T \text{ such that } x \leq y \iff f_t(x) \leq f_t(y) \text{ for all } t \in T\}$.

If all chains L_t have the same order type α we get the definition of the α -*dimension* of G ([5], this cardinal need not exist). By a slight modification we get the definition of the α -*pseudodimension* of G ([7], this cardinal always exists). We describe here especially the definition of the **2**-pseudodimension of G .

Let G be an ordered set, let $T \neq \emptyset$ be a set and let $f_t: G \rightarrow \mathbf{2}$ be a mapping for any $t \in T$. The system $(f_t; t \in T)$ will be called a **2**-*realizer* of G iff for any $x, y \in G$ the following holds:

$$(1) \quad x \leq y \iff f_t(x) \leq f_t(y) \text{ for all } t \in T.$$

Evidently, the condition (1) can be reformulated in the following way:

$$(2) \quad \begin{aligned} \text{(i)} \quad & x < y \Rightarrow f_t(x) \leq f_t(y) \text{ for all } t \in T \text{ and there exists } t_0 \in T \\ & \text{with } f_{t_0}(x) = 0 < 1 = f_{t_0}(y), \\ \text{(ii)} \quad & x \parallel y \Rightarrow \text{there exist } t_1, t_2 \in T \text{ such that } f_{t_1}(x) = 0, f_{t_1}(y) = 1, \\ & f_{t_2}(x) = 1, f_{t_2}(y) = 0. \end{aligned}$$

Let G be an ordered set, let $T \neq \emptyset$ be a set, let $(A_t; t \in T)$ be a system of ideals in G . This system is called an *order base* in G ([10]) iff for any $x, y \in G$ the following holds:

$$(3) \quad \begin{aligned} \text{(i)} \quad & x < y \Rightarrow \text{there exists } t_0 \in T \text{ such that } x \in A_{t_0}, y \notin A_{t_0}, \\ \text{(ii)} \quad & x \parallel y \Rightarrow \text{there exist } t_1, t_2 \in T \text{ such that } x \in A_{t_1}, y \notin A_{t_1}, \\ & x \notin A_{t_2}, y \in A_{t_2}. \end{aligned}$$

The condition (3) can be reformulated in the following way:

$$(4) \quad x \not\leq y \Rightarrow \text{there exists } t_0 \in T \text{ such that } y \in A_{t_0}, x \notin A_{t_0}.$$

Theorem 1.1. *Let G be an ordered set, let $T \neq \emptyset$ be a set. Then the following statements are equivalent:*

(i) For any $t \in T$ there exists a mapping $f_t: G \rightarrow \mathbf{2}$ such that $(f_t; t \in T)$ is a $\mathbf{2}$ -realizer of G .

(ii) For any $t \in T$ there exists an ideal $A_t \subseteq G$ such that $(A_t; t \in T)$ is an order base in G .

Proof. (i) \Rightarrow (ii): Let (i) hold and put $A_t = f_t^{-1}(0)$ for any $t \in T$. If $y \in A_t$, $x \in G$, $x \leq y$ then $f_t(y) = 0$, thus $f_t(x) = 0$ and $x \in A_t$. Hence A_t is an ideal in G . If $x, y \in G$, $x < y$ then by (2) there exists $t_0 \in T$ with $f_{t_0}(x) = 0$, $f_{t_0}(y) = 1$; thus $x \in A_{t_0}$, $y \notin A_{t_0}$. If $x \parallel y$ then there exist $t_1, t_2 \in T$ such that $f_{t_1}(x) = 0$, $f_{t_1}(y) = 1$, $f_{t_2}(x) = 1$, $f_{t_2}(y) = 0$. Then $x \in A_{t_1}$, $y \notin A_{t_1}$, $x \notin A_{t_2}$, $y \in A_{t_2}$. By (3) $(A_t; t \in T)$ is an order base in G and (ii) holds.

(ii) \Rightarrow (i): Let (ii) hold and let $(A_t; t \in T)$ be an order base in G . Let us define a mapping $f_t: G \rightarrow \mathbf{2}$ for any $t \in T$ by $f_t(x) = 0$ if $x \in A_t$, $f_t(x) = 1$ if $x \notin A_t$. We show that $(f_t; t \in T)$ is a $\mathbf{2}$ -realizer of G . Let $x, y \in G$, $x < y$. If $f_t(y) = 0$ then $y \in A_t$ and as A_t is an ideal, $x \in A_t$ so that $f_t(x) = 0$. Thus $f_t(x) \leq f_t(y)$ for all $t \in T$. Further, by (3) there exists $t_0 \in T$ such that $x \in A_{t_0}$, $y \notin A_{t_0}$. Then $f_{t_0}(x) = 0$, $f_{t_0}(y) = 1$. Let $x, y \in G$, $x \parallel y$. Then there exist $t_1, t_2 \in T$ such that $x \in A_{t_1}$, $y \notin A_{t_1}$, $x \notin A_{t_2}$, $y \in A_{t_2}$. Hence $f_{t_1}(x) = 0$, $f_{t_1}(y) = 1$, $f_{t_2}(x) = 1$, $f_{t_2}(y) = 0$. By (2), $(f_t; t \in T)$ is a $\mathbf{2}$ -realizer of G and (i) holds. \square

Corollary. Let G be an ordered set. Then there exists a $\mathbf{2}$ -realizer of G .

Proof. The system of all principal ideals is trivially an order base in G . \square

Definition. Let G be an ordered set. We put

$$\mathbf{2}\text{-pdim } G = \min\{\text{card } T; (f_t; t \in T) \text{ is a } \mathbf{2}\text{-realizer of } G\};$$

this cardinal is called the $\mathbf{2}$ -pseudodimension of G .

Theorem 1.2. Let G be an ordered set, let $T \neq \emptyset$ be a set. Then the following statements are equivalent:

(i) For any $t \in T$ there exists a mapping $f_t: G \rightarrow \mathbf{2}$ such that $(f_t; t \in T)$ is a $\mathbf{2}$ -realizer of G .

(ii) There exists an isomorphic embedding of G into $\mathbf{2}^T$.

Proof. (i) \Rightarrow (ii): Let (i) hold. Define for any $x \in G$ a mapping $\varphi(x): T \rightarrow \mathbf{2}$ by the rule $\varphi(x)(t) = f_t(x)$. We show that φ is an isomorphic embedding of G into $\mathbf{2}^T$. Indeed, for $x, y \in G$ we have $x \leq y \iff f_t(x) \leq f_t(y)$ for all $t \in T \iff \varphi(x)(t) \leq \varphi(y)(t)$ for all $t \in T \iff \varphi(x) \leq \varphi(y)$ in $\mathbf{2}^T$. Therefore (ii) holds.

(ii) \Rightarrow (i): Let (ii) hold and let φ be an isomorphism of G into $\mathbf{2}^T$. Let us define for any $t \in T$ a mapping $f_t: G \rightarrow \mathbf{2}$ by $f_t(x) = \varphi(x)(t)$. For $x, y \in G$ we have:

$x \leq y \iff \varphi(x) \leq \varphi(y) \iff \varphi(x)(t) \leq \varphi(y)(t)$ for all $t \in T \iff f_t(x) \leq f_t(y)$ for all $t \in T$. Thus $(f_t; t \in T)$ is a **2**-realizer of G and (i) holds. \square

Corollary. *Let G be an ordered set. Then the following cardinals are equal:*

- (i) **2**-pdim G ,
- (ii) the least cardinal m such that G can be isomorphically embedded into a set of type $\mathbf{2}^m$,
- (iii) the least cardinal n such that in G there exists an order base of cardinality n .

2. RINGS OF SETS

Let $G \neq \emptyset$ be a set, $A \subseteq G$, $x, y \in G$, $x \neq y$. We say that the set A *separates elements* x, y iff either $x \in A$, $y \notin A$ or $x \notin A$, $y \in A$.

Let \mathcal{A} be a system of subsets of G , $x, y \in G$, $x \neq y$. We say that the system \mathcal{A} *separates elements* x, y iff there exists a set $A \in \mathcal{A}$ which separates x, y .

Let \mathcal{A}, \mathcal{B} be systems of subsets of G . We say that \mathcal{A}, \mathcal{B} *similarly separate elements* of G iff for any two elements $x, y \in G$ the following holds:

$$(5) \quad \mathcal{A} \text{ separates } x, y \iff \mathcal{B} \text{ separates } x, y.$$

Example 2.1. Let $G = \{a, b, c\}$, $\mathcal{A} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$, $\mathcal{B} = \{\{a, b\}, \{a, c\}\}$. Then \mathcal{A}, \mathcal{B} similarly separate elements of G .

Indeed, as \mathcal{A} contains all one-element subsets of G , it separates any two elements of G . Thus it suffices to show that \mathcal{B} separates any two elements of G . The set $\{a, c\} \in \mathcal{B}$ separates elements a, b and the set $\{a, b\} \in \mathcal{B}$ separates both a, c and b, c .

Let \mathcal{A} be a nonempty system of sets. \mathcal{A} is called a *ring of sets* ([2], p. 12) iff $A \cup B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$. If $\bigcup\{X; X \in \mathcal{A}\} = G$ then we will say that \mathcal{A} is a ring of sets on G .

Let \mathcal{B} be a nonempty system of sets and $\bigcup\{X; X \in \mathcal{B}\} = G$. As the system of all rings of sets on G is a closure system on G , there exists the least ring of sets \mathcal{A} on G such that $\mathcal{B} \subseteq \mathcal{A}$. We say that \mathcal{B} *generates* the ring \mathcal{A} .

Theorem 2.1. *Let \mathcal{B} be a nonempty system of sets, $\bigcup\{X; X \in \mathcal{B}\} = G$, let \mathcal{A} be a ring of sets on G and let $\mathcal{B} \subseteq \mathcal{A}$. If \mathcal{B} generates \mathcal{A} then \mathcal{A}, \mathcal{B} similarly separate elements of G .*

Proof. Suppose that \mathcal{B} generates \mathcal{A} and the assertion does not hold. As $\mathcal{B} \subseteq \mathcal{A}$, there must exist $x, y \in G$ such that \mathcal{A} separates them, \mathcal{B} does not. Thus there exists $A \in \mathcal{A}$ which separates x, y and no $B \in \mathcal{B}$ separates x, y . Put

$\mathcal{C} = \{X \in \mathcal{A}; X \text{ does not separate } x, y\}$. Then $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{A}$, $\mathcal{C} \neq \mathcal{A}$ as $A \notin \mathcal{C}$. We show that \mathcal{C} is a ring of sets. Let $X, Y \in \mathcal{C}$. Then $X, Y \in \mathcal{A}$ and we have either $x, y \in X$ or $x, y \notin X$ and also either $x, y \in Y$ or $x, y \notin Y$. If $x, y \in X$ then $x, y \in X \cup Y$; the same holds if $x, y \in Y$. If both $x, y \notin X$ and $x, y \notin Y$ then $x, y \notin X \cup Y$. Thus $X \cup Y \in \mathcal{A}$ and it does not separate x, y , i.e. $X \cup Y \in \mathcal{C}$. If $x, y \notin X$ then $x, y \notin X \cap Y$ and the same if $x, y \notin Y$. If both $x, y \in X$ and $x, y \in Y$ then $x, y \in X \cap Y$. Thus $X \cap Y \in \mathcal{A}$ and $X \cap Y$ does not separate x, y , i.e. $X \cap Y \in \mathcal{C}$. Hence \mathcal{C} is a ring on G , $\mathcal{C} \supseteq \mathcal{B}$, $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{C} \neq \mathcal{A}$, a contradiction with the assumption that \mathcal{B} generates \mathcal{A} . \square

Let $G \neq \emptyset$ be a set, let \mathcal{A}, \mathcal{B} be systems of subsets of G . We will say that \mathcal{B} separates elements of G better than \mathcal{A} iff for any two elements $x, y \in G$ the following holds:

$$(6) \quad \begin{aligned} &\text{there exists } A \in \mathcal{A} \text{ such that } x \in A, y \notin A \implies \text{there exists } B \in \mathcal{B} \\ &\text{such that } x \in B, y \notin B. \end{aligned}$$

We will say that \mathcal{A}, \mathcal{B} equally separate elements of G iff \mathcal{A} separates elements of G better than \mathcal{B} and \mathcal{B} separates elements of G better than \mathcal{A} .

Example 2.2. Let $G = \{a, b, c\}$ and $\mathcal{A} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$, $\mathcal{B} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Then \mathcal{A}, \mathcal{B} equally separate elements of G .

Indeed, as \mathcal{A} contains all one-element subsets of G , it suffices to show: for any $x, y \in G$, $x \neq y$ there exists $B \in \mathcal{B}$ such that $x \in B$, $y \notin B$. This is really so: $a \in \{a, c\}$, $b \notin \{a, c\}$, $b \in \{b, c\}$, $a \notin \{b, c\}$ a.s.o.

The relation of better separating is transitive in the following sense: If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are systems of subsets of a set G such that \mathcal{B} separates elements of G better than \mathcal{A} and \mathcal{C} separates elements of G better than \mathcal{B} then \mathcal{C} separates elements of G better than \mathcal{A} . It is also reflexive. The relation of equal separating is reflexive, symmetric and transitive. Further, we have: If \mathcal{A}, \mathcal{B} equally separate elements of G , \mathcal{C}, \mathcal{D} equally separate elements of G and \mathcal{A} separates elements of G better than \mathcal{C} then \mathcal{B} separates elements of G better than \mathcal{D} .

Let \mathcal{A} be a system of subsets of a set G . We will say that \mathcal{A} is a *complete ring of sets* on G iff for any set I and any $A_i \in \mathcal{A}$ ($i \in I$) we have $\bigcup_{i \in I} A_i \in \mathcal{A}$, $\bigcap_{i \in I} A_i \in \mathcal{A}$.

Note that if \mathcal{A} is a complete ring of sets on G then $\emptyset \in \mathcal{A}$, $G \in \mathcal{A}$.

Let \mathcal{B} be a system of subsets of a set G . Then there exists the least complete ring of sets \mathcal{A} on G such that $\mathcal{B} \subseteq \mathcal{A}$; we will say that \mathcal{B} generates the complete ring \mathcal{A} .

Definition. Let \mathcal{A} be a complete ring of sets on a set G . We put

$$w(\mathcal{A}) = \min\{\text{card } \mathcal{B}; \mathcal{B} \subseteq \mathcal{A}, \mathcal{B} \text{ generates } \mathcal{A}\};$$

this cardinal will be called the *weight* of the complete ring \mathcal{A} .

Theorem 2.2. *Let \mathcal{A} be a complete ring of sets on a set G , let $\mathcal{B} \subseteq \mathcal{A}$ be a system of subsets of G . If \mathcal{B} generates \mathcal{A} then \mathcal{A}, \mathcal{B} equally separate elements of G .*

Proof is similar to the proof of Theorem 2.1. Thus let \mathcal{B} generate \mathcal{A} and suppose that there exist $x, y \in G$, $A \in \mathcal{A}$, $x \in A$, $y \notin A$ such that there exists no $B \in \mathcal{B}$ with $x \in B$, $y \notin B$. Denote $\mathcal{C} = \{X \in \mathcal{A}; \text{neither } x \in X \text{ nor } y \notin X \text{ holds}\} = \{X \in \mathcal{A}; \text{either } x \notin X \text{ or } y \in X \text{ holds}\}$. Then $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{A}$, $\mathcal{C} \neq \mathcal{A}$ as $A \notin \mathcal{C}$ and we show that \mathcal{C} is a complete ring on G . Clearly $\emptyset \in \mathcal{C}$. Let $I \neq \emptyset$ be a set and $X_i \in \mathcal{C}$ for $i \in I$. For any $i \in I$ we have $x \notin X_i$ or $y \in X_i$. If $y \in X_i$ for some $i \in I$, we have $y \in \bigcup_{i \in I} X_i$; in the other case $x \notin X_i$ for all $i \in I$ and then $x \notin \bigcup_{i \in I} X_i$. Thus $\bigcup_{i \in I} X_i \in \mathcal{A}$ and $x \notin \bigcup_{i \in I} X_i$ or $y \in \bigcup_{i \in I} X_i$, i.e. $\bigcup_{i \in I} X_i \in \mathcal{C}$. If $x \notin X_i$ for some $i \in I$, then $x \notin \bigcap_{i \in I} X_i$; otherwise $y \in X_i$ for all $i \in I$ and then $y \in \bigcap_{i \in I} X_i$. Thus $\bigcap_{i \in I} X_i \in \mathcal{A}$, $x \notin \bigcap_{i \in I} X_i$ or $y \in \bigcap_{i \in I} X_i$, i.e. $\bigcap_{i \in I} X_i \in \mathcal{C}$. Further, clearly $G \in \mathcal{C}$. Thus \mathcal{C} is a complete ring on G , $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{A}$, $\mathcal{C} \neq \mathcal{A}$, a contradiction. \square

Theorem 2.3. *Let \mathcal{A}, \mathcal{B} be complete rings of sets on a set G . Then $\mathcal{A} \subseteq \mathcal{B}$ if and only if \mathcal{B} separates elements of G better than \mathcal{A} .*

Proof. If $\mathcal{A} \subseteq \mathcal{B}$ then trivially \mathcal{B} separates elements of G better than \mathcal{A} . Suppose that \mathcal{B} separates elements of G better than \mathcal{A} . For any $x \in G$ there exists the least element $B(x) \in \mathcal{B}$ which contains x , namely $B(x) = \bigcap \{B \in \mathcal{B}; x \in B\}$. Let $A \in \mathcal{A}$ be any element, $A \neq \emptyset$. We show $A = \bigcup \{B(x); x \in A\}$. Trivially, $A \subseteq \bigcup \{B(x); x \in A\}$. Suppose the existence of an element $y \in \bigcup \{B(x); x \in A\} - A$. Then $y \notin A$ and there exists an element $z \in A$ such that $y \in B(z) = \bigcap \{B \in \mathcal{B}; z \in B\}$. As $z \in A$, $y \notin A$, there exists $B \in \mathcal{B}$ such that $z \in B$, $y \notin B$. Then $y \notin \bigcap \{B \in \mathcal{B}; z \in B\} = B(z)$, which is a contradiction. Thus $A = \bigcup \{B(x); x \in A\}$, which implies $A \in \mathcal{B}$. Hence $\mathcal{A} \subseteq \mathcal{B}$. \square

Corollary. *Let \mathcal{A}, \mathcal{B} be complete rings of sets on a set G . Then $\mathcal{A} = \mathcal{B}$ iff \mathcal{A}, \mathcal{B} equally separate elements of G .*

Theorem 2.4. *Let $\mathcal{B}_1, \mathcal{B}_2$ be systems of subsets of a set G , let $\mathcal{A}_1, \mathcal{A}_2$ be complete rings of sets on G and let \mathcal{B}_1 generate \mathcal{A}_1 , \mathcal{B}_2 generate \mathcal{A}_2 . Then $\mathcal{A}_1 \subseteq \mathcal{A}_2$ iff \mathcal{B}_2 separates elements of G better than \mathcal{B}_1 .*

Proof. If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ then \mathcal{A}_2 separates elements of G better than \mathcal{A}_1 . By Theorem 2.2, $\mathcal{A}_1, \mathcal{B}_1$ equally separate elements of G , and $\mathcal{A}_2, \mathcal{B}_2$ equally separate elements of G . Thus \mathcal{B}_2 separates elements of G better than \mathcal{B}_1 . If \mathcal{B}_2 separates

elements of G better than \mathcal{B}_1 then \mathcal{A}_2 separates elements of G better than \mathcal{A}_1 . By Theorem 2.3 we have $\mathcal{A}_1 \subseteq \mathcal{A}_2$. \square

As a corollary, we obtain

Theorem 2.5. *Let $\mathcal{B}_1, \mathcal{B}_2$ be systems of subsets of a set G . Then $\mathcal{B}_1, \mathcal{B}_2$ generate the same complete ring of sets on G iff $\mathcal{B}_1, \mathcal{B}_2$ equally separate elements of G .*

Further, we have

Theorem 2.6. *Let $G \neq \emptyset$ be a set, \mathcal{A} a complete ring of sets on G and $\mathcal{B} \subseteq \mathcal{A}$ a system of subsets of G . Then \mathcal{B} generates \mathcal{A} iff \mathcal{A}, \mathcal{B} equally separate elements of G .*

P r o o f. The necessity of the given condition follows from Theorem 2.2, its sufficiency follows from Theorem 2.5, as trivially \mathcal{A} generates \mathcal{A} . \square

Let G be an ordered set. Then the system of all its ideals is a complete ring of sets on G . Now, we prove

Theorem 2.7. *Let G be an ordered set, let \mathcal{A} be the complete ring of all its ideals and let \mathcal{B} be some system of its ideals. Then \mathcal{B} generates \mathcal{A} iff \mathcal{B} is an order base in G .*

P r o o f. 1. Let \mathcal{B} generate \mathcal{A} . By Theorem 2.6, \mathcal{A}, \mathcal{B} equally separate elements of G . Let $x, y \in G, x \not\leq y$. Then $(y) \in \mathcal{A}, y \in (y), x \notin (y)$. Thus there exists $B \in \mathcal{B}$ such that $y \in B, x \notin B$. By (4), \mathcal{B} is an order base in G .

2. Let \mathcal{B} be an order base in G . Let $x, y \in G, A \in \mathcal{A}$ be such elements that $x \in A, y \notin A$. As A is an ideal in G , necessarily $y \not\leq x$. By (4) there exists $B \in \mathcal{B}$ such that $x \in B, y \notin B$. Thus \mathcal{A}, \mathcal{B} equally separate elements of G and by Theorem 2.6 \mathcal{B} generates \mathcal{A} . \square

A similar result is proved in [11], Hilfsatz 3.2.

Corollary. *Let G be an ordered set. Then the following cardinals are equal:*

- (i) $2\text{-pdim } G$,
- (ii) the least cardinal m such that G can be isomorphically embedded into a set of type 2^m ,
- (iii) the least cardinal n such that in G there exists an order base of cardinality n ,
- (iv) $w(\mathcal{A})$ where \mathcal{A} is the complete ring of all ideals in G .

3. DENSE SUBSETS

Let G be an ordered set and $H \subseteq G$. We will say that H is *dense* in G iff the following holds:

- (7) (i) $x, y \in G, x < y \implies$ there exist $u, v \in H$ such that $x \leq u < v \leq y$,
(ii) $x, y \in G, x \parallel y$ and $z > y$ for any $z \in G, z > x \implies x \in H$.

The condition (i) was formulated already in [4], p. 89, for linearly ordered sets, the condition (ii) can be found—in a modified form—in [9].

Clearly, any ordered set is dense in itself.

Definition. Let G be an ordered set. We put

$$\text{sep } G = \min\{\text{card } H; H \subseteq G \text{ is dense in } G\};$$

this cardinal will be called the *separability* of G .

Lemma 3.1. *Let G be an ordered set, let $H \subseteq G$ be dense in G and let $x, y \in G$. If $u \geq y$ for any $u \in H, u \geq x$, then $x \geq y$.*

Proof. Let the condition be satisfied. If $x \in H$, then $x \geq y$ for $x \geq x$. Thus let $x \notin H$. Assume $x \parallel y$. If $z \in G, z > x$ then by (7) there exist $u, v \in H$ such that $x \leq u < v \leq z$; by assumption then $u \geq y$ and thus $z > y$. By (ii) in (7) we have $x \in H$, a contradiction. Thus the elements x, y must be comparable. If $x < y$, then there exist $u, v \in H$ such that $x \leq u < v \leq y$ so that $u \geq x, u \not\geq y$, a contradiction. Hence $x \geq y$. □

Theorem 3.1. *Let G be an ordered set, let $H \subseteq G$ be dense in G . Then $((u]; u \in H)$ is an order base in G .*

Proof. Let $x, y \in G, x < y$. By (7) there exist $u, v \in H$ such that $x \leq u < v \leq y$. Then $x \in (u], y \notin (u]$ and condition (i) from (3) is satisfied. Let $x, y \in G, x \parallel y$. If $x, y \in H$ then $x \in (x], y \notin (x], x \notin (y], y \in (y]$. Suppose $x \notin H, y \in H$. Then $x \notin (y], y \in (y]$. If $u \geq y$ for any $u \in H$ with $u \geq x$, then by Lemma 3.1 $x \geq y$, a contradiction. Thus there exists $u \in H$ such that $u \geq x, u \not\geq y$ and then $x \in (u], y \notin (u]$. Similarly in the case $x \in H, y \notin H$. Finally, let $x \notin H, y \notin H$. If $u \geq y$ for any $u \in H$ with $u \geq x$, then $x \geq y$ by Lemma 3.1, a contradiction. Hence there exists $u \in H$ such that $u \geq x, u \not\geq y$; then $x \in (u], y \notin (u]$. For the same reason there exists $v \in H$ such that $v \geq y, v \not\geq x$ and then $x \notin (v], y \in (v]$. Thus the condition (ii) from (3) is satisfied and $((u]; u \in H)$ is an order base in G . □

Corollary. *Let G be an ordered set. Then $\mathbf{2}\text{-pdim } G \leq \text{sep } G$.*

By examples we can show that $\mathbf{2}\text{-pdim } G = \text{sep } G$ need not hold. If, e.g., G is a finite chain with m elements, then $\text{sep } G = m$; as the cardinal power $\mathbf{2}^{m-1}$ contains an m -element chain and $\mathbf{2}^{m-2}$ contains no such chain, we see that $\mathbf{2}\text{-pdim } G = m - 1$. If G is a finite m -element antichain, then $\text{sep } G = m$ and $\mathbf{2}\text{-pdim } G = n$ where n is the least positive integer with $\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq m$, for a maximal antichain in $\mathbf{2}^n$ contains $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements ([13], [6]). If G is an infinite antichain, $\text{card } G = m$ then $\text{sep } G = m$ and $\mathbf{2}\text{-pdim } G = n$ where n is the least cardinal with $2^n \geq m$, for the cardinal power $\mathbf{2}^n$ (n infinite) contains an antichain of cardinality 2^n ([12], p. 450, Theorem 1, [8], Theorem 8). We show that if G is an infinite chain then $\mathbf{2}\text{-pdim } G = \text{sep } G$. In contrast to all preceding theorems and lemmas, for a proof of this assertion we need the axiom of choice (AC).

Theorem 3.2. (AC). *Let G be an infinite chain. Then $\mathbf{2}\text{-pdim } G = \text{sep } G$.*

Proof. It suffices to show $\text{sep } G \leq \mathbf{2}\text{-pdim } G$. Let $\mathbf{2}\text{-pdim } G = m$; clearly $m \geq \aleph_0$. By Corollary to Theorem 1.2, in G there exists an order base $(A_t; t \in T)$ with $\text{card } T = m$. Denote $T_0 = \{t \in T; A_t \text{ contains the greatest element, } G - A_t \text{ contains the least element}\}$, $T_{12} = \{(t_1, t_2) \in T^2; A_{t_2} - A_{t_1} \neq \emptyset\}$, $H_0 = \bigcup_{t \in T_0} \{\max A_t, \min(G - A_t)\}$ and let $H_{12} \subseteq \bigcup_{(t_1, t_2) \in T_{12}} (A_{t_2} - A_{t_1})$ be such a set that $H_{12} \cap (A_{t_2} - A_{t_1})$ is a one-point set for any $(t_1, t_2) \in T_{12}$. Put $H = H_0 \cup H_{12}$; then $\text{card } H \leq m$ and we show that H is dense in G . First, we show: If $x, y \in G$, $x \prec y$, then $x, y \in H$. Indeed, there exists $t \in T$ such that $x \in A_t$, $y \notin A_t$. Then necessarily $x = \max A_t$, $y = \min(G - A_t)$ so that $x, y \in H$. Now let $x, y \in G$, $x < y$. If $x \prec y$ or if there exists $z \in G$ such that $x < z \prec y$, then $x, y \in H$ and condition (i) from (7) is satisfied. Let there exist $w, z \in G$ such that $x < w < z < y$. By (i) in (3) there exist $t_1, t_2, t_3 \in T$ such that $x \in A_{t_1}$, $w \notin A_{t_1}$, $w \in A_{t_2}$, $z \notin A_{t_2}$, $z \in A_{t_3}$, $y \notin A_{t_3}$. Thus $A_{t_2} - A_{t_1} \neq \emptyset$, $A_{t_3} - A_{t_2} \neq \emptyset$ and there exist $u, v \in H$ such that $u \in A_{t_2} - A_{t_1}$, $v \in A_{t_3} - A_{t_2}$. Then $x < u < v < y$ and condition (i) in (7) is satisfied. Hence H is dense in G . \square

Corollary. *Let G be an infinite chain. Then the following cardinals are equal:*

- (i) $\mathbf{2}\text{-pdim } G$,
- (ii) the least cardinal m such that G can be isomorphically embedded into a set of type $\mathbf{2}^m$,
- (iii) the least cardinal n such that in G there exists an order base of cardinality n ,
- (iv) $w(\mathcal{A})$ where \mathcal{A} is the complete ring of all ideals in G ,
- (v) $\text{sep } G$.

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