Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 1, 173–176

Persistent URL: `http://dml.cz/dmlcz/127407`

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WEAK BAER MODULES LOCALIZED WITH RESPECT TO A TORSION THEORY

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(Received September 1, 1995)

In [1], Fuchs and Viljoen described the modules B over a valuation domain R such that $\operatorname{Ext}_R(B, X) = 0$ for all bounded torsion and all divisible modules X. This weak form of Baer's splitting problem was considered in [4], [5], [6], and [7] for arbitrary torsion theories over an associative ring. As in the valuation ring case, modules playing the role of B in the "Ext *condition*" above are called B^*-modules. (A precise definition is given later.) Under the hypothesis that τ is of finite type (i.e., the filter associated with τ has a cofinal subset of finitely generated left ideals), results in [5] (and [6]) gave characterizations of torsion theories τ whose τ-torsionfree modules are (flat) B^*-modules. The main purpose of this note is to prove a result (Theorem 2) that allows us to remove the restrictive overall hypothesis that τ is of finite type from all the main results of [5] and [6].

Let R be an associative ring with 1, let τ be a torsion theory of left R-modules and let $\mathcal{L}_\tau$ be the filter of left ideals of R associated to τ. By $\tau(M)$ we denote the τ-torsion submodule of a module M, and by Q_τ we denote the localization of R relative to τ; Q_τ has a natural ring structure that extends the ring structure of $R/\tau(R)$. For the basic properties of τ and other torsion theoretic terms used in this note, see Golan [2].

Recall that a left R-module E is called *τ-injective* if $\operatorname{Ext}_R(T, E) = 0$ for each τ-torsion module T. As in [7], a module D is called *τ-divisible* if D is a homomorphic image of a direct sum of τ-injective modules. A module M is called a *D^*-module* if $\operatorname{Ext}_R(M, D) = 0$ for each τ-divisible module D. A module M is said to have *τ-bounded order* if M is a submodule of a module N with a set of generators annihilated by a left ideal I in $\mathcal{L}_\tau$. A module M is called a *B^*-module* if $\operatorname{Ext}_R(M, X) = 0$ for each τ-divisible X and each X with τ-bounded order.

[1] This paper was supported in part by the TGRC-KOSEF, 1994, and BSRI-95-1402.

Before stating our main result, we need the following minor generalization of [7, Lemma 2.6].

Lemma 1. *If a Q_τ-module B is a D^*-module, then $Q_\tau \otimes_R B \cong B$ and B is a projective Q_τ-module.*

Proof. Let $m\colon Q_\tau \otimes_R B \longrightarrow B$ be the multiplication map. If $k = \sum q_i \otimes b_i \in \ker m$, then $\bigcap(R/\tau(R)\colon q_i)k = 0$; hence $\ker m \subseteq \tau(Q_\tau \otimes_R B)$. But $Q_\tau \otimes_R B$ is a projective Q_τ-module by [7, Lemma 2.5]. Consequently, $Q_\tau \otimes_R B$ is τ-torsionfree, and hence $\ker m = 0$. □

As in [2], we say that τ is an *exact* torsion theory if the localization functor for τ is exact, and we say that τ is *perfect* if the localization of each module M is given by $Q_\tau \otimes_R M$.

We can now give our main result.

Theorem 2. *If every τ-torsionfree Q_τ-module is a D^*-module, then τ is a perfect torsion theory and Q_τ is a semisimple artinian ring.*

Proof. Since every τ-torsionfree Q_τ-module is assumed to be a D^*-module, then every τ-torsionfree Q_τ-module is projective as a Q_τ-module by Lemma 1. Since $\tau(Q_\tau) = 0$, it follows that every nonsingular left Q_τ-module must be projective. Hence Q_τ is a left nonsingular ring, and thus Q_τ is a left noetherian ring by [3, Theorem 5.23].

Next we show that τ is an exact torsion theory. Let E be a τ-torsionfree τ-injective module, and consider the exact sequence

$$0 \longrightarrow \ker f \longrightarrow E \xrightarrow{f} F \longrightarrow 0,$$

where F is τ-torsionfree. Since $\ker f$ must be τ-torsionfree and τ-injective in this situation, then $\ker f$ is a Q_τ-module by [2, Proposition 26.33]. Hence F is a Q_τ-module. By Lemma 1, F is a projective Q_τ-module; so, as a direct summand of E, F must be τ-injective. Thus τ is exact by [2, Proposition 44.1].

From [2, Corollary 45.6 and Theorem 45.1] and the two preceding paragraphs, we see that τ is perfect. But for a perfect torsion theory, every Q_τ-module is τ-torsionfree; so in this case, every Q_τ-module is projective. Therefore, Q_τ is a semisimple artinian ring. □

In [5] the question, "When is every τ-torsionfree module a B^*-module?" is considered. Similarly, in [6] the question, "When is every τ-torsionfree module a flat B^*-module?" is studied. These questions are answered under the hypothesis that τ

is of finite type. The answers to these questions show that τ must be closely related to the Goldie torsion theory τ_g; the τ_g-torsionfree modules are precisely the nonsingular modules. The finiteness property of τ is used to prove the following key lemma of [5]:

[5, Lemma 4.] Let τ be of finite type. If every τ-torsionfree module is a B^*-module, then Q_τ is a semisimple artinian ring and τ induces the Goldie torsion theory on $R/\tau(R) - mod$.

When Q_τ is semisimple and τ is perfect, then τ automatically induces the Goldie torsion theory on $R/\tau(R) - mod$. Hence Theorem 2 shows that [5, Lemma 4] is true without the hypothesis that τ is of finite type. Since [5, Lemma 4] is the only source of the use of the hypothesis that τ is of finite type throughout [5] and [6], all of the main results of [5] and [6] are true without the assumption that τ is of finite type. (In results on the Goldie theory, such as [5, Proposition 11 and Theorem 12] or [6, Theorem 10], this means that the overall hypothesis that R has finite left uniform dimension is not needed.)

Example 3. Let $\mathbb{Z}$ denote the integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{R}$ the real numbers. Consider R to be either matrix ring:

$$R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix} \quad \text{or} \quad R = \begin{pmatrix} \mathbb{Z} & \mathbb{R}[x] \\ 0 & \mathbb{Q} \end{pmatrix}.$$

The old versions of the results in [5] and [6] do not apply to Goldie torsion theory for R, as R does not have finite left uniform dimension. But since R has many properties similar to the matrix rings in [5, Theorem 18] and [6, Theorem 14], one might have wondered if every τ_g-torsionfree R-module is a B^*-module. Our Theorem 2 shows immediately that this is not the case.

In addition to generalizing results from [5] and [6], we illustrate the use of Theorem 2 with the following application. We use $hd_R M$ to denote the homological dimension of a left R-module M.

Corollary 4. *If $\tau(R) = 0$, the following statements are equivalent:*

(1) *Every τ-torsionfree Q_τ-module is a D^*-module,*
(2) *Every Q_τ-module is a D^*-module,*
(3) *$hd_R Q_\tau \leqslant 1$ and Q_τ is a semisimple artinian ring.*

P r o o f. (1) $\iff$ (2). From Theorem 2, Q_τ is semisimple artinian; so every Q_τ-module must be τ-torsionfree.

(1) $\Longrightarrow$ (3). This is immediate from Theorem 2 and [7, Lemma 2.1].

(3) $\Longrightarrow$ (1). Let B be a Q_τ-module, and let D be τ-divisible. We need to show that $\operatorname{Ext}_R(B, D) = 0$. Since Q_τ is semisimple artinian, we may assume that $B = Q_\tau$.

Let $\bigoplus E_\alpha \longrightarrow D$ be an epimorphism, where each E_α is τ-injective. Let F_α be a free R-module with $F_\alpha \longrightarrow E_\alpha$ an epimorphism. Since $\tau(R) = 0$, then $F_\alpha \subseteq \bigoplus Q_\tau$; so the τ-injectivity of each E_α gives rise to the epimorphism

$$\bigoplus_\alpha \Bigl(\bigoplus Q_\tau\Bigr) \longrightarrow \bigoplus E_\alpha \longrightarrow D.$$

Since $hd_R Q_\tau \leqslant 1$, we have an exact sequence

$$\operatorname{Ext}_R(Q_\tau, \bigoplus Q_\tau) \longrightarrow \operatorname{Ext}_R(Q_\tau, D) \longrightarrow 0.$$

But $(Q_\tau)_R$ is a flat and $Q_\tau \otimes_R Q_\tau \cong Q_\tau$; so $\operatorname{Ext}_R(Q_\tau, \bigoplus Q_\tau) \cong \operatorname{Ext}_{Q_\tau}(Q_\tau, \bigoplus Q_\tau) = 0$. Therefore, $\operatorname{Ext}_R(Q_\tau, D) = 0$, as desired. $\square$

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