Czechoslovak Mathematical Journal

Dao Rong Tong

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Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 2, 243–252

Persistent URL: http://dml.cz/dmlcz/127414

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THE $\mathscr{A}r$-FREE PRODUCTS OF ARCHIMEDEAN l-GROUPS

Dao-Rong Ton, Nanjing

(Received July 10, 1995)

Abstract. The objective of this paper is to give two descriptions of the $\mathscr{A}r$-free products of archimedean ℓ-groups and to establish some properties for the $\mathscr{A}r$-free products. Specifically, it is proved that $\mathscr{A}r$-free products satisfy the weak subalgebra property.

1. Introduction

We use the standard terminology and notation of [1, 3, 4]. All groups in this paper are abelian. The group operation of an l-group is written by additive notation. We use $\mathbb{N}$ and $\mathbb{Z}$ for the natural numbers and the integers, respectively. The symbol $\oplus$ refers to the group theoretic direct sum while $\boxplus$ denotes the cardinal sum of l-groups.

A po-group is a partially ordered group $[G, P]$ where $P = \{x \in G \mid x \geqslant 0\}$ is the positive semigroup of G. A totally ordered group is called an 0-group. Let G and H be two po-groups. A map φ from G into H is called a po-group homomorphism, if φ is a group homomorphism and $x \geqslant y$ implies $\varphi(x) \geqslant \varphi(y)$ for any x, $y \in G$. A po-group homomorphism φ is called a po-group isomorphism if φ is an injection and φ^{-1} is also a po-group homomorphism from $\varphi(G)$ to G.

Let $\mathscr{U}$ be a class of l-groups and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathscr{U}$. The $\mathscr{U}$-free product of G_λ is an l-group $G \in \mathscr{U}$, denoted by ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$, together with a family of injective l-homomorphisms $\alpha_\lambda\colon G_\lambda \to G$ (call coprojections) such that

1. $\bigcup\limits_{\lambda\in\Lambda} \alpha_\lambda(G_\lambda)$ generates G as an l-group;

2. if $H \in \mathscr{U}$ and $\{\beta_\lambda\colon G_\lambda \to H \mid \lambda \in \Lambda\}$ is a family of l-homomorphisms, then there exists a (necessarily) unique l-homomorphism $\gamma\colon G \to H$ satisfying $\beta_\lambda = \gamma\alpha_\lambda$ for all $\lambda \in \lambda$.

We often identify each free factor G_λ with its image $\alpha_\lambda(G_\lambda)$ in ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$ and thus view each G_λ as an l-subgroup of ${}^{\mathscr{U}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$. By the Sikorski existence theorem [6],

$\mathscr{U}$-free products always exist if $\mathscr{U}$ is a class of l-groups closed under l-subgroups and direct products. Consequently, if $\mathscr{U}$ is a variety of l-groups, $\mathscr{U}$-free products always exist. Let $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{A}$ be the varieties of all l-groups, representable l-groups and abelian l-groups, respectively. In [10–13] Powell and Tsinakis have given several descriptions and some properties for free products in the varieties $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{A}$. Let $\mathscr{A}r$ be the class of all archimedean l-groups. Clearly, $\mathscr{A}r$ is closed under taking l-subgroups and direct products. Hence $\mathscr{A}r$-free products always exist. In this paper we will give two descriptions of the $\mathscr{A}r$-free products of archimedean l-groups and discuss some of their properties.

2. Descriptions for $\mathscr{A}r$-free products

First of all we consider $\mathscr{A}r$-free products of archimedean 0-groups (which, by Hölder's Theorem, are subgroups of the additive reals).

We recall some definitions. Let $\mathscr{U}$ be a class of l-groups and $[G,P]$ a po-group. The $\mathscr{U}$-free extension of G is an l-group $\mathscr{F}_{\mathscr{U}}(G)\in\mathscr{U}$ for which there exists an injective po-group homomorphism $\alpha\colon\ G\to\mathscr{F}_{\mathscr{U}}(G)$ such that

1. $\alpha(G)$ generates $\mathscr{F}_{\mathscr{U}}(G)$ as an l-group;
2. if $H\in\mathscr{U}$ and $\beta\colon\ G\to H$ is a po-group homomorphism, then there exists an l-homomorphism $\gamma\colon\ \mathscr{F}_{\mathscr{U}}(G)\to H$ satisfying $\gamma\alpha=\beta$.

The $\mathscr{U}$-free extension $\mathscr{F}_{\mathscr{U}}(G)$ of a po-group $[G,P]$ is called the $\mathscr{U}$-free l-group generated by $[G,P]$, denoted by $\mathscr{F}_{\mathscr{U}}\big([G,P]\big)$, if the mapping α in the above definition is a po-group isomorphism between G and $\alpha(G)$. By Grätzer existence theorem on a free algebra generated by a partial algebra (Theorem 28.2 of [5]) we have.

Lemma 2.1. *There exists an $\mathscr{A}r$-free l-group $\mathscr{F}_{\mathscr{A}r}\big([G,P]\big)$ generated by a po-group $[G,P]$ if and only if $[G,P]$ is a po-group isomorphic to a po-subgroup of an archimedean l-group.*

Let $\{R_\lambda\mid\lambda\in\Lambda\}$ be a family of archimedean 0-groups. $H=\oplus_{\lambda\in\Lambda}R_\lambda$ is the abelian group free product of this family. Let H^+ be the set of all sums of conjugates in H of $\bigcup\limits_{\lambda\in\Lambda}R_\lambda^+$. Then $[H,H^+]=\mathop{\boxplus}\limits_{\lambda\in\Lambda}R_\lambda$ and $\mathop{\boxplus}\limits_{\lambda\in\Lambda}R_\lambda\in\mathscr{A}r$. By Theorem 11.2.4 of [5] and the above Lemma 2.1 we see that

$$^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda}R_\lambda\cong\mathscr{F}_{\mathscr{A}r}\big(\mathop{\boxplus}_{\lambda\in\Lambda}R_\lambda\big).\tag{1}$$

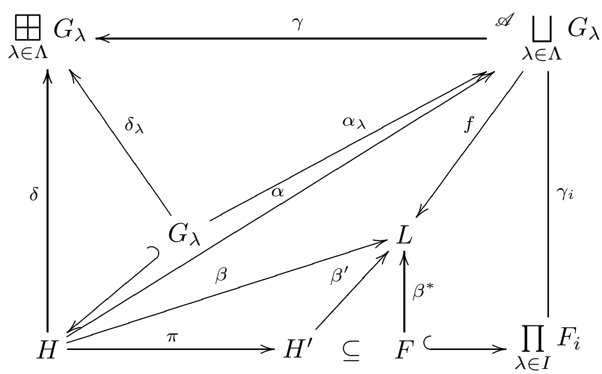
We now consider the description for $\mathscr{A}r$-free products of arbitrary archimedean l-groups. Let $\{G_\lambda\mid\lambda\in\Lambda\}$ be a family of archimedean l-groups. Then the $\mathscr{A}$-free product $G={}^{\mathscr{A}}\bigsqcup\limits_{\lambda\in\Lambda}G_\lambda$ exists with the coprojections α_λ, and we have several descriptions for G. Let $H = \oplus_{\lambda\in\Lambda} G_\lambda$ be the abelian group free product of G_λ. By the proof of Theorem 2.4 of [11] there exists a group isomorphism $\alpha\colon H \to \alpha(H) \subseteq {}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ such that the restriction of α onto each individual G_λ is α_λ. $G_\lambda(\lambda \in \Lambda)$ can be naturally embedded into the cardinal sum $\boxplus_{\lambda\in\Lambda} G_\lambda$ as l-groups with the embedding $\delta_\lambda\colon G_\lambda \to \boxplus_{\lambda\in\Lambda} G_\lambda$. Hence there exists a group homomorphism $\delta\colon H \to \boxplus_{\lambda\in\Lambda} G_\lambda$ which extends $\delta_\lambda(\lambda \in \Lambda)$, and there exists an l-homomorphism $\gamma\colon G \to \boxplus_{\lambda\in\Lambda} G_\lambda$ such that $\gamma\alpha_\lambda = \delta_\lambda(\lambda \in \Lambda)$. We declare two $\mathscr{A}r$-surjections $\beta_i\colon G \to F_i(i = 1, 2)$ to be equivalent if there exists an l-isomorphism $\gamma\colon F_1 \to F_2$ such that $\gamma\beta_1 = \beta_2$. Let

$$D = \{\gamma_i\colon G \to F_i \mid i \in I\}$$

be the set of representatives of equivalence classes of Ar-surjections out of G. Thus, $\gamma \in D$ and D is not empty. For each $\lambda \in \Lambda$ and each $i \in I$, $\gamma_i\alpha_\lambda$ is an l-homomorphism of G_λ into F_i. The direct product $\prod_{i\in I} F_i$ is an archimedean l-group. For each $\lambda \in \Lambda$, let π_λ be the natural l-homomorphism of G_λ onto the l-subgroup G'_λ of $\prod_{i\in I} F_i$. That is,

$$\pi_\lambda(g_\lambda) = \big(\ldots, \gamma_i\alpha_\lambda(g_\lambda), \ldots\big)$$

for $g_\lambda \in G_\lambda$. Let H be the subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} G'_\lambda$. Let π be the group homomorphism of H onto H' which extends each π_λ $(\lambda \in \Lambda)$.



That is,

$$\pi(h) = \big(\ldots, \gamma_i\alpha(h), \ldots\big)$$

for $h \in H$. Because $\gamma \in D$ and each $\delta_\lambda(\lambda \in \Lambda)$ is an l-isomorphism, π is a group isomorphism of H onto H' and π_λ is an l-isomorphism for $\lambda \in \Lambda$. Let F be the sublattice of $\prod_{i\in I} F_i$ generated by H'. For each $h \in H$, let $h' = \pi(h)$. Since $\prod_{i\in I} F_i$ is a distributive lattice,

$$F\Big\{ \bigvee_{j\in J} \bigwedge_{k\in K} h'_{jk} \mid h_{jk} \in H,\ J \text{ and } K \text{ finite}\Big\}.$$

Thus we have the following result.

Proposition 2.2. *Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of archimedean l-groups. Then the $\mathscr{A}r$-free product ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ is the sublattice F of the direct product $\prod_{i\in I} F_i$ generated by the group isomorphic image H' of the abelian group free product H of G_λ, where $D = \{\gamma_i\colon G \to F_i \mid i \in I\}$ is the set of representatives of the equivalence classes of all $\mathscr{A}r$-surjections out of ${}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$.*

P r o o f. Suppose that $L \in \mathscr{A}r$ and that $\{\beta_\lambda\colon G_\lambda \to L \mid \lambda \in \Lambda\}$ is a family of l-homomorphisms. We shall show that there exists a unique l-homomorphism $\beta^*\colon F \to L$ such that $\beta^*\pi_\lambda = \beta_\lambda$ for each $\lambda \in \Lambda$. By the universal property of the group free product, there exists a group homomorphism $\beta\colon H \to L$ which extends each $\beta_\lambda(\lambda \in \Lambda)$. For any $h' = \pi(h) \in H'$, put

$$\beta'(h') = \beta(h).$$

Then β' is a group homomorphism of H into L. By the universal property of the $\mathscr{A}$-free product, there exists a unique l-homomorphism $f\colon G \to L$ such that $\beta_\lambda = f\alpha_\lambda$ for each $\lambda \in \Lambda$. Then $f\alpha = \beta'\pi = \beta$. By Lemma 11.3.1 of [4] we need only to show that for each finite subset $\{h_{jk} \mid j \in J,\ k \in K\} \subseteq H$, $\bigvee_{j\in J} \bigwedge_{k\in K} \beta'\pi(h_{jk}) \neq 0$ implies $\bigvee_{j\in J} \bigwedge_{k\in K} \pi(h_{jk}) \neq 0$. In fact, $\bigvee_{j\in J} \bigwedge_{k\in K} f\alpha(h_{jk}) \neq 0$. Because $f \in D$, $\bigvee_{j\in J} \bigwedge_{k\in K} \gamma_i\alpha(h_{jk}) \neq 0$ for some $i \in I$. So

$$\bigvee_{j\in J} \bigwedge_{k\in K} \pi(h_{jk}) = \bigvee_{j\in J} \bigwedge_{k\in K} (\ldots, \gamma_i\alpha(h_{jk}), \ldots) = (\ldots, \bigvee_{j\in J} \bigwedge_{k\in K} \gamma_i\alpha(H_{jk}), \ldots) \neq 0.$$

Therefore β' can be uniquely extended to an l-homomorphism $\beta^*\colon F \to L$. □

Below we will give another description for $\mathscr{A}r$-free products. Given $G \in \mathscr{A}r$, an l-ideal K of G will be called an archimedean kernel if $G/K \in \mathscr{A}r$. Let $AK(G)$ be the set of all archimedean kernels of G. For any $0 \neq g \in G$, there exists an archimedean kernel K_g of G such that $g \overline{\in} K_g$. K_g is called an AK excluding g. For example, 0 is always an AK excluding $g \neq 0$, because $G \in \mathscr{A}r$.

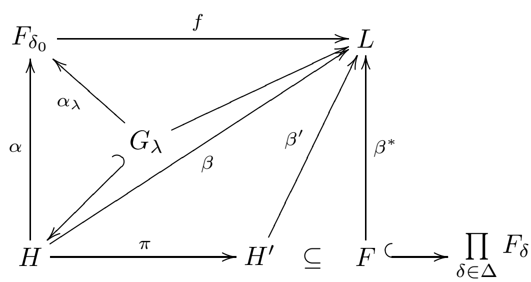
Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be family of l-groups in $\mathscr{A}r$. Let

$$\Gamma = \bigcup_{\lambda\in\Lambda} AK(G_\lambda)$$

and consider the set Δ of all choice functions $\delta\colon\ \Lambda \to \Gamma$. For each $\delta \in \Delta$ and each $\lambda \in \mathscr{L}$, let $K_{\delta(\lambda)} \in AK(G_\lambda)$. Then $\boxplus_{\lambda\in\Lambda} \big(G_\lambda/K_{\delta(\lambda)}\big) \in \mathscr{A}r$. $\boxplus_{\lambda\in\Lambda} \big(G_\lambda/K_{\delta(\lambda)}\big)$ can be also naturally viewed as a po-group. By Lemma 2.1 there exists an $\mathscr{A}r$-free l-group $F_\delta = \mathscr{F}_{\mathscr{A}r}\big(\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta(\lambda)})\big)$ generated by the po-group $\boxplus_{\lambda\in\Lambda}\big(G_\lambda/K_{\delta(\lambda)}\big)$ for each $\delta \in \Delta$. Then $\prod_{i\in I} F_\delta$ is an archimedean l-group. We denote by ϱ_δ the projection of $\prod_{i\in I} F_\delta$ onto F_δ for each $\delta \in \Delta$. For each $\lambda \in \Lambda$, let π_λ be the l-homomorphism of G_λ onto the l-subgroup G'_λ of $\prod_{i\in I} F_\delta$ satisfying $\varrho_\delta\pi_\lambda(g_\lambda) = g_\lambda + K_{\delta(\lambda)}$ for $g_\lambda \in G_\lambda$. π_λ is an l-isomorphism for each $\lambda \in \Lambda$. In fact, for $0 \neq g_\lambda$ we take $K_{\delta(\lambda)} = K_{g_\lambda}$, an AK excluding g_λ. Then $g_\lambda + K_{g_\lambda} \neq K_{g_\lambda}$, and so $\varrho_\delta\pi_\lambda(g_\lambda) \neq 0$. Let H' be the subgroup of $\prod_{i\in I} F_\delta$ generated by $\bigcup_{\lambda\in\Lambda} G_\lambda$ and let π be the group homomorphism of $H = \oplus_{\lambda\in\Lambda} G_\lambda$ onto H' which extends each $\pi_\lambda(\lambda \in \Lambda)$. It is easy to see that π is a group isomorphism. Then we have the following description of $\mathscr{A}r$-free products.

Theorem 2.3. *Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of archimedean l-groups. Then the $\mathscr{A}r$-free product ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ is the sublattice F of the direct product $\prod_{\delta\in\Delta} F_\delta$ generated by the group isomorphic image H' of the group free product H of G_λ.*

Proof. We show the universal property. Suppose that $L \in \mathscr{A}r$ and that $\{\beta_\lambda\colon\ G_\lambda \to L \mid \lambda \in \Lambda\}$ is a family of l-homomorphisms. We shall show that there exists a unique l-homomorphism $\beta^*\colon\ F \to L$ such that $\beta^*\pi_\lambda = \beta_\lambda$. Clearly, there exists a group homomorphism $\beta\colon\ H \to L$ which extends each



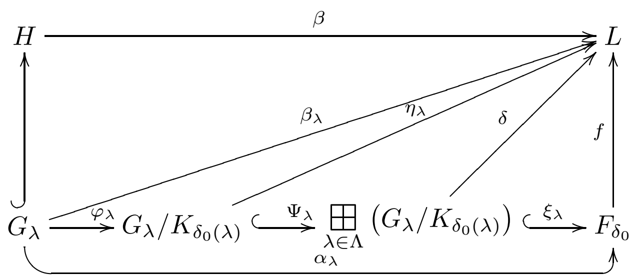
$\beta_\lambda(\lambda \subset \Lambda)$. For any $h' = \pi(h) \in H'(h \in H)$, put

$$\beta'(h') = \beta(h).$$

By Lemma 11.3.1 of [4] we need only to show that for each finite subset $\{h_{jk} \mid j \in J, k \in K\} \subseteq H$, $\bigvee_{j\in J}\bigwedge_{k\in K} \beta'\pi(h_{jk}) \neq 0$ implies $\bigvee_{j\in J}\bigwedge_{k\in K} \pi(h_{jk}) \neq 0$. For each $\lambda \in \Lambda$, put $K_{\delta_0(\lambda)} = \beta_\lambda^{-1}(0)$ and $F_{\delta_0} = \mathscr{F}_{\mathscr{A}r}\big(\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta_0(\lambda)})\big)$. Let φ_λ be the natural l-homomorphism of G_λ onto $G_\lambda/K_{\delta_0(\lambda)}$, let η_λ be the l-isomorphism of $G_\lambda/K_{\delta_0(\lambda)}$ into L such that $\eta_\lambda\varphi_\lambda = \beta_\lambda$, let Ψ_λ be the embedding of $G_\lambda/K_{\delta_0(\lambda)}$ into $\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta_0(\lambda)})$, let γ be the group homomorphism of $\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta_0(\lambda)})$ into L such that $\gamma\Psi_\lambda = \eta_\lambda$ (γ is also a po-group homomorphism), and let ξ_λ be the po-group isomorphism of $\boxplus_{\lambda\in\Lambda}(G_\lambda/K_{\delta_0(\lambda)})$ into F_{δ_0}. Then there exists an l-homomorphism f of F_{δ_0} into L such that $f\xi_\lambda = \gamma$. Let $\alpha_\lambda = \xi_\lambda\Psi_\lambda\varphi_\lambda$. Then

$$\beta_\lambda = \eta_\lambda\varphi_\lambda = \gamma\Psi_\lambda\varphi_\lambda = f\xi_\lambda\Psi_\lambda\varphi_\lambda = f\alpha_\lambda$$

and $\varrho_{\delta_0}\pi_\lambda = \alpha_\lambda$ for each $\lambda \in \Lambda$. Let α be the unique group homomorphism



of H into F_{δ_0} which extends each $\alpha_\lambda(\lambda \in \Lambda)$. It follows that

$$\beta'\pi = \beta = f\alpha \text{ and } \varrho_{\delta_0}\pi = \alpha.$$

Thus, $\bigvee_{j\in J}\bigwedge_{k\in K} f\alpha(h_{jk}) \neq 0$. That is, $f\big(\bigvee_{j\in J}\bigwedge_{k\in K}\alpha(h_{jk})\big) \neq 0$. Hence $\bigvee_{j\in J}\bigwedge_{k\in K}\alpha(h_{jk}) \neq 0$. So

$$\begin{aligned}\bigvee_{j\in J}\bigwedge_{k\in K}\pi(h_{jk}) &= \bigvee_{j\in J}\bigwedge_{k\in K}\big(\dots,\varrho_{\delta_0}\pi(h_{jk}),\dots\big) = \Big(\dots,\bigvee_{j\in J}\bigwedge_{k\in K}\varrho_{\delta_0}\pi(h_{jk}),\dots\Big)\\ &= \Big(\dots,\bigvee_{j\in J}\bigwedge_{k\in K}\alpha(h_{jk}),\dots\Big) \neq 0.\end{aligned}$$

Therefore β' can be uniquely extended to an l-homomorphism $\beta^*\colon F \to L$. □

3. The relation between $\mathscr{A}$-free products and $\mathscr{A}r$-free products

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of archimedean l-groups. By universal properties there exists an l-homomorphism φ of ${}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ onto ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$. If ${}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ is archimedean, then ${}^{\mathscr{A}}\bigsqcup_{\lambda\in\Lambda} G_\lambda \cong {}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$. Now we consider the $\mathscr{A}r$-free product of two archimedean 0-groups R_1 and R_2. By Corollary 1.9.1 [7] and the above formula (1) we have

$$R_1{}^{\mathscr{A}} \sqcup R_2 \cong \mathscr{F}_{\mathscr{A}}(R_1 \boxplus R_2),$$
$$R_1{}^{\mathscr{A}r} \sqcup R_2 \cong \mathscr{F}_{\mathscr{A}r}(R_1 \boxplus R_2),$$

where $\mathscr{F}_{\mathscr{A}}(R_1 \boxplus R_2)$ and $\mathscr{F}_{\mathscr{A}r}(R_1 \boxplus R_2)$ are respectively the $\mathscr{A}$-free l-group and the $\mathscr{A}r$-free l-group generated by $R_1 \boxplus R_2$. So the problem is reduced to the following under what condition the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}\big([G,P]\big)$ generated by a po-group $[G,P]$ is archimedean. In [2] S.J. Bernau established a necessary and sufficient condition under which the $\mathscr{A}$-free l-group generated by a po-group is archimedean. However, his proof contains an error. Namely, $[G,P]$ is a po-group and need not be a partially ordered vector space (see [14] for derails). The correct result is given in the following theorem. First we introduce some concepts.

Let $[G,P]$ be a po-group and S a nonempty subset of G. S is said to be positively independent if for any finite subset $\{x_1,\ldots,x_k\}$ of S and non-negative integers $\{\lambda_1,\ldots,\lambda_k\}$, $\sum\limits_{i=1}^{k} \lambda_i x_i \in -P$ only if $\lambda_i = 0$ $(i = 1,\ldots,k)$. A po-group $[G,P]$ is said to be strongly uniformly archimedean if, given $u \in G$ and a positively independent subset $\{v_1,\ldots,v_k\}$ of G, there exists $n \in \mathbb{N}$ such that if $\lambda_1,\ldots,\lambda_k$ are non-negative integers and $\sum\limits_{i=1}^{k} \lambda_i \geqslant mn$ with $m \in \mathbb{N}$, then $\sum\limits_{i=1}^{k} \lambda_i v_i \nleqslant mu$. It is well known that if a po-group $[G,P]$ is semi-closed, then the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}\big([G,P]\big)$ generated by $[G,P]$ exists (cf. [16]).

Theorem 3.1. *The $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}\big([G,P]\big)$ generated by a semi-closed po-group $[G,P]$ is archimedean if and only if $[G,P]$ is strongly uniformly archimedean.*

The proof of this theorem is similar to that of Theorem 4.3 of [2].

Now let R_1 and R_2 be two archimedean 0-groups. We call two nonzero elements (a,b) and (c,d) in $R_1 \times R_2$ separated if $(a,b) + \nu(c,d) = 0$ for a positive real number ν. It is clear that $R_1 \boxplus R_2$ is semi-closed. So Theorem 2.6 of [8] and Theorem 3.1 yield.

Theorem 3.2. *The following are equivalent:*

1. $R_1{}^{\mathscr{A}} \sqcup R_2$ *is archimedean,*
2. $R_1 \boxplus R_2$ *is strongly uniformly archimedean,*
3. $R_1 \boxplus R_2$ *has no separated, positively independent pairs,*
4. $R_1{}^{\mathscr{A}} \sqcup R_2 \cong R_1{}^{\mathscr{A}r} \sqcup R_2$.

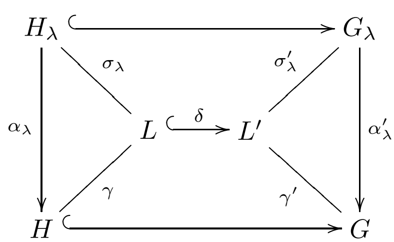
4. The weak subalgebra property

Let $\mathscr{U}$ be a class of l-groups closed under l-subgroups and direct products. $\mathscr{U}$-free products are said to have the subalgebra property if for any family $\{G_\lambda \mid \lambda \in \Lambda\}$ in $\mathscr{U}$ with l-subgroups $H_\lambda \subseteq G_\lambda$, ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} H_\lambda$ is simply the l-subgroup of ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup\limits_{\lambda\in\Lambda} H_\lambda$. It is well known that $\mathscr{A}$-free products satisfy the subalgebra property [11]. $\mathscr{U}$-free products are said to have the weak subalgebra property if, whenever $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of l-groups in $\mathscr{U}$ with l-subgroups $H_\lambda \subseteq G_\lambda$ and any family of l-homomorphisms $\sigma_\lambda\colon H_\lambda \to L \in \mathscr{U}$ can be extended to a family of l-homomorphisms $\sigma'_\lambda\colon G_\lambda \to L' \in \mathscr{U}$ and there exists a $\mathscr{U}$-injection $\delta\colon L \to L'$ such that $\sigma'_\lambda\big|_{H_\lambda} = \delta\sigma_\lambda$, then ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} H_\lambda$ is the l-subgroup of ${}^{\mathscr{U}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup\limits_{\lambda\in\Lambda} H_\lambda$.

Theorem 4.1. *$\mathscr{A}r$-free products satisfy the weak subalgebra property.*

Proof. Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of l-groups in $\mathscr{A}r$ with l-subgroups $H_\lambda \subseteq G_\lambda$, any family of l-homomorphisms $\sigma_\lambda\colon H_\lambda \to L \in \mathscr{A}r$ can be extended to a family of l-homomorphisms $\sigma'_\lambda\colon G_\lambda \to L' \in \mathscr{A}r$ and there exists an $\mathscr{A}r$-injection $\delta\colon L \to L'$ such that $\sigma'_\lambda\big|_{H_\lambda} = \delta\sigma_\lambda$. We see that $H = {}^{\mathscr{A}}\bigsqcup\limits_{\lambda\in\Lambda} H_\lambda$ is the l-subgroup of $G = {}^{\mathscr{A}}\bigsqcup\limits_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup\limits_{\lambda\in\Lambda} H_\lambda$.

(1) First we show that any l-homomorphism $\gamma\colon H \to L \in \mathscr{A}r$ can be extended to an l-homomorphism $\gamma'\colon G \to L' \in \mathscr{A}r$ and there exists an $\mathscr{A}r$-injection $\delta\colon L \to L'$ such that $\gamma'\big|_H = \delta\gamma$. In fact, any l-homomorphism $\sigma_\lambda\colon H_\lambda \to L \in \mathscr{A}r$ induces a family of l-homomorphisms $\sigma_\lambda\colon H_\lambda \to L \in \mathscr{A}r$ such that $\gamma\alpha_\lambda = \sigma_\lambda$ for each $\lambda \in \Lambda$ where α_λ is the inclusion map. Then σ_λ can

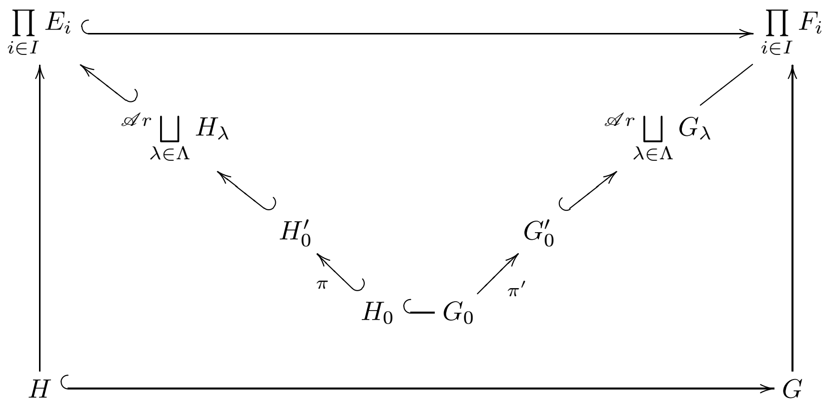


be extended to a family of l-homomorphisms $\sigma'_\lambda\colon G_\lambda \to L' \in \mathscr{A}r$ and there exists an $\mathscr{A}r$-injection $\delta\colon L \to L'$ such that $\sigma'_\lambda\big|_{H_\lambda} = \delta\sigma_\lambda$. By the universal property there exists an l-homomorphism $\gamma'\colon G \to L'$ such that $\gamma'\alpha'_\lambda = \sigma'_\lambda$ for each $\lambda \in \Lambda$ where α'_λ is the inclusion map. Hence

$$\delta\sigma_\lambda = \sigma'_\lambda\big|_{H_\lambda} = (\gamma'\alpha'_\lambda)\big|_{H_\lambda} = \gamma'\big|_{H_\lambda}$$

for each $\lambda \in \Lambda$. By virtue of the uniqueness, $\gamma'\big|_{H_\lambda} = \delta\gamma$.

(2) Now we show that ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is the l-subgroup of ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda$. Let $G_0 = \oplus_{\lambda\in\Lambda} G_\lambda$, $H_0 = \oplus_{\lambda\in\Lambda} H_\lambda$. Then G_0 and H_0 are subgroups of G and H, respectively, and H_0 is a subgroup of G_0, H is an l-subgroup



of G. Let

$$D = \left\{\gamma'_i\colon G \to F_i \mid i \to I\right\}$$

be the set of representatives of equivalence classes of $\mathscr{A}r$-surjections out of G. For each $i \in I$, $\gamma'_i\big|_H$ is an $\mathscr{A}r$-surjection out of H. Conversely, for an arbitrary $\mathscr{A}r$-surjection $\gamma\colon H \to E$ there exists by paragraph (1) an $i \in I$ and an $\mathscr{A}r$-injection $\delta\colon E \to F_i$ such that $\delta\gamma = \gamma'_i\big|_H$. Hence the set

$$C = \left\{\gamma'_i\big|_H\colon H \to E_i \leqslant F_i \mid i \in I\right\}$$

contains at least one element of each equivalence class of $\mathscr{A}r$-surjections out of H. But many different γ'_s may give rise to the same γ. If C contains more than one representative of some of the classes then the result of the construction is still the $\mathscr{A}r$-coproduct. So redundancy in C does not harm the result. By Proposition 2.2 we see that the $\mathscr{A}r$-free product ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ is the sublattice of the direct product $\prod_{i\in I} F_i$

generated by the group isomorphic image G_0' of G_0 with the group isomorphism π', and the $\mathscr{A}r$-free product ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is the sublattice of the direct product $\prod_{i\in I} E_i$ generated by the group isomorphic image H_0' of H_0 with the group isomorphism π. $\pi'\big|_{G_\lambda}$ and $\pi\big|_{H_\lambda}$ are all l-isomorphisms for each $\lambda \in \Lambda$. Hence ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ is the l-subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} G_\lambda$ where $G_\lambda' = \pi'(G_\lambda) \cong G_\lambda$ and ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is the l-subgroup of $\prod_{i\in I} E_i$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$ where $H_\lambda' = \pi(H_\lambda) \cong H_\lambda$. From the above we see that $\prod_{i\in I} E_i$ is an l-subgroup of $\prod_{i\in I} F_i$ and $\pi'\big|_{H_0} = \pi$. Therefore ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is the l-subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$, and so ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} H_\lambda$ is also the l-subgroup of ${}^{\mathscr{A}r}\bigsqcup_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$. □

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Author's address: 29=305, 3 Dikang Road, Nanjing, 210024, P.R. China.