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SINGULAR DIRICHLET BOUNDARY VALUE PROBLEMS II: RESONANCE CASE

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Abstract. Existence results are established for the resonant problem $y'' + \lambda_m\, a\, y = f(t,y)$ a.e. on $[0,1]$ with y satisfying Dirichlet boundary conditions. The problem is singular since f is a Carathéodory function, $a \in L^1_{\mathrm{loc}}(0,1)$ with $a > 0$ a.e. on $[0,1]$ and $\int_0^1 x(1-x)a(x)\,\mathrm{d}x < \infty$.

1. Introduction

This paper presents existence results for the Dirichlet resonant second order problem

$$(1.1)\qquad \begin{cases} y'' + \lambda_m\, a(t)\, y = f(t,y) \quad \text{a.e. on} \quad [0,1],\\ y(0) = y(1) = 0 \end{cases}$$

where $f\colon\ [0,1]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, $a \in L^1_{\mathrm{loc}}(0,1)$ with $a > 0$ a.e. on $[0,1]$ and $\int_0^1 x(1-x)a(x)\,\mathrm{d}x < \infty$.

Remark. λ_m (which is the $(m+1)^{st}$ eigenvalue of an appropriate problem) will be described later in the introduction.

Equations of the type (1.1), with $a \in L^1[0,1]$, have been studied extensively in the literature [2–3, 5–8, 10–11]. However very little attention has been given to the case when $a \notin L^1[0,1]$ (see [3, 5, 11] for results concerning upper and lower solutions). We remark here that the eigenvalue problem (which is singular) has been studied [1, 4, 9]. In this paper we use a well known technique, initiated by Mawhin and Ward [8] in the early 1980's and extended by Iannacci and Nkashama [6] in the late 1980's, to establish some new existence results for (1.1). The results here rely on a new existence principle established by the author in [12]. For convenience we now recall the results in [12] which will be used in this paper.

Our first result is an existence principle for

$$(1.2)\qquad \begin{cases} y'' + \mu\, a(t)\, y = f(t,y) \quad \text{a.e. on } [0,1], \\ y(0) = y(1) = 0, \ \mu \ \text{ a constant} \end{cases}$$

which was established using fixed point methods. First recall $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function if

(i) $t \mapsto f(t,y)$ is measurable for all $y\in\mathbb{R}$,

(ii) $y \mapsto f(t,y)$ is continuous for a.e. $t\in[0,1]$.

Theorem 1.1. *Let $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function with*

(1.3) *for any $r>0$ there exists $h_r\in L^1_{\text{loc}}(0,1)$ with $|f(t,u)|\leqslant h_r(t)$ for almost all $t\in[0,1]$ and all $|u|\leqslant r$; also $\int_0^1 x(1-x)h_r(x)\,\mathrm{d}x<\infty$ with $\lim\limits_{t\to 0^+} t^2(1-t)h_r(t)=0$ if $\int_0^1(1-x)h_r(x)\,\mathrm{d}x=\infty$ and $\lim\limits_{t\to 1^-} t(1-t)^2h_r(t)=0$ if $\int_0^1 xh_r(x)\,\mathrm{d}x=\infty$*

satisfied. Also assume

(1.4) *$a\in L^1_{\text{loc}}(0,1)$ with $a>0$ a.e. on $(0,1)$ and $\int_0^1 x(1-x)a(x)\,\mathrm{d}x<\infty$; also $\lim\limits_{t\to 0^+} t^2(1-t)a(t)=0$ if $\int_0^1(1-x)a(x)\,\mathrm{d}x=\infty$ and $\lim\limits_{t\to 1^-} t(1-t)^2a(t)=0$ if $\int_0^1 xa(x)\,\mathrm{d}x=\infty$*

holds and suppose μ is such that

$$(1.5)\qquad \begin{cases} y'' + \mu\, a(t)\, y = 0 \quad \text{a.e. on } [0,1], \\ y(0) = y(1) = 0 \end{cases}$$

has only the trivial solution. In addition assume there is a constant M_0, independent of λ, with $|y|_0=\sup\limits_{[0,1]}|y(t)|\neq M_0$ for any solution y (here $y\in AC[0,1]\cap C^1(0,1)$ with $y'\in AC_{\text{loc}}(0,1)$) to

$$(1.6)_\lambda\qquad \begin{cases} y'' + \mu\, a(t)\, y = \lambda\, f(t,y) \quad \text{a.e. on } [0,1], \\ y(0) = y(1) = 0 \end{cases}$$

for each $\lambda\in(0,1)$. Then (1.2) has at least one solution $u\in AC[0,1]\cap C^1(0,1)$ with $u'\in AC_{\text{loc}}(0,1)$.

Remark. In [12] we showed that if $h_r\in L^1[0,1]$ in (1.3) then in fact the solution u described above has the additional properties $au^2\in L^1[0,1]$, $u'\in L^2[0,1]$ and $\lim\limits_{t\to 0^+} u(t)u'(t)=\lim\limits_{t\to 1^-} uu'=0$.

We now establish a more general result. Suppose the conditions of theorem 1.1 are satisfied and in addition assume

(1.7) there exist p, q, with $0 \leqslant p, q < \frac{1}{2}$, and $x^p(1-x)^q h_r \in L^1[0,1]$; here h_r is as described in (1.3)

and

$$x^{1-p}(1-x)^{1-q}\, a \in L^1[0,1] \tag{1.8}$$

hold. Then the solution u described in theorem 1.1 satisfies $au^2 \in L^1[0,1]$, $u' \in L^2[0,1]$ and $\lim_{t\to 0^+} u(t)u'(t) = \lim_{t\to 1^-} u(t)u'(t) = 0$.

Recall [12],

$$u(t) = c_0\, w_2(t)(1-t) \int_0^t s w_1(s) f(s,u(s))\, \mathrm{d}s + c_0\, w_1(t)\, t \int_t^1 (1-s) w_2(s) f(s,u(s))\, \mathrm{d}s$$

(c_0, w_1 and w_2 are as described in theorem 2.1 in [12]). If we show

$$\int_0^1 a(t)(1-t)^2 \left(\int_0^t s h_r(s)\, \mathrm{d}s \right)^2 \mathrm{d}t < \infty$$

and

$$\int_0^1 a(t)\, t^2 \left(\int_t^1 (1-s) h_r(s)\, \mathrm{d}s \right)^2 \mathrm{d}t < \infty$$

(here h_r is as described in (1.3)) then of course $au^2 \in L^1[0,1]$. Immediately we have

$$\begin{aligned} &\int_0^1 a(t)(1-t)^2 \left(\int_0^t s h_r(s)\, \mathrm{d}s \right)^2 \mathrm{d}t \\ &\leqslant \int_0^1 a(t)\, t^{2(1-p)}(1-t)^{2(1-q)} \left(\int_0^t s^p (1-s)^q h_r(s)\, \mathrm{d}s \right)^2 \mathrm{d}t < \infty \end{aligned}$$

so $au^2 \in L^1[0,1]$.

Also recall for $t \in (0,1)$ that

$$u'(t) = c_0\, y_2'(t) \int_0^t s w_1(s) f(s,u(s))\, \mathrm{d}s + c_0 y_1'(t) \int_t^1 (1-s) w_2(s) f(s,u(s))\, \mathrm{d}s.$$

We show

$$\int_0^1 \left(y_2'(t) \int_0^t s h_r(s)\, \mathrm{d}s \right)^2 \mathrm{d}t < \infty \tag{1.9}$$

where

$$y_2'(t) = 1 + \mu \int_t^1 (1-x)a(x)w_2(x)\,\mathrm{d}x. \tag{1.10}$$

Now (1.9) is immediate since

$$\begin{aligned}
&\int_0^1 \left(y_2'(t) \int_0^t s h_r(s)\,\mathrm{d}s \right)^2 \mathrm{d}t \\
&\leqslant 2 \int_0^1 \left(\int_0^t s h_r(s)\,\mathrm{d}s \right)^2 \mathrm{d}t \\
&\quad + 2\mu^2 |w_2|_0^2 \int_0^1 \left(\int_t^1 (1-x)a(x)\,\mathrm{d}x \int_0^t s h_r(s)\,\mathrm{d}s \right)^2 \mathrm{d}t \\
&\leqslant 2 \int_0^1 \frac{1}{(1-t)^{2q}} \left(\int_0^t s(1-s)^q h_r(s)\,\mathrm{d}s \right)^2 \mathrm{d}t \\
&\quad + 2\mu^2 |w_2|_0^2 \int_0^1 \left(\int_t^1 (1-x)^{1-q} x^{1-p} a(x)\,\mathrm{d}x \int_0^t s^p (1-s)^q h_r\,\mathrm{d}s \right)^2 \mathrm{d}t.
\end{aligned}$$

Similarly

$$\int_0^1 \left(y_1'(t) \int_t^1 (1-s)h_r(s)\,\mathrm{d}s \right)^2 \mathrm{d}t < \infty$$

so $u' \in L^1[0,1]$. Finally we show $\lim\limits_{t\to 0^+} u(t)u'(t) = 0$. This is immediate once we show

$$\lim_{t\to 0^+} y_2(t) \int_0^t s h_r(s)\,\mathrm{d}s \; y_2'(t) \int_0^t s h_r(s)\,\mathrm{d}s = 0, \tag{1.11}$$

$$\lim_{t\to 0^+} y_1(t) \int_t^1 (1-s)h_r(s)\,\mathrm{d}s \; y_2'(t) \int_0^t s h_r(s)\,\mathrm{d}s = 0, \tag{1.12}$$

$$\lim_{t\to 0^+} y_2(t) \int_0^t s h_r(s)\,\mathrm{d}s \; y_1'(t) \int_t^1 (1-s)h_r(s)\,\mathrm{d}s = 0 \tag{1.13}$$

and

$$\lim_{t\to 0^+} y_1(t) \int_t^1 (1-s)h_r(s)\,\mathrm{d}s \; y_1'(t) \int_t^1 (1-s)h_r(s)\,\mathrm{d}s = 0. \tag{1.14}$$

Now (1.11) is true since (see (1.10))

$$\begin{aligned}
\left(\int_0^t s h_r(s)\,\mathrm{d}s \right)^2 \int_t^1 (1-x)a(x)\,\mathrm{d}x &\leqslant \left(\int_0^t s^p h_r(s)\,\mathrm{d}s \right)^2 \int_t^1 x^{2(1-p)} (1-x)a(x)\,\mathrm{d}x \\
&\leqslant \left(\int_0^t s^p h_r(s)\,\mathrm{d}s \right)^2 \int_t^1 x^{1-p} (1-x)^{1-q} a(x)\,\mathrm{d}x
\end{aligned}$$

and (1.7) implies $\lim_{t\to 0^+} \int_0^t s^p h_r(s)\,\mathrm{d}s = 0$.

Also we know (1.4) implies $\lim_{t\to 0^+} y_1(t) \int_t^1 (1-x)h_r(x)\,\mathrm{d}x = 0$ and thus (1.12) is immediately true since

$$\int_t^1 (1-x)a(x)\,\mathrm{d}x \int_0^t sh_r(s)\,\mathrm{d}s \\ \leqslant \int_t^1 x^{1-p}(1-x)^{1-q}a(x)\,\mathrm{d}x \int_0^t s^p(1-s)^q h_r(s)\,\mathrm{d}s < \infty.$$

Now (1.13) follows from the fact that

$$\int_0^t xh_r(x)\,\mathrm{d}s \int_t^1 (1-s)h_r(s)\,\mathrm{d}s \leqslant \int_0^t x^p h_r(x)\,\mathrm{d}s \int_t^1 s^{1-p}(1-s)h_r(s)\,\mathrm{d}s \\ \leqslant \int_0^t x^p h_r(x)\,\mathrm{d}s \int_t^1 s^p(1-s)^q h_r(s)\,\mathrm{d}s.$$

Finally (1.14) follows once we show $\lim_{t\to 0^+} y_1(t)\left(\int_t^1 (1-s)h_r(s)\,\mathrm{d}s\right)^2 = 0$. This is immediate since

$$t|w_1(t)|\left(\int_t^1 (1-s)h_r(s)\,\mathrm{d}s\right)^2 = t^{1-2p}|w_1(t)|\left(\int_t^1 s^p(1-s)h_r(s)\,\mathrm{d}s\right)^2 \\ \leqslant t^{1-2p}|w_1(t)|\left(\int_t^1 s^p(1-s)^q h_r(s)\,\mathrm{d}s\right)^2.$$

Thus $\lim_{t\to 0^+} u(t)u'(t) = 0$. Similarly $\lim_{t\to 1^-} u(t)u'(t) = 0$.

Next we recall some results [12] concerning the eigenvalue problem. In particular consider

$$Ly = \lambda y, \text{ a.e. on } [0,1], \\ y(0) = y(1) = 0$$

where $Ly = -\frac{1}{a}\,y''$. Assume (1.4) holds for the remainder of this section. Now $L_a^2[0,1]$ denotes the space of functions u with $\int_0^1 a(t)|u(t)|^2\,\mathrm{d}t < \infty$; also for $u, v \in L_a^2[0,1]$ define $\langle u, v\rangle = \int_0^1 a(t)u(t)\overline{v(t)}\,\mathrm{d}t$. Let

$$D(L) = \Big\{u \in C[0,1]\colon\ u \in AC[0,1],\ u' \in AC_{\mathrm{loc}}(0,1) \text{ with } \frac{1}{a}\,u'' \in L_a^2[0,1] \\ u(0) = u(1) = 0 \text{ and } \lim_{t\to 0^+} u(t)u'(t) = \lim_{t\to 1^-} u(t)u'(t) = 0\Big\}.$$

In [12] we showed using the spectral theorem for compact self adjoint operators that L has a countably infinite number of real eigenvalues λ_i with corresponding eigenfunctions $\varphi_i \in D(L)$. Also the eigenvalues λ_i are simple and $\lambda_i > 0$ for all i, so we may arrange the eigenvalues so that

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots.$$

In addition the eigenfunctions φ_i may be chosen so that they form an orthonormal set. Also $\{\varphi_i\}$ form a basis for $L^2_a[0,1]$ so if $r \in L^2_a[0,1]$ then r has a fourier series representation

$$r = \sum_{i=0}^{\infty} \langle r, \varphi_i \rangle \, \varphi_i$$

and r satisfies Parseval's equality

$$\int_0^1 a|r|^2 \, \mathrm{d}t = \sum_{i=0}^{\infty} |\langle r, \varphi_i \rangle|^2.$$

Next consider functions $u \in AC[0,1]$, $u(0) = u(1) = 0$ with $u \in L^2_a[0,1]$ and $u' \in L^2[0,1]$. Then $u = \sum\limits_{i=0}^{\infty} c_i \varphi_i$ where $c_i = \langle u, \varphi_i \rangle$. In [12] we showed

$$u' = \sum_{i=0}^{\infty} c_i \, \varphi_i'$$

(convergence is understood to be in L^2).

2. Existence theory

In this section we use theorem 1.1 to establish existence results for

$$\text{(2.1)} \qquad \begin{cases} y'' + \lambda_m \, a \, y + y \, g(t,y) = h(t,y) + v(t) & \text{a.e. on } [0,1], \\ y(0) = y(1) = 0 \end{cases}$$

where $m \in \{0, 1, 2, \dots\}$ and λ_m is as described in section 1.

Remark. For the remainder of this paper let $f(t,y) = h(t,y) - y\,g(t,y)$.

Theorem 2.1. *Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with*

(2.2) *for any $r > 0$ there exists $h_r \in L^1[0,1]$ with $|f(t,u)| \leqslant h_r(t)$ for almost all $t \in [0,1]$ and all $|u| \leqslant r$*

holding. In addition assume (1.4) *holds with*

$$v \in L^1[0,1] \text{ and there exists } p,\ 0 \leqslant p < \frac{1}{2} \text{ with } x^p(1-x)^p\, a \in L^1[0,1] \tag{2.3}$$

(2.4) *there exists* $\tau \in C[0,1]$ *with* $a\tau \in L^1[0,1]$ *and* $0 \leqslant g(t,u) \leqslant \tau(t)a(t)$ *for a.e.* $t \in [0,1]$ *and* $u \in \mathbb{R}$*; here* $\tau(t) \leqslant \lambda_{m+1} - \lambda_m$ *for a.e.* $t \in [0,1]$ *with* $\tau(t) < \lambda_{m+1} - \lambda_m$ *on a subset of* $[0,1]$ *of positive measure*

$$|h(t,y)| \leqslant q_1(t) + q_2(t)|y|^\gamma \text{ for a.e } t \in [0,1] \text{ with } 0 \leqslant \gamma < 1 \tag{2.5}$$

and

$$q_i \in L^1[0,1], \quad i = 1,2 \tag{2.6}$$

are satisfied.

(i) *Suppose there exists a constant* $k > \gamma$ *with* $1 > k = \frac{\alpha}{\beta}$, *where* β *is odd and* α *is even, and*

$$\begin{aligned} 0 < &\int_{I^+} [A\varphi_m(t)]^{k+1} \liminf_{x\to\infty} \left(x^{1-k} g(t,x)\right) \mathrm{d}t \\ &+ \int_{I^-} [A\varphi_m(t)]^{k+1} \limsup_{x\to-\infty} \left(x^{1-k} g(t,x)\right) \mathrm{d}t \end{aligned} \tag{2.7}$$

for all constants $A \neq 0$*; here* $I^+ = \{t \in [0,1]\colon A\varphi_m(t) > 0\}$ *and* $I^- = \{t\colon A\varphi_m(t) < 0\}$.

Then (2.1) *has at least one solution.*

(ii) *Suppose* $\gamma = 0$ *and*

$$\begin{aligned} A\int_0^1 v(t)\varphi_m(t)\,\mathrm{d}t < A &\int_{I^+} \varphi_m(t) \liminf_{x\to\infty} (x\, g(t,x))\,\mathrm{d}t \\ &+ A\int_{I^+} \varphi_m(t) \liminf_{x\to\infty} [-h(t,x)]\,\mathrm{d}t \\ &+ A\int_{I^-} \varphi_m(t) \limsup_{x\to-\infty} (x\, g(t,x))\,\mathrm{d}t \\ &+ A\int_{I^-} \varphi_m(t) \limsup_{x\to-\infty} [-h(t,x)]\,\mathrm{d}t \end{aligned} \tag{2.8}$$

for all constants $A \neq 0$.

Then (2.1) *has at least one solution.*

P r o o f. Choose μ such that $\lambda_m < \mu < \lambda_{m+1}$ and look at the boundary value problem

$$(2.9)_\delta \qquad \begin{cases} y'' + \mu\, a\, y = \delta[h(t,y) + v(t) - y\, g(t,y) + (\mu - \lambda_m) a\, y] & \text{a.e. on } [0,1], \\ y(0) = y(1) = 0 \end{cases}$$

for $\delta \in (0,1)$. Suppose y is a solution of $(2.9)_\delta$. If we show that there exists a constant M_0, independent of λ, with $|y|_0 \leqslant M_0$ then existence of a solution to (2.1) will be guaranteed from theorem 1.1.

Now $L^2_a[0,1] = \Omega \oplus \Omega^\perp$ where $\Omega = \{\varphi_0, \ldots, \varphi_m\}$. Let

$$w = \sum_{i=m+1}^{\infty} c_i \varphi_i, \quad u = \sum_{i=0}^{m} c_i \varphi_i, \quad u_0 = \sum_{i=0}^{m-1} c_i \varphi_i, \quad u_1 = c_m \varphi_m \quad \text{and} \quad \tilde{y} = w + u_0$$

where $c_i = \langle y, \varphi_i \rangle$.

Remarks. (i) Notice from section 1 that $ay^2 \in L^1[0,1]$ and $y' \in L^2[0,1]$.
(ii) Note $\tilde{y} = y - u_1$ and $y = w + u$.

Multiply the differential equation in $(2.9)_\delta$ by $u - w$ and integrate from 0 to 1 to obtain

$$\begin{aligned} &\int_0^1 ([w']^2 - [\lambda_m\, a + (1-\delta)(\mu - \lambda_m) a + \delta g(t,y)] w^2)\, \mathrm{d}t \\ &\qquad - \int_0^1 \left([u']^2 - [\lambda_m\, a + (1-\delta)(\mu - \lambda_m) a + \delta g(t,y)] u^2\right)\, \mathrm{d}t \\ &\qquad = \delta \int_0^1 [h(t,y) + v(t)](u - w)\, \mathrm{d}t. \end{aligned}$$

Remark. Now $w - u = y - 2u$ and we claim $\lim_{t \to 0^+} y'(y - 2u) = 0$. To see this we show $\lim_{t \to 0^+} u(t) y'(t) = 0$. This will be established by showing $\lim_{t \to 0^+} \varphi_i(t) y'(t) = 0$ for $i = 0, 1, \ldots, m$. Fix $i \in \{0, 1, \ldots, m\}$. Then

$$\varphi_i(t) = \lambda_i\, (1-t) \int_0^t s a(s) \varphi_i(s)\, \mathrm{d}s + \lambda_i\, t \int_t^1 (1-s) a(s) \varphi_i(s)\, \mathrm{d}s$$

and

$$\begin{aligned} y'(t) = \delta\, c_0\, y_2'(t) &\int_0^t s w_1(s)[f(s,y(s)) + v(s) + (\mu - \lambda_m) a(s) y(s)]\, \mathrm{d}s \\ &+ \delta\, c_0\, y_1'(t) \int_t^1 (1-s) w_2(s)[f(s,y(s)) + v(s) + (\mu - \lambda_m) a(s) y(s)]\, \mathrm{d}s. \end{aligned}$$

Now $\lim\limits_{t\to 0^+} \varphi_i(t)y'(t) = 0$ if we show

$$\lim_{t\to 0^+} \int_0^t xa(x)\,\mathrm{d}x \int_t^1 (1-s)a(s)\,\mathrm{d}s = 0 \quad \text{and} \quad \lim_{t\to 0^+} t\left(\int_t^1 (1-x)a(x)\,\mathrm{d}x\right)^2 = 0.$$

The first limit is immediate since

$$\begin{aligned}\int_0^t xa(x)\,\mathrm{d}x \int_t^1 (1-s)a(s)\,\mathrm{d}s &\leqslant \int_0^t x^p a(x)\,\mathrm{d}x \int_t^1 s^{1-p}(1-s)a(s)\,\mathrm{d}s \\ &\leqslant \int_0^t x^p a(x)\,\mathrm{d}x \int_0^1 s^p(1-s)^p a(s)\,\mathrm{d}s\end{aligned}$$

and noting (2.3) implies $\lim\limits_{t\to 0^+} \int_0^t x^p a(x)\,\mathrm{d}x = 0$; the second limit follows since

$$t\left(\int_t^1 (1-x)a(x)\,\mathrm{d}x\right)^2 \leqslant t^{1-2p}\left(\int_0^1 x^p(1-x)a(x)\,\mathrm{d}x\right)^2$$

Thus $\lim\limits_{t\to 0^+} y'(y-2u) = 0$. Similarly $\lim\limits_{t\to 1^-} y'(y-2u) = 0$.

Now use (2.4) to obtain

$$\begin{aligned}(2.10)\quad &\int_0^1 ([w']^2 - [\lambda_m + \tau(t)]a\,w^2)\,\mathrm{d}t - (\mu - \lambda_m)\int_0^1 a\,w^2\,\mathrm{d}t \\ &\quad - \int_0^1 \left([u_0']^2 - \lambda_m\,a\,u_0^2\right)\,\mathrm{d}t \leqslant \int_0^1 |[h(t,y)+v(t)](u-w)|\,\mathrm{d}t.\end{aligned}$$

Let

$$R(\tilde{y}) = \int_0^1 \left([w']^2 - [\lambda_m + \tau(t)]a\,w^2\right)\,\mathrm{d}t - \int_0^1 \left([u_0']^2 - \lambda_m\,a\,u_0^2\right)\,\mathrm{d}t.$$

We *claim* that there exists $\varepsilon > 0$ with

$$(2.11)\qquad R(\tilde{y}) \geqslant \varepsilon\left(\|w\|_a^2 + \|w'\|^2 + \|u_0\|_a^2 + \|u_0'\|^2\right);$$

here $\|z\|_a^2 = \int_0^1 a\,z^2\,\mathrm{d}t$ and $\|z'\|^2 = \int_0^1 [z']^2\,\mathrm{d}t$ where $z = w$ or u_0. Next notice a standard argument, using (2.4), implies $R(\tilde{y}) \geqslant 0$ and if $R(\tilde{y}) = 0$ then $\tilde{y} = 0$.

If (2.11) is not true then there exists a sequence $\{\tilde{y}_n\} = \{w_n + u_{0,n}\}$ with

$$(2.12)\qquad \|w_n\|_a^2 + \|w_n'\|^2 + \|u_{0,n}\|_a^2 + \|u_{0,n}'\|^2 = 1$$

and

$$(2.13)\qquad R(\tilde{y}_n) \to 0 \quad \text{as} \quad n\to\infty.$$

Now (2.12) implies that there is a subsequence S of integers with

(2.14) $$w_n \to w \ \text{ in } \ C[0,1] \ \text{ and } \ w_n' \rightharpoonup w' \ \text{ in } \ L^2[0,1] \ \text{ as } \ n \to \infty \ \text{ in } \ S$$

and

(2.15) $$u_{0,n} \to u_0 \ \text{ in } \ C[0,1] \ \text{ and } \ u_{0,n}' \rightharpoonup u_0' \ \text{ in } \ L^2[0,1] \ \text{ as } \ n \to \infty \ \text{ in } \ S;$$

here $\rightharpoonup$ denotes weak convergence.

Since weak and strong convergence are the same in finite dimensional spaces we have

(2.16) $$u_{0,n} \to u_0 \ \text{ in } \ W^{1,2}[0,1] \ \text{ as } \ n \to \infty \ \text{ in } \ S.$$

Also

(2.17) $$\int_0^1 [w']^2 \,\mathrm{d}t \leqslant \liminf \int_0^1 [w_n']^2 \,\mathrm{d}t.$$

Now (2.13), (2.14), (2.15), (2.16), (2.17) (and the Lebesgue dominated convergence theorem with $ay^2 \in L^1[0,1]$ and $\tau \in C[0,1]$) and the fact that $\liminf[s_n + t_n] \geqslant \liminf s_n + \liminf t_n$ for sequences s_n and t_n, yields (with $\tilde{y} = w + u_0$),

$$R(\tilde{y}) \leqslant \liminf \int_0^1 [w_n']^2 \,\mathrm{d}t + \lim \int_0^1 [-a(t)](\lambda_m + \tau(t)) w_n^2 \,\mathrm{d}t - \lim \int_0^1 [u_n']^2 \,\mathrm{d}t + \lim \int_0^1 a(t)(\lambda_m) u_n^2 \,\mathrm{d}t \leqslant \liminf R(\tilde{y}_n) \leqslant 0.$$

Hence $\tilde{y} = 0$. However

$$\begin{aligned} &\|w_n\|_a^2 + \|w_n'\|^2 + \|u_{0,n}\|_a^2 + \|u_{0,n}'\|^2 \\ &\quad = R(\tilde{y}_n) + \int_0^1 [2(u_{0,n}')^2 + a\, u_{0,n}^2 - \lambda_m\, a\, u_{0,n}^2 + a\, w_n^2 \\ &\qquad + (\lambda_m + \tau) a\, w_n^2] \,\mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty \quad \text{in} \quad S, \end{aligned}$$

which is impossible. Hence (2.11) holds for some $\varepsilon > 0$ (note ε is independent of μ). Put (2.11) into (2.10) to obtain

$$\varepsilon \left(\|w\|_a^2 + \|w'\|^2 + \|u_0\|_a^2 + \|u_0'\|^2 \right) - (\mu - \lambda_m) \int_0^1 a\, w^2 \,\mathrm{d}t \leqslant \int_0^1 |h(t,y) + v(t)| |u - w| \,\mathrm{d}t.$$

Now *choose* $\mu = \lambda_m + \frac{\varepsilon\lambda_{m+1}}{2}$ where ε is chosen sufficiently small so that $\lambda_m + \frac{\varepsilon\lambda_{m+1}}{2} < \lambda_{m+1}$. Then since $(\mu - \lambda_m)\|w\|_a^2 \leqslant \frac{(\mu-\lambda_m)}{\lambda_{m+1}}\|w'\|^2$ we have

$$\text{(2.18)} \qquad \frac{\varepsilon}{2}\left(\|w\|_a^2 + \|w'\|^2 + \|u_0\|_a^2 + \|u_0'\|^2\right) \leqslant \int_0^1 |h(t,y) + v(t)||u - w|\,\mathrm{d}t.$$

Notice also for $t \in [0,1]$ that

$$|w(t) - u(t)| = \left|\int_0^t [w'(s) - u'(s)]\,\mathrm{d}s\right| \leqslant \|w' - u'\| \quad \text{and} \quad |y(t)| \leqslant \|w' + u'\|.$$

Thus

$$\text{(2.19)} \qquad \begin{aligned} \int_0^1 q_1(t)|w - u|\,\mathrm{d}t &\leqslant |w - u|_0 \int_0^1 q_1\,\mathrm{d}t \leqslant \|w' - u'\| \int_0^1 q_1\,\mathrm{d}t \\ &= \left(\|w'\|^2 + \|u_0'\|^2 + \|u_1'\|^2\right)^{\frac{1}{2}} \int_0^1 q_1\,\mathrm{d}t \end{aligned}$$

$$\text{(2.20)} \qquad \int_0^1 |v(t)||w - u|\,\mathrm{d}t \leqslant \left(\|w'\|^2 + \|u_0'\|^2 + \|u_1'\|^2\right)^{\frac{1}{2}} \int_0^1 |v|\,\mathrm{d}t$$

and

$$\text{(2.21)} \qquad \begin{aligned} \int_0^1 q_2(t)|w - u||y|^\gamma\,\mathrm{d}t &\leqslant |w - u|_0|y|_0^\gamma \int_0^1 q_2\,\mathrm{d}t \\ &\leqslant \left(\|w'\|^2 + \|u_0'\|^2 + \|u_1'\|^2\right)^{\frac{\gamma+1}{2}} \int_0^1 q_2\,\mathrm{d}t \end{aligned}$$

since

$$\|w' - u'\|^2 = \|w'\|^2 + \|u'\|^2 = \|y'\|^2 = \|\tilde{y}'\|^2 + \|u_1'\|^2 = \|w'\|^2 + \|u_0'\|^2 + \|u_1'\|^2.$$

Put (2.19), (2.20) and (2.21) into (2.18) to obtain

$$\begin{aligned} &\frac{\varepsilon}{2}\left(\|w\|_a^2 + \|w'\|^2 + \|u_0\|_a^2 + \|u_0'\|^2\right) \\ &\qquad \leqslant \left(\|w'\|^2 + \|u_0'\|^2 + \|u_1'\|^2\right)^{\frac{1}{2}} \left(\int_0^1 q_1\,\mathrm{d}t + \int_0^1 |v|\,\mathrm{d}t\right) \\ &\qquad\quad + \left(|w'\|^2 + \|u_0'\|^2 + \|u_1'\|^2\right)^{\frac{\gamma+1}{2}} \int_0^1 q_2\,\mathrm{d}t. \end{aligned}$$

Now since $0 \leqslant \gamma < 1$ there exist constants A_1 and A_2 with

$$\text{(2.22)} \qquad \|w\|_a^2 + \|u_0\|_a^2 + \|w'\|^2 + \|u_0'\|^2 \leqslant A_1 + A_2\|u_1'\|^{\gamma+1}.$$

Consequently

$$\|\tilde{y}\|_a^2 + \|\tilde{y}'\|^2 \leqslant A_1 + A_2 \|u_1'\|^{\gamma+1}. \tag{2.23}$$

We next claim that (2.23) implies that there is a constant $M_0 > 0$ with

$$\|y\|_a^2 + \|y'\|^2 \leqslant M_0. \tag{2.24}$$

Suppose the claim is false. Then there is a sequence (δ_n) in $(0,1)$ and a sequence (y_n) with for a.e. $t \in [0,1]$,

$$y_n'' + \lambda_m\, a\, y_n + (1-\delta_n)(\mu - \lambda_m) a\, y_n + \delta_n y_n\, g(t, y_n) = \delta_n [h(t, y_n) + v(t)] \tag{2.25}$$

and

$$\|y_n\|_a^2 + \|y_n'\|^2 \to \infty. \tag{2.26}$$

From (2.23), with $y_n = \tilde{y}_n + u_{1,n}$, we have

$$\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2 \to \infty. \tag{2.27}$$

Also (2.23) and (2.27) imply

$$\frac{\|\tilde{y}_n\|_a^2 + \|\tilde{y}_n'\|^2}{\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2} \to 0. \tag{2.28}$$

Let

$$r_n = \frac{y_n}{\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{1}{2}}} = \tilde{r}_n + r_{1,n}$$

where

$$\tilde{r}_n = \frac{\tilde{y}_n}{\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{1}{2}}} \quad \text{and} \quad r_{1,n} = \frac{u_{1,n}}{\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{1}{2}}}.$$

Now (2.28) implies there is a subsequence $S^\star$ of integers with

$$\tilde{r}_n \to 0 \ \text{ in } \ C[0,1] \ \text{ and } \ \tilde{r}_n' \rightharpoonup 0 \ \text{ in } \ L^2[0,1] \ \text{ as } \ n \to \infty \ \text{ in } \ S^\star. \tag{2.29}$$

Now

$$\frac{\|y_n\|_a^2 + \|y_n'\|^2}{\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2} = 1 + \frac{\|\tilde{y}_n\|_a^2 + \|\tilde{y}_n'\|^2}{\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2}$$

together with (2.28) and (2.29) implies that there is a subsequence S of integers and a constant $A \neq 0$ with

$$r_{1,n} \to A\varphi_m \text{ in } C[0,1] \text{ as } n \to \infty \text{ in } S \tag{2.30}$$

and

$$r_n \to A\varphi_m \text{ in } C[0,1] \text{ as } n \to \infty \text{ in } S. \tag{2.31}$$

Remark. Note as well that $r_n \rightharpoonup A\varphi_m$ in $W^{1,2}[0,1]$ as $n \to \infty$ in S and $r_{1,n} \to A\varphi_m$ in $W^{1,2}[0,1]$ as $n \to \infty$ in S.

Multiply (2.25) by $\frac{r_{1,n}}{\delta_n}$ and integrate from 0 to 1 (noting that $\int_0^1 [r_{1,n} y_n'' + \lambda_m r_{1,n} a\, y_n]\, \mathrm{d}t = 0$), to obtain

$$\begin{aligned}\int_0^1 v\, r_{1,n}\, \mathrm{d}t = & -\int_0^1 r_{1,n}\, h(t,y_n)\, \mathrm{d}t + \int_0^1 r_{1,n}\, y_n\, g(t,y_n)\, \mathrm{d}t \\ & + \frac{(1-\delta_n)(\mu-\lambda_m)}{\delta_n \left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{1}{2}}} \int_0^1 a\, u_{1,n}^2\, \mathrm{d}t\end{aligned} \tag{2.32}$$

and so

$$\int_0^1 v\, r_{1,n}\, \mathrm{d}t \geqslant -\int_0^1 r_{1,n}\, h(t,y_n)\, \mathrm{d}t + \int_0^1 r_{1,n}\, y_n\, g(t,y_n)\, \mathrm{d}t. \tag{2.33}$$

Case (i). Suppose (2.7) holds.

Multiply (2.33) by $\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{-k}{2}}$ to obtain

$$\begin{aligned}\frac{1}{\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{k}{2}}} \int_0^1 v\, r_{1,n}\, \mathrm{d}t \geqslant & -\int_0^1 \frac{r_{1,n}\, h(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{k}{2}}}\, \mathrm{d}t \\ & + \int_0^1 \frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{k}{2}}}\, \mathrm{d}t.\end{aligned}$$

Now since (2.30) holds then there exists a constant B_0 with $|r_{1,n}|_0 \leqslant B_0$ for all n so we may apply the Lebesgue dominated convergence theorem to deduce

$$\liminf \int_0^1 v\, r_{1,n}\, \mathrm{d}t = \lim \int_0^1 v\, r_{1,n}\, \mathrm{d}t = A \int_0^1 v\varphi_m\, \mathrm{d}t. \tag{2.34}$$

Now (2.27) and (2.34) imply

$$\liminf \int_0^1 \frac{v\, r_{1,n}}{\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{k}{2}}}\, \mathrm{d}t = 0.$$

and so
(2.35)
$$0 \geqslant \liminf \int_0^1 \frac{r_{1,n}[-h(t,y_n)]}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}}\,\mathrm{d}t + \liminf \int_0^1 \frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}}\,\mathrm{d}t.$$

Remark. In (2.35) we have $n \to \infty$ in S.

Now

$$\liminf \int_0^1 \frac{r_{1,n}[-h(t,y_n)]}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}}\,\mathrm{d}t = 0 \tag{2.36}$$

since

$$\begin{aligned} 0 &\leqslant \int_0^1 \frac{|r_{1,n} h(t,y_n)|}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}}\,\mathrm{d}t \\ &\leqslant \frac{1}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k+1}{2}}} \left(\int_0^1 q_1 |u_{1,n}|\,\mathrm{d}t + \int_0^1 q_2 |u_{1,n}||y_n|^\gamma\,\mathrm{d}t\right) \end{aligned}$$

together with

$$\int_0^1 q_1|u_{1,n}|\,\mathrm{d}t \leqslant |u_{1,n}|_0 \int_0^1 q_1\,\mathrm{d}t \leqslant \left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{1}{2}} \int_0^1 q_1\,\mathrm{d}t$$

and (here we use (2.23)),

$$\begin{aligned} \int_0^1 q_2|y_n|^\gamma|u_{1,n}|\,\mathrm{d}t &\leqslant \left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{1}{2}} \left(\|y_n\|_a^2 + \|y'_n\|^2\right)^{\frac{\gamma}{2}} \int_0^1 q_2\,\mathrm{d}t \\ &\leqslant \left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2 + \|\tilde{y}_n\|_a^2 + \|\tilde{y}'_n\|^2\right)^{\frac{\gamma+1}{2}} \int_0^1 q_2\,\mathrm{d}t \\ &\leqslant \left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2 + A_1 + A_2\|u'_{1,n}\|^{\gamma+1}\right)^{\frac{\gamma+1}{2}} \int_0^1 q_2\,\mathrm{d}t \end{aligned}$$

implies that (2.36) is true (note $k > \gamma$ and $\gamma < 1$). This together with (2.35) yields
(2.37)
$$0 \geqslant \liminf \int_{I^+} \frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}}\,\mathrm{d}t + \liminf \int_{I^-} \frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}}\,\mathrm{d}t.$$

Remark. Notice since φ_m has at most a finite number of zero's [1,4] and $a\tau \in L^1[0,1]$ then, with $J = [0,1]/\{I^+ \cup I^-\}$, we have

$$\int_J |u_{1,n}\, y_n|\, a\,\tau\,\mathrm{d}t \leqslant |u_{1,n}|_0\, |y_n|_0 \int_J a\,\tau\,\mathrm{d}t = 0 \quad \text{for each} \quad n \in S.$$

Consequently

$$\int_J \frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}}\, \mathrm{d}t = 0 \quad \text{for each} \quad n \in S.$$

We next want to apply Fatou's lemma in (2.37). To justify this we need to show that there exists $\varrho \in L^1[0,1]$ with

$$\frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}} \geqslant \varrho(t) \quad \text{for a.e.} \quad t \in [0,1]. \tag{2.38}$$

To see this first notice

$$[\tilde{y}_n(t)]^2 \leqslant \|\tilde{y}_n\|^2 + \|\tilde{y}'_n\|^2 \leqslant A_1 + A_2 \|u'_{1,n}\|^{\gamma+1}$$

so there exists $A_3 > 0$ (since $k > \gamma$ and (2.27) holds) with

$$\frac{[\tilde{y}_n(t)]^2}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k+1}{2}}} \leqslant A_3. \tag{2.39}$$

Also for a.e. $t \in [0,1]$,

$$\begin{aligned}\frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}} &= \frac{1}{2}\,\frac{g(t,y_n)[y_n^2 + u_{1,n}^2 - \tilde{y}_n^2]}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k+1}{2}}} \geqslant -\frac{1}{2}\,\frac{g(t,y_n)\,\tilde{y}_n^2}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k+1}{2}}} \\ &\geqslant -\frac{1}{2}\,\frac{a(t)\,\tau(t)\,\tilde{y}_n^2}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k+1}{2}}} \geqslant -\frac{1}{2} A_3\, a(t)\,\tau(t).\end{aligned}$$

Thus (2.38) is true since $a\,\tau \in L^1[0,1]$. Apply Fatou's lemma to (2.37) to obtain

$$\begin{aligned} 0 &\geqslant \int_{I^+} \liminf \left(\frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}} \right) \mathrm{d}t + \int_{I^-} \liminf \left(\frac{r_{1,n}\, y_n\, g(t,y_n)}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{k}{2}}} \right) \mathrm{d}t \\ &= \int_{I^+} \liminf \left(r_{1,n} \left(\frac{y_n}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{1}{2}}} \right)^k y_n^{1-k}\, g(t,y_n) \right) \mathrm{d}t \\ &\quad + \int_{I^-} \liminf \left(r_{1,n} \left(\frac{-y_n}{\left(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\right)^{\frac{1}{2}}} \right)^k [-[-y_n]^{1-k} g(t,y_n)] \right) \mathrm{d}t. \end{aligned}$$

This together with (2.30) and (2.31) implies

$$\begin{aligned} 0 \geqslant &\int_{I^+} [A\varphi_m(t)]^{k+1} \liminf \left(y_n^{1-k}\, g(t,y_n) \right) \mathrm{d}t \\ &+ \int_{I^-} [A\varphi_m(t)]^{k+1} \limsup \left(y_n^{1-k}\, g(t,y_n) \right) \mathrm{d}t. \end{aligned} \tag{2.40}$$

Remark. Note $(-1)^k = (-1)^{\frac{\alpha}{\beta}} = 1$ and $-(-1)^{1-k} = 1$.

Let $t \in I^+$. Then (2.31) implies there is an integer n_1 (i.e. $n_1(t)$) with

$$y_n(t) \geqslant \frac{1}{2}\, A\, \varphi_m(t) \left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{1}{2}} \quad \text{for} \quad n \geqslant n_1. \tag{2.41}$$

Let $t \in I^-$. There is an integer n_2 with

$$y_n(t) \leqslant \frac{1}{2}\, A\, \varphi_m(t) \left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{1}{2}} \quad \text{for} \quad n \geqslant n_2. \tag{2.42}$$

This together with (2.40) implies

$$\begin{aligned} 0 \geqslant & \int_{I^+} [A\varphi_m(t)]^{k+1} \liminf_{x \to \infty} \left(x^{1-k}\, g(t,x)\right)\, \mathrm{d}t \\ & + \int_{I^-} [A\varphi_m(t)]^{k+1} \limsup_{x \to -\infty} \left(x^{1-k}\, g(t,x)\right)\, \mathrm{d}t, \end{aligned}$$

which contradicts (2.7). Thus (2.24) holds and so $|y|_0^2 \leqslant M_0$. Theorem 1.1 now guarantees that (2.1) has a solution.

Case (ii). Suppose (2.8) holds.

Now (2.33) together with (2.34) yields

$$\begin{aligned} A \int_0^1 v\, \varphi_m\, \mathrm{d}t \geqslant & \liminf \int_{I^+} r_{1,n}\, y_n\, g(t,y_n)\, \mathrm{d}t + \liminf \int_{I^-} r_{1,n}\, y_n\, g(t,y_n)\, \mathrm{d}t \\ & + \liminf \int_{I^+} r_{1,n}[-h(t,y_n)]\, \mathrm{d}t + \liminf \int_{I^-} r_{1,n}[-h(t,y_n)]\, \mathrm{d}t. \end{aligned}$$

As in case (i) there exists $\varrho \in L^1[0,1]$ with

$$r_{1,n}\, y_n\, g(t,y_n) \geqslant \varrho(t) \quad \text{for a.e.} \quad t \in [0,1]$$

and so Fatou's lemma together with (2.41) and (2.42) implies

$$\begin{aligned} A \int_0^1 v\, \varphi_m\, \mathrm{d}t \geqslant & A \int_{I^+} \varphi_m(t) \liminf_{x \to \infty} (xg(t,x))\, \mathrm{d}t + A \int_{I^-} \varphi_m(t) \limsup_{x \to -\infty} (xg(t,x))\, \mathrm{d}t \\ & + \liminf \int_{I^+} r_{1,n}[-h(t,y_n)]\, \mathrm{d}t + \liminf \int_{I^-} r_{1,n}[-h(t,y_n)]\, \mathrm{d}t. \end{aligned}$$

Also $|h(t,y)| \leqslant q_1(t) + q_2(t)$ a.e. so we may apply Fatou's lemma (together with (2.41) and (2.42)) to deduce

$$\begin{aligned} A\int_0^1 v(t)\varphi_m(t)\,\mathrm{d}t \geqslant A\int_{I+} &\varphi_m(t)\liminf_{x\to\infty}\,(xg(t,x))\,\,\mathrm{d}t \\ &+ A\int_{I-} \varphi_m(t)\limsup_{x\to-\infty}\,(xg(t,x))\,\,\mathrm{d}t \\ &+ A\int_{I+} \varphi_m(t)\liminf_{x\to\infty}[-h(t,x)]\,\mathrm{d}t \\ &+ A\int_{I-} \varphi_m(t)\limsup_{x\to-\infty}[-h(t,x)]\,\mathrm{d}t, \end{aligned}$$

which contradicts (2.8). Thus (2.24) holds. □

Remark. One can obtain in addition a result if $k=\gamma$ in case (i). Of course (2.36) does not necessarily hold in this case so we need to adjust (2.7) using the ideas in case (ii).

We now obtain an extra existence result if $\int_0^1 v\varphi_m\,\mathrm{d}t = 0$ and $a \in L^1[0,1]$.

Theorem 2.2. *Let* $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ *be a Carathéodory function and assume* (1.4), (2.2), (2.3), (2.4), (2.5) *and* (2.6) *are satisfied. In addition assume*

(2.43) $$\int_0^1 v(t)\varphi_m(t)\,\mathrm{d}t = 0 \quad \textit{and} \quad a\in L^1[0,1],$$

(2.44) $$h(t,u)\leqslant 0 \ \ \textit{for a.e. } t\in[0,1],\ u>0$$

and

(2.45) $$h(t,u)\geqslant 0 \ \ \textit{for a.e. } t\in[0,1],\ u<0$$

hold. Then (2.1) *has a solution.*

P r o o f. Suppose y is a solution to $(2.9)_\delta$. Follow the arguement in theorem 2.1 to equation (2.32). In this case (since (2.43) is true) we have

(2.46) $$\begin{aligned} 0 = &-\int_0^1 r_{1,n}\,h(t,y_n)\,\mathrm{d}t + \int_0^1 r_{1,n}\,y_n\,g(t,y_n)\,\mathrm{d}t \\ &+ \frac{(1-\delta_n)(\mu-\lambda_m)}{\delta_n\left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac12}}\int_0^1 a\,u_{1,n}^2\,\mathrm{d}t. \end{aligned}$$

Now [1,4 section 4] implies φ_m has a finite number of zeros so

$$(2.47)\qquad 0 > \int_{I^+} (\, r_{1,n}\, y_n\, g(t, y_n) + r_{1,n}[-h(t, y_n)])\, \mathrm{d}t + \int_{I^-} (\, r_{1,n}\, y_n\, g(t, y_n) + r_{1,n}[-h(t, y_n)])\, \mathrm{d}t.$$

We now claim that for $t \in I^+$ there exists an integer n_1 (independent of t) with

$$(2.48)\qquad r_{1,n}(t) > 0 \quad \text{and} \quad y_n(t) > 0 \quad \text{for} \quad n \geqslant n_1 \quad \text{and} \quad n \in S.$$

Similarly we claim for $t \in I^-$ there is a integer n_2 (independent of t) with

$$(2.49)\qquad r_{1,n}(t) < 0 \quad \text{and} \quad y_n(t) < 0 \quad \text{for} \quad n \geqslant n_2 \quad \text{and} \quad n \in S.$$

To see the first part of (2.48) let $r_{1,n} = \beta_{1,n}\varphi_m$. Fix $t_1 \in I^+$. Then (2.30) implies that there is an integer m_1 (i.e $m_1(t_1)$) with

$$r_{1,n}(t_1) \geqslant \frac{1}{2}\, A\, \varphi_m(t_1)\, \left(\|u_{1,n}\|_a^2 + \|u_{1,n}'\|^2\right)^{\frac{1}{2}} \quad \text{for} \quad n \geqslant m_1.$$

Thus $\frac{\beta_{1,n}}{A} > 0$ for $n \geqslant m_1$ since $\frac{\beta_{1,n}}{A}\, A\varphi_m(t_1) = r_{1,n}(t_1) > 0$ for $n \geqslant m_1$. Consequently for any $s \in I^+$ we have $r_{1,n}(s) = \frac{\beta_{1,n}}{A}\, A\varphi_m(s) > 0$ for $n \geqslant m_1$. The proof of the second part of (2.48) is more involved. Notice (2.25), (2.27), (2.31) and the ideas used in (2.19), (2.20) and (2.21) immediately guarantee the existence of $\omega \in L^1[0,1]$ (ω independent of n) with, for n sufficiently large,

$$(2.50)\qquad |r_n''(t)| \leqslant \omega(t) \quad \text{for a.e.} \quad t \in [0,1].$$

Remark. $a \in L^1[0,1]$ is needed to guarantee $\omega \in L^1[0,1]$.

Notice also [10] since $a \in L^1[0,1]$ that $\varphi_m \in C^1[0,1]$. Now (2.50) implies (see also (2.31)) that there is a subsequence $S^\star$ of integers with

$$(2.51)\qquad r_n \to A\varphi_m \quad \text{in} \quad C^1[0,1] \quad \text{as} \quad n \to \infty \quad \text{in} \quad S^\star.$$

Notice $\varphi_m'(0) \neq 0$ (i.e. if $\varphi_m'(0) = 0$ then $\varphi_m'' = -\lambda_m\, a\, \varphi_m$, $\varphi_m(0) = \varphi_m'(0) = 0$ which implies $\varphi_m \equiv 0$, a contradiction). Also we know φ_m has a finite number of zero's in $(0,1)$; let s_i denote these zero's and let s_0 be the smallest one. Without loss of generality assume $A\varphi_m > 0$ on $(0, s_0)$ with $\varphi_m(0) = \varphi_m(s_0) = 0$. Since $\varphi_m'' = -\lambda_m\, a\, \varphi_m$ a.e. then $A\, \varphi_m'(0) > 0$. Thus there exists $t_1 \in (0, s_0)$ with $A\varphi_m'(t) \geqslant A\frac{1}{2}\, \varphi_m'(0)$ for $t \in [0, t_1]$. Also (2.51) implies that there is an integer k_1 with $r_n'(t) -$

$A\varphi'_m(t) > -\frac{1}{4}\,A\,\varphi'_m(0)$ for $n \geqslant k_1$. Thus $r'_n(t) > \frac{1}{4}\,A\,\varphi'_m(0)$ for $n \geqslant k_1$ and $t \in [0,t_1]$. Consequently

$$\begin{aligned} y_n(t) &= \int_0^t y'_n(s)\,\mathrm{d}s \\ &= \int_0^t r'_n(s)\,\mathrm{d}s\,\bigl(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2\bigr)^{\frac{1}{2}} > 0 \quad \text{for} \quad n \geqslant k_1 \quad \text{and} \quad t \in (0,t_1]. \end{aligned}$$

Similarly there exists t_2 with $t_1 < t_2 < s_0$ and there exists an integer k_2 with $y_n(t) > 0$ for $n \geqslant k_2$ and $t \in [t_2, s_0)$ (note in this case for a fixed $t_3 \in (t_2, s_0)$ then (2.41) implies that $y_n(t_3) > 0$ for n sufficiently large; now consider $t \in [t_2, t_3)$ and $t \in (t_3, s_0)$). Finally since $r_n \to A\,\varphi_m$ in $C[0,1]$ as $n \to \infty$ in $S^\star$ there exists (since $\min_{[t_1,t_2]} A\,\varphi_m(t) > 0$) an integer k_3 with $y_n(t) > 0$ for $n \geqslant k_3$ and $t \in [t_1, t_2]$. Consequently for $n \geqslant \max\{k_1, k_2, k_3\}$ we have $y_n(t) > 0$ on $(0, s_0)$. Since φ_m has only a finite number of zero's s_i in $(0,1)$ then there exists an integer n_1 with $y_n(t) > 0$ for $n \geqslant n_1$ and $t \in I^+$, so (2.48) is true. A similar argument shows (2.49) is also true.

Now (2.44), (2.45), (2.48) and (2.49) yield for $n \in S^\star$ and $n \geqslant \max\{n_1, n_2\}$,

$$\begin{aligned} 0 \leqslant &\int_{I^+} \bigl(\,r_{1,n}\,y_n\,g(t,y_n) + r_{1,n}[-h(t,y_n]\bigr)\,\mathrm{d}t \\ &+ \int_{I^-} \bigl(\,r_{1,n}\,y_n\,g(t,y_n) + r_{1,n}[-h(t,y_n)]\bigr)\,\mathrm{d}t. \end{aligned}$$

This contradicts (2.47). □

The above results have "dual" versions. We will just give the dual version of theorem 2.1.

Theorem 2.3. *Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and assume* (1.4), (2.2), (2.3), (2.5) *and* (2.6) *hold. In addition suppose*

(2.52) *there exists $\tau \in C[0,1]$ with $a\tau \in L^1[0,1]$ and $-\tau(t)a(t) \leqslant g(t,u) \leqslant 0$ for a.e. $t \in [0,1]$ and $u \in \mathbb{R}$; here $\tau(t) \leqslant \lambda_m - \lambda_{m-1}$ for a.e. $t \in [0,1]$ with $\tau(t) < \lambda_m - \lambda_{m-1}$ on a subset of $[0,1]$ of positive measure*

is satisfied.

(i) *Suppose there exists a constant $k > \gamma$ with $1 > k = \frac{\alpha}{\beta}$, where β is odd and α is even, and*

$$\begin{aligned} 0 > &\int_{I^+} [A\varphi_m(t)]^{k+1} \limsup_{x\to\infty} \bigl(x^{1-k}\,g(t,x)\bigr)\,\mathrm{d}t \\ &+ \int_{I^-} [A\varphi_m(t)]^{k+1} \liminf_{x\to-\infty} \bigl(x^{1-k}\,g(t,x)\bigr)\,\mathrm{d}t \end{aligned} \tag{2.53}$$

for all constants $A \neq 0$*; here* $I^+ = \{t \in [0,1]\colon A\varphi_m(t) > 0\}$ *and* $I^- = \{t\colon A\varphi_m(t) < 0\}$.

Then (2.1) *has at least one solution.*

(ii) *Suppose* $\gamma = 0$ *and*

$$A\int_0^1 v(t)\varphi_m(t)\,\mathrm{d}t > A\int_{I^+} \varphi_m(t)\limsup_{x\to\infty}\,(x\,g(t,x))\,\mathrm{d}t + A\int_{I^+} \varphi_m(t)\limsup_{x\to\infty}[-h(t,x)]\,\mathrm{d}t + A\int_{I^-} \varphi_m(t)\liminf_{x\to-\infty}\,(xg\,(t,x))\,\mathrm{d}t + A\int_{I^-} \varphi_m(t)\liminf_{x\to-\infty}[-h(t,x)]\,\mathrm{d}t \tag{2.54}$$

for all constants $A \neq 0$.

Then (2.1) *has at least one solution.*

Proof. In this case choose μ such that $\lambda_{m-1} < \mu < \lambda_m$ and suppose y is a solution to

$$y'' + \mu\, a\, y = \delta[h(t,y) + v(t) - y\,g(t,y) + (\mu - \lambda_m)a\,y] \quad \text{a.e. on } [0,1],$$
$$y(0) = y(1) = 0$$

for $\delta \in (0,1)$. Let

$$w = \sum_{i=m}^{\infty} c_i\varphi_i, \quad u = \sum_{i=0}^{m-1} c_i\varphi_i, \quad w_0 = \sum_{i=m+1}^{\infty} c_i\varphi_i, \quad \text{and} \quad w_1 = c_m\varphi_m$$

in this case. The same type of analysis as that in theorem 2.1 establishes the result.

□

Example. Let $f\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ and $v, a\colon [0,\infty)\to\mathbb{R}$. Suppose $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with (2.2) holding. In addition assume (1.4) and (2.3) hold and also that f has the decomposition $f(t,u) = h(t,u) - y\,g(t,u)$. Now suppose (2.4), (2.5) and (2.6) are satisfied and that there exists a constant $k > \gamma$, where $1 > k = \frac{\alpha}{\beta}$ with β odd and α even, with (2.7) holding. Finally assume

$$f(t,0) + v(t) = 0 \quad \text{and} \quad a(t) \in \mathbb{R} \quad \text{for a.e.} \quad t \geqslant 1 \tag{2.55}$$

is satisfied. Then

$$y'' + \lambda_m\, a\, y = f(t,y) + v(t) \quad \text{a.e. on } [0,\infty), \qquad y(0) = y(\infty) = 0 \tag{2.56}$$

has a solution in $C[0,\infty)$.

To see this notice theorem 2.1 (i) guarantees that

$$\begin{aligned} &y'' + \lambda_m\, a\, y = f(t,y) + v(t) \quad \text{a.e. on } [0,1],\\ &y(0) = y(1) = 0 \end{aligned}$$

has a solution $y \in C[0,1]$ (the other smoothness properties in theorem 2.1 (i) also hold). Let

$$y^\star = \begin{cases} y, & 0 \leqslant t \leqslant 1,\\ 0, & t \geqslant 1. \end{cases}$$

Notice $y^\star \in C[0,\infty)$ and $y^\star$ satisfies $y'' + \lambda_m a y = f(t,y) + v(t)$ a.e. on $[0,\infty)$.

Remark. We can also obtain analogue results for (2.56) when the conditions of theorem 2.1 (ii) or theorem 2.2 are satisfied.

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