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POINTWISE CONVERGENCE FAILS TO BE STRICT

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Abstract. It is known that the ring $B(\mathbb{R})$ of all Baire functions carrying the pointwise convergence yields a sequential completion of the ring $C(\mathbb{R})$ of all continuous functions. We investigate various sequential convergences related to the pointwise convergence and the process of completion of $C(\mathbb{R})$. In particular, we prove that the pointwise convergence fails to be strict and prove the existence of the categorical ring completion of $C(\mathbb{R})$ which differs from $B(\mathbb{R})$.

1.

Consider the set $\mathbb{R}^{\mathbb{R}}$ of all real functions carrying the pointwise (sequential) convergence. If we start with the ring $C(\mathbb{R})$ of all continuous functions, then the ring $B_1(\mathbb{R})$ of all 1-st Baire class functions is the first sequential closure of $C(\mathbb{R})$, the ring $B_\alpha(\mathbb{R})$ of all α -th Baire class functions, $\alpha < \omega_1$, is the α -th sequential closure of $C(\mathbb{R})$, and the ring $B(\mathbb{R})$ of all Baire functions is the smallest subset of $\mathbb{R}^{\mathbb{R}}$ containing $C(\mathbb{R})$ and closed with respect to the pointwise convergence, hence sequentially complete, see [NOV], [LAC].

Let \mathbb{L} be a sequential convergence on $B_1(\mathbb{R})$ such that, for each sequence $\langle f_n \rangle$ of continuous functions, $\langle f_n \rangle$ converges to $f \in B_1(\mathbb{R})$ under \mathbb{L} iff it converges to f pointwise. Then \mathbb{L} is said to be *admissible*. If \mathbb{L} is compatible with the group or ring structure of $B_1(\mathbb{R})$, then each Cauchy sequence of continuous functions converges under \mathbb{L} and we get $B_1(\mathbb{R})$ as a group or ring (sequential) *precompletion* of $C(\mathbb{R})$. Observe that \mathbb{L} , for example the pointwise convergence, need not be complete. To get a completion, in such cases we have to iterate the precompletion process. In case of the pointwise convergence the usual sequential completion of $C(\mathbb{R})$ is the ring

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$B(\mathbb{R})$. In the present paper we investigate admissible convergences and alternative ways of (pre)completing $C(\mathbb{R})$.

Strictness is a natural way how to control the convergence of sequences of ideal points in an extension of a convergence space or a precompletion of a sequential group or ring ([FZS], [FKE], [PAU]).

An admissible convergence \mathbb{L} on $B_1(\mathbb{R})$ is said to be *strict* if the following condition is satisfied (see Definition 1.2 in [FZS]):

- (s) Let $\langle f_n \rangle$ be a sequence ranging in $B_1(\mathbb{R}) \setminus C(\mathbb{R})$ which converges under \mathbb{L} to $f \in B_1(\mathbb{R})$. Then there are a subsequence $\langle f'_n \rangle$ of $\langle f_n \rangle$ and sequences $\langle g_n^{(k)} \rangle$, $k \in \mathbb{N}$, of continuous functions such that the sequence $\langle g_n^{(k)} \rangle$ pointwise converges to f'_k and each diagonal sequence $\langle g_{d(n)}^{(n)} \rangle$, $d: \mathbb{N} \rightarrow \mathbb{N}$, pointwise converges to f .

In [FZS] the authors asked whether the pointwise convergence is strict. We prove that the answer is “NO”.

Theorem 1.1. *The pointwise convergence on $B_1(\mathbb{R})$ fails to be strict.*

P r o o f. Let p_1, p_2, p_3, \dots denote the increasing sequence of all prime numbers. For each $n \in \mathbb{N}$, let $A_n = \{k/p_n; k = 1, 2, \dots, p_n - 1\}$ and let f_n denote the characteristic function of A_n . Let f denote the constant zero function. Then $f_n \in B_1(\mathbb{R}) \setminus C(\mathbb{R})$ for each $n \in \mathbb{N}$ and the sequence $\langle f_n \rangle$ pointwise converges to f . For each $k \in \mathbb{N}$, let $\langle g_n^{(k)} \rangle$ be a sequence of continuous functions which pointwise converges to f_k . We show that there exists a mapping u of \mathbb{N} into \mathbb{N} such that for each mapping v of \mathbb{N} into \mathbb{N} , $v(k) > u(k)$ for each $k \in \mathbb{N}$, and for each strictly increasing mapping s of \mathbb{N} into \mathbb{N} the subsequence $\langle g_{v(s(n))}^{(s(n))} \rangle$ of the diagonal sequence $\langle g_{v(n)}^{(n)} \rangle$ does not pointwise converge to f . Clearly, then the pointwise convergence on $B_1(\mathbb{R})$ fails to be strict.

So, since all sets A_k are finite, for each $k \in \mathbb{N}$ choose $u(k) \in \mathbb{N}$ such that $g_n^{(k)}(x) > 1/2$ for each $n > u(k)$ and each $x \in A_k$. Let v be a mapping of \mathbb{N} into \mathbb{N} such that $v(k) > u(k)$ for each $k \in \mathbb{N}$ and let s be a strictly increasing mapping of \mathbb{N} into \mathbb{N} . From $g_{v(s(1))}^{(s(1))}(1/p_{s(1)}) > 1/2$ it follows that there exists a closed interval $I_1 \subset (0, 1)$ such that $1/p_{s(1)} \in \text{int } I_1$ and $g_{v(s(1))}^{(s(1))}(I_1) > 1/2$. Put $t(1) = 1$. By induction, define a strictly increasing mapping t of \mathbb{N} into \mathbb{N} and a sequence $\langle I_n \rangle$ of closed intervals such that $\text{int } I_n \supset I_{n+1} \neq \emptyset$ and $g_{v(s(t(n)))}^{(s(t(n)))}(I_n) > 1/2$ for all $n \in \mathbb{N}$. Choose $t(2) \in \mathbb{N}$ such that $s(t(2)) \in \{s(2), s(3), \dots\}$ and $A_{s(t(2))} \cap \text{int } I_1 \neq \emptyset$. Choose a closed interval I_2 such that $\text{int } I_2 \neq \emptyset$, $\text{int } I_1 \supset I_2$ and $g_{v(s(t(2)))}^{(s(t(2)))}(I_2) > 1/2$. Analogously define $t(3)$ and $I_3, \dots, t(n)$ and I_n , and so on. Now, choose $x_0 \in \bigcap_{i=1}^{\infty} I_i \neq \emptyset$. Since

$g_{v(s(t(n)))}^{(s(t(n)))}(x_0) > 1/2$ for all $n \in \mathbb{N}$, the sequence $\langle g_{v(s(n))}^{(s(n))} \rangle$ does not pointwise converge to f . This completes the proof. \square

2.

In this section we prove some simple facts about strict admissible convergences.

Let f and f_n , $n \in \mathbb{N}$, be functions in $\mathbb{R}^{\mathbb{R}}$. We say (cf. [FKE]) that the sequence $\langle f_n \rangle$ and the function f are *linked* if there are sequences $\langle g_n^{(k)} \rangle$, $k \in \mathbb{N}$, in $\mathbb{R}^{\mathbb{R}}$ such that for each $k \in \mathbb{N}$ the sequence $\langle g_n^{(k)} \rangle$ pointwise converges to f_k and each diagonal sequence $\langle g_{d(n)}^{(n)} \rangle$, $d: \mathbb{N} \rightarrow \mathbb{N}$, pointwise converges to f . Note: condition (s) can be reformulated as “if $\langle f_n \rangle$ converges to f under \mathbb{L} and $f_n \in B_1(\mathbb{R}) \setminus C(\mathbb{R})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\langle f'_n \rangle$ of $\langle f_n \rangle$ which is linked to f via a double sequence, i.e. sequence of sequences, of continuous functions”.

Lemma 2.1. *Let $\langle f_n \rangle$ be a sequence of functions linked to a function f . Then $\langle f_n \rangle$ converges pointwise to f .*

Proof. Assume that, on the contrary, for some $x \in \mathbb{R}$ the sequence $\langle f_n(x) \rangle$ does not converge to $f(x)$. Then there exists a positive number ε such that $|f_n(x) - f(x)| > \varepsilon$ for infinitely many $n \in \mathbb{N}$. Clearly, this is a contradiction with the assumption that $\langle f_n \rangle$ and f are linked. \square

Corollary 2.2. *Every strict convergence on $B_1(\mathbb{R})$ is finer than the pointwise convergence.*

Next, we describe the coarsest and the finest strict convergences on $B_1(\mathbb{R})$ compatible with the ring structures of $B_1(\mathbb{R})$.

Construction 2.3. Denote by \mathbb{L}_s the set of all pairs $(\langle f_n \rangle, f)$ such that $\langle f_n \rangle$ is a sequence of functions of $B_1(\mathbb{R})$, $f \in B_1(\mathbb{R})$ and for each subsequence $\langle f'_n \rangle$ of $\langle f_n \rangle$ there exists its subsequence $\langle f''_n \rangle$ which is linked to f via a double sequence of continuous functions. As a rule, $(\langle f_n \rangle, f) \in \mathbb{L}_s$ means that the sequence $\langle f_n \rangle$ converges to f under \mathbb{L}_s .

Claim 2.3.1. \mathbb{L}_s is a strict \mathcal{L}_0^* -ring convergence.

Proof. It follows easily from Lemma 2.1 that each sequence \mathbb{L}_s -converges to at most one limit. The remaining axioms of convergence follow directly from the definition of \mathbb{L}_s . Indeed, each constant sequence converges, each subsequence of a convergent sequence converges to the same limit, and \mathbb{L}_s satisfies the Urysohn axiom: if $\langle f_n \rangle$ and f are such that for each subsequence $\langle f'_n \rangle$ of $\langle f_n \rangle$ there exists its

subsequence $\langle f'_n \rangle$ such that $(\langle f''_n \rangle, f) \in \mathbb{L}_s$, then $(\langle f_n \rangle, f) \in \mathbb{L}_s$. Further, sums and products of convergent sequences converge to the corresponding sums and products of their limits, hence \mathbb{L}_s is compatible with the ring structure of $B_1(\mathbb{R})$. It follows from Lemma 2.1 that \mathbb{L}_s is admissible and since \mathbb{L}_s is clearly strict, the proof is complete. \square

Claim 2.3.2. Let \mathbb{L} be a strict \mathcal{L}_0^* -ring convergence on $B_1(\mathbb{R})$. Then \mathbb{L}_s is coarser than \mathbb{L} .

P r o o f. If $\langle f_n \rangle$ converges to f under \mathbb{L} , then some subsequence $\langle f'_n \rangle$ of $\langle f_n \rangle$ is linked to f via a double sequence of continuous functions and hence $\langle f'_n \rangle$ converges to f under \mathbb{L}_s . Since \mathbb{L}_s satisfies the Urysohn axiom, it follows that $\mathbb{L} \subset \mathbb{L}_s$. \square

Construction 2.4. Denote by \mathcal{N} the set of all sequences in $B_1(\mathbb{R})$ of the form $\langle \sum_{i=1}^k (f_{in} - f_i)g_i \rangle$, where $k \in \mathbb{N}$, $\langle f_{in} \rangle$ is a sequence of continuous functions pointwise converging to $f_i \in B_1(\mathbb{R})$, $i = 1, \dots, k$. Trivially, \mathcal{N} is closed with respect to subsequences and finite sums. Since

$$\begin{aligned} \langle (f_{1n} - f_1)g_1 \rangle \langle (f_{2n} - f_2)g_2 \rangle &= \langle (f_{1n}f_{2n} - f_1f_2)g_1g_2 \rangle \\ &\quad - \langle (f_{1n} - f_1)f_2g_1g_2 \rangle - \langle (f_{2n} - f_2)f_1g_1g_2 \rangle, \end{aligned}$$

it follows that \mathcal{N} is closed with respect to finite products, too. By Lemma 2 in [FZE], there exists a unique \mathcal{L} -ring convergence under which a sequence $\langle f_n \rangle$ converges to the constant zero function \mathbb{O} iff $\langle f_n \rangle \in \mathcal{N}$. Denote by \mathbb{L}_r its Urysohn modification. Recall that $\langle f_n \rangle$ converges to \mathbb{O} under \mathbb{L}_r iff for each subsequence $\langle f'_n \rangle$ of $\langle f_n \rangle$ there exists its subsequence $\langle f''_n \rangle$ belonging to \mathcal{N} .

Claim 2.4.1. \mathbb{L}_r is a strict \mathcal{L}_0^* -ring convergence.

P r o o f. Obviously, \mathbb{L}_r is finer than the pointwise convergence on $B_1(\mathbb{R})$ and hence the limits of \mathbb{L}_r -convergent sequences are uniquely determined. Thus \mathbb{L}_r is an \mathcal{L}_0^* -ring convergence. If $\langle f_n \rangle$ is a sequence of continuous functions pointwise converging to $f \in B_1(\mathbb{R})$, then $(\langle f_n \rangle, f) \in \mathbb{L}_r$. Consequently, \mathbb{L}_r is admissible. The proof of strictness of \mathbb{L}_r is straightforward. *Hint:* if $(\langle h_n \rangle, h) \in \mathbb{L}_r$ and $h_n \in B_1(\mathbb{R}) \setminus C(\mathbb{R})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\langle h'_n \rangle$ of $\langle h_n \rangle$ such that $\langle h'_n - h \rangle \in \mathcal{N}$; hence $\langle h'_n \rangle$ is of the form $\langle \sum_{i=1}^k (f_{in} - f_i)g_i + h \rangle$, where $f_i, g_i, h \in B_1(\mathbb{R})$, and $\langle f_{in} \rangle$ is a sequence of continuous functions pointwise converging to f_i , $i = 1, \dots, k$; the rest is trivial. \square

Claim 2.4.2. Let \mathbb{L} be a strict \mathcal{L}_0^* -ring convergence on $B_1(\mathbb{R})$. Then $\mathbb{L}_r \subset \mathbb{L}$.

Proof. Since \mathbb{L} is admissible and compatible with the ring structure of $B_1(\mathbb{R})$, $\langle (f_n - f)g \rangle$ converges under \mathbb{L} to \mathbb{O} whenever $\langle f_n \rangle$ is a sequence of continuous functions pointwise converging to $f \in B_1(\mathbb{R})$ and $g \in B_1(\mathbb{R})$. Hence $\mathbb{L}_r \subset \mathbb{L}$.

It is known that each commutative \mathcal{L}_0^* -group can have many nonequivalent \mathcal{L}_0^* -group completions and its Novák completion ([NOV]) yields its categorical \mathcal{L}_0^* -group completion ([FKO]). We show that the Novák \mathcal{L}_0^* -group completion of $C(\mathbb{R})$ fails to be an \mathcal{L}_0^* -ring completion. \square

Example 2.5. Let \mathbb{L} denote the pointwise convergence on $C(\mathbb{R})$. Then the Novák \mathcal{L}_0^* -group completion of $C(\mathbb{R})$ has $B_1(\mathbb{R})$ as its underlying group and is equipped with an \mathcal{L}_0^* -group convergence \mathbb{L}_1^* defined as follows: $\langle f_n \rangle$ converges to f under \mathbb{L}_1^* iff for each subsequence $\langle f'_n \rangle$ of $\langle f_n \rangle$ there exists its subsequence $\langle f''_n \rangle$ such that $f''_n - f = g_n - g$, $n \in \mathbb{N}$, where $\langle g_n \rangle$ is a sequence of continuous functions pointwise converging to $g \in B_1(\mathbb{R})$. Let h be the characteristic function of the singleton $\{0\}$ and let f_n be the constant function with value $1/n$, $n \in \mathbb{N}$. Then $h \in B_1(\mathbb{R})$ and the sequence of continuous functions $\langle f_n \rangle$ pointwise converges to \mathbb{O} , but their product $\langle hf_n \rangle$ fails to converge under \mathbb{L}_1^* . Hence \mathbb{L}_1^* fails to be an \mathcal{L}_0^* -ring convergence and clearly $\mathbb{L}_1^* \not\subseteq \mathbb{L}_r$.

Note: it is known that an \mathcal{L}_0^* -ring need not have an \mathcal{L}_0^* -ring completion ([FZE]) and there are known sufficient conditions guaranteeing the existence of the categorical \mathcal{L}_0^* -ring completion ([FKO]); $C(\mathbb{R})$ fails to be a field and hence does not satisfy the conditions.

We finish this section by mentioning some problems. First, we do not know whether \mathbb{L}_s or \mathbb{L}_r is complete. Second, if not, then we can ask whether $B_1(\mathbb{R})$ equipped with \mathbb{L}_s or \mathbb{L}_r has an \mathcal{L}_0^* -ring completion, at all.

3.

Our final goal is to construct an \mathcal{L}_0^* -ring completion of $C(\mathbb{R})$ having a universal extension property or, in categorical terms, an epireflection of $C(\mathbb{R})$ into complete \mathcal{L}_0^* -rings. Since $C(\mathbb{R})$ is not a field, we cannot use the construction due to J. Novák. The interested reader is referred to [FKO] for the background information about \mathcal{L}_0^* -ring completions and to [HES] about categorical notions. To make the paper more self-contained, we briefly recall some related notions.

Let \mathbb{K} be an \mathcal{L}_0^* -convergence on a set $Y \neq \emptyset$. For $A \subset Y$, denote by $\text{cl } A$ the set of all \mathbb{K} -limits of sequences ranging in A . Define $0\text{-cl } A = A$ and, by induction, for each ordinal number $\alpha \leq \omega_1$ define $\alpha\text{-cl } A = \bigcup_{\beta < \alpha} \text{cl}(\beta\text{-cl } A)$. Then $1\text{-cl } A = \text{cl } A$ and

each α -cl yields a closure operator on Y . Further, ω_1 -cl is idempotent and hence a topology; it is the finest of all topologies on Y coarser than cl.

Note: if $\omega_1\text{-cl } A = Y$ and f, g are continuous maps of (Y, \mathbb{K}) into an \mathcal{L}_0^* -space (Y', \mathbb{K}') such that $f(x) = g(x)$ for each $x \in A$, then $f = g$; consequently a morphism with a topologically dense range is an epimorphism in \mathcal{L}_0^* -spaces.

Let Y be a ring (commutative, not necessarily possessing a unit element) and let \mathbb{K} be an \mathcal{L}_0^* -ring convergence on Y . A sequence $\langle x_n \rangle$ is said to be *Cauchy* if $\langle x'_n - x''_n \rangle$ converges under \mathbb{K} to zero whenever $\langle x'_n \rangle$ and $\langle x''_n \rangle$ are subsequences of $\langle x_n \rangle$. If each Cauchy sequence converges, then we speak of a complete \mathcal{L}_0^* -ring. Let X be a subring of Y . Then, for each ordinal number $\alpha \leq \omega_1$, the set $\alpha\text{-cl } X$ is a subring of Y and if Y is complete, then the subring $\omega_1\text{-cl } X$ is complete, too. If Y is complete and $Y = \omega_1\text{-cl } X$, then (Y, \mathbb{K}) is said to be an \mathcal{L}_0^* -ring *completion* of X carrying the restriction of \mathbb{K} to X . Finally, for \mathcal{L}_0^* -convergences the coordinatewise convergence on products is the categorical one and the product of complete \mathcal{L}_0^* -rings is a complete \mathcal{L}_0^* -ring.

Let \mathcal{A}, \mathcal{B} , and \mathcal{C} denote the categories of all \mathcal{L}_0^* -rings, all \mathcal{L}_0^* -rings having a completion, and all complete \mathcal{L}_0^* -rings, respectively. A straightforward application of the usual categorical tricks yields the following

Theorem 3.1. *\mathcal{C} is an epireflective subcategory of \mathcal{B} .*

Proof. Let X be a ring carrying an \mathcal{L}_0^* -ring convergence \mathbb{L} and let $(\overline{X}, \overline{\mathbb{L}})$ be its \mathcal{L}_0^* -ring completion. We show that there exists its \mathcal{L}_0^* -ring completion $\varrho(X, \mathbb{L}) = (\hat{X}, \hat{\mathbb{L}})$ having the following universal extension property: for each continuous homomorphism f of (X, \mathbb{L}) into a complete \mathcal{L}_0^* -ring (Y, \mathbb{K}) there exists a unique continuous homomorphism \hat{f} of $(\hat{X}, \hat{\mathbb{L}})$ into (Y, \mathbb{K}) such that $f(x) = \hat{f}(x)$ for each $x \in X$. Since \hat{X} is the smallest sequentially closed subset containing X , the embedding $\text{id}: (X, \mathbb{L}) \rightarrow (\hat{X}, \hat{\mathbb{L}})$ is an epimorphism and ϱ yields an epireflector of \mathcal{B} into \mathcal{C} . The construction of $(\hat{X}, \hat{\mathbb{L}})$ is divided into two parts. The first has an auxiliary character.

Part 1. There exists a nonempty set $\mathcal{S} = \{f_a: (X, \mathbb{L}) \rightarrow (X_a, \mathbb{L}_a); a \in A\}$ of continuous homomorphisms such that each (X_a, \mathbb{L}_a) is a complete \mathcal{L}_0^* -ring and if f is a continuous homomorphism of (X, \mathbb{L}) into a complete \mathcal{L}_0^* -ring (Y, \mathbb{K}) , then there exists $a \in A$ and a homeomorphic isomorphism g of (X_a, \mathbb{L}_a) onto a subring $(Y_f, \mathbb{K} \upharpoonright Y_f)$ of (Y, \mathbb{K}) such that f is a composition of $g \circ f_a$ and the embedding of $(Y_f, \mathbb{K} \upharpoonright Y_f)$ into (Y, \mathbb{K}) . Indeed, each f determines a complete \mathcal{L}_0^* -subring of (Y, \mathbb{K}) the underlying set of which is $Y_f = \omega_1\text{-cl } f(X)$. Since $\text{card}(1\text{-cl } f(X))$ cannot exceed the cardinality of the set of all countable infinite subsets of $f(X)$ and $\omega_1\text{-cl } f(X) = \bigcup_{\beta < \omega_1} \beta\text{-cl } f(X)$, it follows that $\text{card}(Y_f) \leq \exp \text{card}(X)$. Hence there is a set $\{(X_b, \mathbb{L}_b); b \in B\}$ of complete \mathcal{L}_0^* -rings such that $\text{card}(X_b) \leq \exp \text{card}(X)$ and

each $(Y_f, \mathbb{K} \upharpoonright Y_f)$ is homeomorphic and isomorphic to some (X_b, \mathbb{L}_b) , $b \in B$. Note: \mathcal{S} yields a so-called solution set for (X, \mathbb{L}) with respect to the inclusion functor of \mathcal{C} into \mathcal{B} .

Part 2. The product $\prod_{a \in A} (X_a, \mathbb{L}_a)$ is a complete \mathcal{L}_0^* -ring and, via the canonical embedding sending $x \in X$ into $\varphi(x) = (f_a(x), a \in A)$, (X, \mathbb{L}) can be view as the corresponding \mathcal{L}_0^* -subring of $\prod_{a \in A} (X_a, \mathbb{L}_a)$ (remember, (X, \mathbb{L}) is an \mathcal{L}_0^* -subring of its completion $(\overline{X}, \overline{\mathbb{L}})$). Denote by $(\hat{X}, \hat{\mathbb{L}})$ the smallest sequentially closed \mathcal{L}_0^* -subring of $\prod_{a \in A} (X_a, \mathbb{L}_a)$ containing (X, \mathbb{L}) . It is easy to see that $(\hat{X}, \hat{\mathbb{L}})$ has the desired properties.

This completes the proof. \square

Lemma 3.2. *Let (Y, \mathbb{K}) be a complete \mathcal{L}_0^* -ring and let X be a subring of Y . Put $\overline{X} = \omega\text{-cl } X$ and define $\overline{\mathbb{L}} \subset \overline{X}^{\mathbb{N}} \times \overline{X}$ as follows: $(\langle x_n \rangle, x) \in \overline{\mathbb{L}}$ whenever $(\langle x_n \rangle, x) \in \mathbb{K}$ and there exists a natural number k such that $x_n \in (k\text{-cl } X)$ for each $n \in \mathbb{N}$. Then $(\overline{X}, \overline{\mathbb{L}})$ is a complete \mathcal{L}_0^* -ring and the identity mapping on \overline{X} is a continuous isomorphism of $(\overline{X}, \overline{\mathbb{L}})$ onto $(\overline{X}, \mathbb{K} \upharpoonright \overline{X})$.*

Proof. It is easy to verify that $\overline{\mathbb{L}}$ is an \mathcal{L}_0^* -ring convergence on \overline{X} finer than $\mathbb{K} \upharpoonright \overline{X}$. If $\langle x_n \rangle$ is a Cauchy sequence in $(\overline{X}, \overline{\mathbb{L}})$, then there exists $k \in \mathbb{N}$ such that $x \in (k\text{-cl } X)$ for each $n \in \mathbb{N}$. Thus $(\overline{X}, \overline{\mathbb{L}})$ is complete. \square

Theorem 3.3. *Let (X, \mathbb{L}) be an \mathcal{L}_0^* -ring in \mathcal{B} and let $(\hat{X}, \hat{\mathbb{L}})$ be its categorical \mathcal{L}_0^* -ring completion. Then $\omega\text{-cl } X = \hat{X}$.*

Proof. The assertion follows from Lemma 3.2. Putting $\hat{X} = Y$ and $\hat{\mathbb{L}} = \mathbb{K}$, we easily infer that $\overline{X} = \hat{X}$ and $\overline{\mathbb{L}} = \hat{\mathbb{L}}$. \square

Corollary 3.4. *$B(\mathbb{R})$ carrying the pointwise convergence fails to be the categorical completion of $C(\mathbb{R})$.*

Proof. Indeed, $\omega\text{-cl } C(\mathbb{R}) = B_\omega(\mathbb{R}) \subsetneq B(\mathbb{R})$, while $C(\mathbb{R})$ is ω -dense in its categorical \mathcal{L}_0^* -ring completion. \square

Problem. Describe the categorical \mathcal{L}_0^* -ring completion of $C(\mathbb{R})$.

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