Czechoslovak Mathematical Journal

H. K. Pathak; Shin Min Kang; Yeol Je Cho

Coincidence and fixed point theorems for nonlinear hybrid generalized contractions

Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 2, 341–357

Persistent URL: `http://dml.cz/dmlcz/127421`

Czechoslovak Mathematical Journal, 48 (123) (1998), 341–357

COINCIDENCE AND FIXED POINT THEOREMS FOR NONLINEAR HYBRID GENERALIZED CONTRACTIONS

H. K. Pathak, Bhilai Nagar, S. M. Kang and Y. J. Cho, Chinju

(Received November 13, 1995)

Abstract. In this paper we first prove some coincidence and fixed point theorems for nonlinear hybrid generalized contractions on metric spaces. Secondly, using the concept of an asymptotically regular sequence, we give some fixed point theorems for Kannan type multi-valued mappings on metric spaces. Our main results improve and extend several known results proved by other authors.

MSC 2000: 54H25, 47H10

Keywords: Weakly commuting, compatible and weakly compatible mappings, asymptotically regular sequence, coincidence point and fixed point, Kannan mapping

1. Introduction and preliminaries

Let (X,d) be a metric space and let f and g be mappings from X into itself. In [19], Sessa defined f and g to be *weakly commuting* if $d(gfx, fgx) \leqslant d(gx, fx)$ for all x in X. It can be seen that commuting mappings are weakly commuting, but the converse is false as shown by Example in [21].

Recently, Jungck [6] extended the concept of weak commutativity in the following way:

Definition 1.1. Let f and g be mappings from a metric space (X,d) into itself. The mappings f and g are said to be *compatible* if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = z$ for some z in X.

It is obvious that weakly commuting mappings are compatible, but the converse is not true. Some examples of this fact can be found in [6].

Recently, Kaneko [9] and Singh et al. [24] extended the concepts of weak commutativity and compatibility for single-valued mappings to the setting of single-valued and multi-valued mappings, respectively.

Let (X,d) be a metric space and let $CB(X)$ denote the family of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric on $CB(X)$ induced by the metric d, i.e.,

$$H(A,B)=\max\Big\{\sup_{x\in A} d(x,B),\sup_{y\in B} d(y,A)\Big\}$$

for all $A,B\in CB(X)$, where $d(x,A)=\inf\limits_{y\in A} d(x,y)$.

It is well-known that $(CB(X),H)$ is a metric space, and if a metric space (X,d) is complete, then $(CB(X),H)$ is complete.

Lemma 1.1. [19] *Let $A,B\in CB(X)$ and $k>1$. Then for each $a\in A$, there exists a point $b\in B$ such that $d(a,b)\leqslant kH(A,B)$.*

Let (X,d) be a metric space and let $f\colon X\to X$ and $T\colon X\to CB(X)$ be single-valued and multi-valued mappings, respectively.

Definition 1.2. The mappings f and T are said to be *weakly commuting* if, for all $x\in X$, $fTx\in CB(X)$ and $H(Tfx,fTx)\leqslant d(fx,Tx)$, where H is the Hausdorff metric defined on $CB(X)$.

Definition 1.3. The mappings f and T are said to be *compatible* if and only if $fTx\in CB(X)$ for all $x\in X$ and $H(Tfx_n,fTx_n)\to 0$ as $n\to\infty$ whenever $\{x_n\}\subset X$ such that $Tx_n\to M\in CB(X)$ and $fx_n\to t\in M$ as $n\to\infty$.

Remark 1.1. In [10], Kaneko and Sessa gave an example that weak commutativity implies compatibility, but the converse is not true.

Recently, Pathak [16] introduced the concept of weak compatible mappings for single-valued and multi-valued mappings on a metric space as follows:

Definition 1.4. The mappings f and T are said to be *f-weak compatible* if $fTx\in CB(X)$ for all $x\in X$ and the following limits exist and satisfy the relevant inequalities:

(i) $$\lim_{n\to\infty} H(fTx_n,Tfx_n)\leqslant \lim_{n\to\infty} H(Tfx_n,Tx_n),$$

(ii) $$\lim_{n\to\infty} d(fTx_n,fx_n)\leqslant \lim_{n\to\infty} H(Tfx_n,Tx_n),$$

whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to M \in CB(X)$ and $fx_n \to t \in M$ as $n \to \infty$.

It can be seen that compatible mappings f and T are weak compatible, but the converse is not true. Example in [16] and Example 2.1 of this paper support this fact.

Note that if T is a single-valued mapping in Definitions 1.2, 1.3 and 1.4, we obtain the concepts of weak commutativity [21], compatibility [6] and weak compatibility [15], [17], [18] for single-valued mappings.

Remark 1.2. We have only f-weak compatibility for single-valued and multi-valued mappings in contrast to single-valued mappings for which we can define f-weak as well as T-weak compatibility.

On the other hand, in [20], Rhoades et al. introduced the concept of asymptotically regular sequences in metric spaces and proved a fixed point theorem using this concept.

Let $T\colon X \to CB(X)$ be a multi-valued mapping and $\{x_n\}$ a sequence in X. Let $f\colon X \to X$ be a mapping such that $T(X) \subset f(X)$. Then $\{x_n\}$ is said to be asymtotically T-regular with respect to f if $d(fx_n, Tx_n) \to 0$ as $n \to \infty$. In the case when f is the identity mapping on X, we simply say that the sequence $\{x_n\}$ is asymptotically T-reqular [18]. A point x is said to be a fixed point of a single-valued mapping f (a multi-valued mapping T) if $x = fx$ ($x \in Tx$). The point x is called a coincidence point of f and T if $fx \in Tx$.

In this paper we give some concidence and fixed point theorems for nonlinear hybrid generalized contractions, i.e., the generalized contractive conditions including single-valued and multi-valued mappings on metric spaces. Our main results improve and generalize many results proved by many authors, Kaneko [8], [9], Kaneko and Sessa [10], Kubiak [13], Pathak [16], Nadler [19] and Singh et al. [24].
Finally, we prove some fixed point theorems for Kannan type multi-valued mappings by using the concept of asymptotically regular sequences in metric spaces.

2. Coincidence theorems and fixed point theorems

In this section we give some coincidence and fixed point theorems for nonlinear hybrid generalized contractions.

Theorem 2.1. *Let (X, d) be a complete metric space, let $f\colon X \to X$ and $T\colon X \to CB(X)$ be f-weak compatible continuous mappings such that $T(X) \subset$*

$f(X)$ and

$$H(Tx,Ty) \leqslant h \cdot [a \cdot L(x,y) + (1-a) \cdot N(x,y)] \tag{2.1}$$

for all x, y in X, where $0 \leqslant h < 1$, $0 \leqslant a \leqslant 1$,

$$L(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \tfrac{1}{2}[d(fx,Ty) + d(fy,Tx)]\}$$

and

$$N(x,y) = [\max\{d^2(fx,fy), d(fx,Tx) \cdot d(fy,Ty), d(fx,Ty) \cdot d(fy,Tx), \\ \tfrac{1}{2}d(fx,Tx) \cdot d(fy,Tx), \tfrac{1}{2}d(fx,Ty) \cdot d(fy,Ty)\}]^{1/2}.$$

Then there exists a point $t \in X$ such that $ft \in Tt$, i.e., the point t is a coincidence point of f and T.

Proof. Pick x_0 in X and choose $x_1 \in X$ such that $fx_1 \in Tx_0$. This is possible since $Tx_0 \subset f(X)$. If $h = 0$, we obtain $d(fx_1, Tx_1) \leqslant k \cdot H(Tx_0, Tx_1) = 0$, i.e., $fx_1 \in Tx_1$ since Tx_1 is closed. Assume that $0 < h < 1$ and set $k = 1/\sqrt{h}$. By the definition of H, there exists a point $y_1 \in Tx_1$ such that $d(y_1, fx_1) \leqslant k \cdot Hd(Tx_1, Tx_0)$. Observe that this inequality may be in the reversed direction if $k \leqslant 1$ by Lemma 1.1. Since $Tx_1 \subset f(X)$, let $x_2 \in X$ be such that $y_1 = fx_2$. In general, having chosen $x_n \in X$, we may choose $x_{n+1} \in X$ such that

$$y_n = fx_{n+1} \in Tx_n \quad \text{and} \quad d(y_n, fx_n) \leqslant k \cdot H(Tx_n, Tx_{n-1})$$

for each $n \geqslant 1$. Using (2.1), we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leqslant k \cdot H(Tx_{n-1}, Tx_n) \\ &\leqslant \sqrt{h} \cdot [a \cdot L(x_{n-1}, x_n) + (1-a) \cdot N(x_{n-1}, x_n)], \end{aligned}$$

where

$$\begin{aligned} L(x_{n-1}, x_n) &\leqslant \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad \tfrac{1}{2}[d(fx_{n-1}, fx_{n+1}) + 0]\} \\ &\leqslant \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \tfrac{1}{2}[d(fx_{n-1}, fx_n) \\ &\qquad + d(fx_n, fx_{n+1})]\} \\ &\leqslant \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\} \end{aligned}$$

and

$$\begin{aligned}N(x_{n-1},x_n) &\leqslant [\max\{d^2(fx_{n-1},fx_n), d(fx_{n-1},fx_n)\cdot d(fx_n,fx_{n+1}),0,0,\\ &\qquad \tfrac{1}{2}d(fx_{n-1},fx_{n+1})\cdot d(fx_n,fx_{n+1})\}]^{1/2}\\ &\leqslant [\max\{d^2(fx_{n-1},fx_n), d(fx_{n-1},fx_n)\cdot d(fx_n,fx_{n+1}),\\ &\qquad \tfrac{1}{2}[d(fx_{n-1},fx_n)+d(fx_n,fx_{n+1})]\cdot d(fx_n,fx_{n+1})\}]^{1/2}\\ &\leqslant [\max\{d^2(fx_{n-1},fx_n), d(fx_{n-1},fx_n)\cdot d(fx_n,fx_{n+1}),\\ &\qquad d^2(fx_n,fx_{n+1})\}]^{1/2}.\end{aligned}$$

Suppose that $d(fx_n,fx_{n+1}) > \sqrt{h}\cdot d(fx_{n-1},fx_n)$ for some $n\in\mathbb{N}$. Then we obtain $d(fx_n,fx_{n+1}) < d(fx_n,fx_{n+1})$, which is a contradiction, and so

$$d(fx_n,fx_{n+1}) \leqslant \sqrt{h}\cdot d(fx_{n-1},fx_n) \tag{2.2}$$

for all $n\in\mathbb{N}$. Since $\sqrt{h}<1$ and X is complete, it follows from (2.2) that $\{fx_n\}$ is a Cauchy sequence converging to a point $t\in X$. Also, the fact that $H(Tx_{n-1},Tx_n)\leqslant h\cdot d(fx_{n-1},fx_n)$ and $\{fx_n\}$ is a Cauchy sequence in X implies that $\{Tx_n\}$ is a Cauchy sequence in the complete metric space $(CB(X),H)$. So, letting $Tx_n\to M\in CB(X)$, we have

$$\begin{aligned}d(t,M) &\leqslant d(t,fx_n)+d(fx_n,M)\\ &\leqslant d(t,fx_n)+H(Tx_{n-1},M)\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Thus, since M is closed, we have $t\in M$. Also, the f-weak compatibility of f and T implies that

$$\lim_{n\to\infty} H(fTx_n,Tfx_n) \leqslant \lim_{n\to\infty} H(Tfx_n,Tx_n)$$

and

$$\lim_{n\to\infty} d(fTx_n,fx_n) \leqslant \lim_{n\to\infty} H(Tfx_n,Tx_n).$$

Using the above second inequality, we obtain

$$\begin{aligned}\lim_{n\to\infty} d(ffx_n,fx_n) &\leqslant \lim_{n\to\infty} d(ffx_n,fTx_n)+\lim_{n\to\infty} d(fTx_n,fx_n)\\ &\leqslant \lim_{n\to\infty} d(ffx_n,fTx_n)+\lim_{n\to\infty} H(Tfx_n,Tx_n).\end{aligned}$$

Since f and T are continuous, we have

$$H(f(M),Tt)\leqslant H(Tt,M) \quad \text{and} \quad d(ft,t)\leqslant H(Tt,M). \tag{2.3}$$

On the other hand, we have

$$d(ft,Tt) \leqslant d(ft,f^2x_{n+1}) + d(f^2x_{n+1},Tt)$$

$$(2.4) \qquad \begin{aligned} &\leqslant d(ft,f^2x_{n+1}) + H(fTx_n,Tt) \\ &\leqslant d(ft,f^2x_{n+1}) + H(fTx_n,Tfx_n) + H(Tfx_n,Tt) \\ &\leqslant d(ft,f^2x_{n+1}) + H(Tfx_n,Tx_n) + \varepsilon_n, \end{aligned}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Thus, letting $n \to \infty$ in (2.4), we have $d(ft,Tt) \leqslant H(Tt,M)$. Now using (2.1), we have

$$(2.5) \qquad H(Tx_n,Tt) \leqslant h \cdot [a \cdot L(x_n,t) + (1-a) \cdot N(x_n,t)],$$

where

$$(2.6) \qquad \begin{aligned} L(x_n,t) &= \max\{d(fx_n,ft), d(fx_n,Tx_n), d(ft,Tt), \\ &\qquad \tfrac{1}{2}[d(fx_n,Tt) + d(ft,Tx_n]\} \\ &\leqslant \max\{d(fx_n,ft), d(fx_n,Tx_n), d(ft,Tt), \\ &\qquad \tfrac{1}{2}[d(fx_n,Tt) + d(ft,fx_n) + d(fx_n,Tx_n)]\} \end{aligned}$$

and

$$(2.7) \qquad \begin{aligned} N(x_n,t) \leqslant [\max\{&d^2(fx_n,ft), d(fx_n,Tx_n) \cdot d(ft,Tt), \\ &d(fx_n,Tt) \cdot [d(ft,fx_n) + d(fx_n,Tx_n)], \\ &\tfrac{1}{2}d(fx_n,Tx_n) \cdot [d(ft,fx_n) + d(fx_n,Tx_n)], \\ &\tfrac{1}{2}d(fx_n,Tt) \cdot d(ft,Tt)\}]^{1/2}. \end{aligned}$$

Passing to the limits in (2.6) and (2.7) as $n \to \infty$, we have

$$(2.8) \qquad \begin{aligned} \lim_{n\to\infty} L(x_n,t) &\leqslant \max\{d(t,ft), d(t,M), d(ft,Tt), \\ &\qquad \tfrac{1}{2}[d(t,Tt) + d(ft,t) + d(t,M)]\} \\ &\leqslant \max\{H(Tt,M), 0, H(Tt,M), \tfrac{1}{2}[H(M,Tt) + H(Tt,M)]\} \\ &= H(M,Tt) \end{aligned}$$

and

$$(2.9) \qquad \begin{aligned} \lim_{n\to\infty} N(x_n,t) &\leqslant [\max\{d^2(t,ft), d(t,M) \cdot d(ft,Tt), \\ &\qquad d(t,Tt) \cdot [d(ft,t) + d(t,M)], \\ &\qquad \tfrac{1}{2}d(t,M) \cdot [d(ft,t) + d(t,M)], \\ &\qquad \tfrac{1}{2}d(t,Tt) \cdot d(ft,Tt)\}]^{1/2} \\ &\leqslant [\max\{H^2(Tt,M), 0, H(Tt,M)[H(Tt,M) + 0], 0, \\ &\qquad \tfrac{1}{2}H^2(Tt,M)\}]^{1/2} \\ &= H(M,Tt), \end{aligned}$$

respectively. Thus, we have, from (2.5), (2.8) and (2.9),

$$H(M,Tt) \leqslant h \cdot [a \cdot H(M,Tt) + (1-a) \cdot H(M,Tt)] = h \cdot H(M,Tt),$$

which implies that $H(M,Tt)=0$. Therefore, $d(ft,Tt)=0$ and so $ft \in Tt$ since Tt is closed. This completes the proof. □

Remark 2.1. We recall that a non-empty subset S of X is proximinal if for each $x \in X$, there exists a point $y \in S$ such that $d(x,y)=d(x,S)$. Let $PB(X)$ be the family of all bounded proximinal subsets of X. If $T\colon X \to PB(X)$ and we have chosen $x_n \in X$, let $x_{n+1} \in X$ be such that

$$y_n = fx_{n+1} \in Tx_n \quad \text{and} \quad d(fx_n, y) = d(fx_n, Tx_n).$$

We include here an iteration scheme of Smithson [23], where $T(X)$ is compact and hence proximinal. Since a proximinal set is closed, we have $PB(X) \subset CB(X)$, and it can be observed that the results of [8] and [9] follow as corollaries.

Corollary 2.2. *Let (X,d) be a complete metric space, let $f\colon X \to X$ and $T\colon X \to PB(X)$ be continuous mappings such that $fTx \in PB(X)$ and $H(Tfx, fTx) \leqslant h \cdot d(fx,Tx)$ for all x,y in X. If $T(X) \subset f(X)$ and (2.1) is satisfied for all x,y in X, where $0 \leqslant h < 1$, then there exists $t \in X$ such that $ft \in Tt$.*

Corollary 2.3. *Let (X,d) be a complete metric space, $T\colon X \to CB(X)$ and let f be a continuous self-mapping of X such that $H(Tx,Ty) \leqslant h \cdot d(fx,fy)$ for all x,y in X, where $0 \leqslant h < 1$ and $Tfx = fTx$. If $T(X) \subset f(X)$, then there exists $t \in X$ such that $ft \in Tt$.*

Remark 2.2. In Corollary 2.3, the continuity of f implies the continuity of T.

Corollary 2.4. [16] *Let (X,d) be a complete metric space, let $f\colon X \to X$ and $T\colon X \to CB(X)$ be f-weak compatible continuous mappings such that $T(X) \subset f(X)$ and*

$$H(Tx,Ty) \leqslant h \cdot L(x,y) \tag{2.10}$$

for all x,y in X, where $0 \leqslant h < 1$. Then there exists a point $t \in X$ such that $ft \in Tt$.

Corollary 2.5. *Let (X,d) be a complete metric space, let $f\colon X \to X$ and $T\colon X \to CB(X)$ be f-weak compatible continuous mappings such that $T(X) \subset f(X)$ and*

$$H(Tx,Ty) \leqslant h \cdot N(x,y) \tag{2.11}$$

for all x, y in X, where $0 \leqslant h < 1$. Then there exists a point $t \in X$ such that $ft \in Tt$.

The following example shows that Theorem 2.1 is indeed a proper generalization of Theorem 2 in [10], Corollaries 2.2 and 2.3.

Example 2.1. Let $X = [0, \infty)$ be endowed with the Euclidean metric $d(x, y) = |x - y|$. Let $f(x) = \frac{3}{2}(x^2 + x)$ and $Tx = [0, x^2 + 2]$ for each $x \geqslant 0$. Then T and f are clearly continuous and $T(X) = f(X) = X$. Since $Tx_n \to [0, 3]$ and $fx_n = \frac{3}{2}(x_n^2 + x_n) \to 3 \in [0, 3]$ if $x_n \to 1$, we easily conclude that

$$d(fTx_n, fx_n) \to 0, \quad H(fTx_n, Tfx_n) \to 7, \quad H(Tfx_n, Tx_n) \to 8$$

if $x_n \to 1$ and therefore f and T are f-weak compatible, but they are not compatible. Thus, Theorem 2 in [10] is not applicable. Corollaries 2.2 and 2.3 are not applicable either since f and T are not weakly commuting (for $x = 3$) and hence they are not commuting. Again, since

$$\begin{aligned} H(Tx, Ty) &= |x^2 - y^2| \\ &= \frac{2(x+y)}{3(x+y+1)} \left(\frac{3}{2}|x - y|(x + y + 1)\right) \\ &= \frac{2(x+y)}{3(x+y+1)} \left(\frac{3}{2}|x^2 - y^2 + x - y|\right) \\ &\leqslant \frac{2}{3} d(fx, fy) \\ &\leqslant h \cdot [a \cdot L(x, y) + (1 - a) \cdot N(x, y)] \end{aligned}$$

for all $x, y \in X$, where $h \in \left[\frac{2}{3}, 1\right)$ and $0 \leqslant a \leqslant 1$, all conditions of Theorem 2.1 are satisfied and for each $t \in [0, 1]$, we have $ft \in Tt$.

In the sequel, we use the following lemma for our main theorem, which is a generalization of lemmas from [10] and [16].

Lemma 2.6. *Let $T\colon X \to CB(X)$ and $f\colon X \to X$ be f-weak compatible mappings. If $fw \in Tw$ for some $w \in X$ and (2.1) holds for all x, y in X, then $fTw = Tfw$.*

P r o o f. Let $x_n = w$ for all $n \in \mathbb{N}$. Then $fx_n = fw \to fw$ and $Tx_n \to M = Tw$ as $n \to \infty$. Hence, if $fw \in Tw$, then, by the f-weak compatibility of f and T,

$$\begin{aligned} &H(fTw, Tfw) \leqslant H(Tfw, Tw), \\ &d(f^2w, fw) \leqslant d(f^2w, fTw) + d(fTw, fw) \leqslant H(Tfw, Tw). \end{aligned} \tag{2.12}$$

By (2.1) we obtain

$$H(Tfw, Tw) \leqslant h \cdot [a \cdot L(fw, w) + (1-a) \cdot N(fw, w)], \tag{2.13}$$

where

$$\begin{aligned} L(fw, w) &= \max\{d(f^2w, fw), d(fw, Tfw), d(fw, Tw), \\ &\qquad \tfrac{1}{2}[d(f^2w, Tw) + d(fw, Tfw)]\} \\ &\leqslant \max\{H(Tfw, Tw), H(Tfw, Tw), 0, \\ &\qquad \tfrac{1}{2}[d(f^2w, fw) + d(fw, Tw) + H(Tw, Tfw)]\} \\ &\leqslant \max\{H(Tfw, Tw), H(Tfw, Tw), 0, H(Tfw, Tw)\} \\ &= H(Tfw, Tw) \end{aligned}$$

and

$$\begin{aligned} N(fw, w) &= [\max\{d^2(f^2w, fw), d(f^2w, Tfw) \cdot d(fw, Tw), \\ &\qquad d(f^2w, Tw) \cdot d(fw, Tfw), \tfrac{1}{2}d(f^2w, Tfw) \cdot d(fw, Tfw), \\ &\qquad \tfrac{1}{2}d(f^2w, Tw) \cdot d(fw, Tw)\}]^{1/2} \\ &\leqslant [\max\{H^2(Tfw, Tw), 0, H^2(Tfw, Tw), \tfrac{1}{2}H^2(Tfw, Tw), 0\}]^{1/2} \\ &= H(Tfw, Tw). \end{aligned}$$

Hence, (2.13) implies that

$$\begin{aligned} H(Tfw, Tw) &\leqslant h \cdot [a \cdot H(Tfw, Tw) + (1-a) \cdot H(Tfw, Tw)] \\ &= h \cdot H(Tfw, Tw), \end{aligned}$$

which is a contradiction. Therefore, we have $Tfw = Tw$ and hence (2.12) implies $Tfw = fTw$. This completes the proof. □

To obtain a fixed point theorem, we need additional assumptions as those given in [6], [9] and [10]. In the sequel, by applying the proof technique of [10] and using the above lemma, we have the following theorem:

Theorem 2.7. *Let f and T have the same meanings as in Theorem* 2.1. *Assume also that for each $x \in X$ either* (i) *$fx \neq f^2x$ implies $fx \in Tx$ or* (ii) *$fx \in Tx$ implies $f^nx \to z$ for some $z \in X$. Then f and T have a common fixed point in X.*

Remark 2.3. It is not yet known whether the continuity of both f and T is necessary or not in Corollary 2.4. However, simple examples prove that the conditions $f(X) \subset T(X)$ and the f-weak compatibility of f and T are necessary in Corollary 2.4. Nevertheless, by weakening the inequality (2.10) for single-valued mappings f and T using the continuity of at least one of them, we can extend Theorem 2.1 of [3].

Theorem 2.8. *Let (X,d) be a complete metric space and let $f,T\colon X\to X$ be f-weak compatible mappings such that $T(X)\subset f(X)$ and*

$$d(Tx,Ty)\leqslant h\cdot[a\cdot L(x,y)+(1-a)\cdot N(x,y)] \tag{2.14}$$

for all x,y in X, where $0\leqslant h<1$, $0\leqslant a\leqslant 1$,

$$L(x,y)=\max\{d(fx,fy),d(fx,Tx),d(fy,Ty),d(fx,Ty),d(fy,Tx)\}$$

and

$$\begin{aligned}N(x,y)=[\max\{&d^2(fx,fy),d(fx,Tx)\cdot d(fy,Ty),d(fx,Ty)\cdot d(fy,Tx),\\&d(fx,Tx)\cdot d(fy,Tx),d(fy,Ty)\cdot d(fx,Ty)\}]^{1/2}.\end{aligned}$$

If one of f or T is continuous, then there exists a unique common fixed point of f and T.

P r o o f. Following the technique of Das and Naik [3], it can easily be seen that the sequence $\{Tx_n\}$, where $Tx_n=fx_{n+1}$ for all $n\in\mathbb{N}$, is a Cauchy sequence in X and, since X is complete, it follows that $\{Tx_n\}$ converges to some point $z\in X$. Assume that T is continuous. Then $T^2x_n\to Tz$ and $Tfx_n\to Tz$ as $n\to\infty$. By the f-weak compatibility of f and T, we have

$$\begin{aligned}\lim_{n\to\infty}d(fTx_n,Tfx_n)&\leqslant\lim_{n\to\infty}d(Tfx_n,Tx_n),\\\lim_{n\to\infty}d(fTx_n,fx_n)&\leqslant\lim_{n\to\infty}d(Tfx_n,Tx_n).\end{aligned}\tag{2.15}$$

Now using (2.14), (2,15) and the continuity of T, we have

$$d(T^2x_n,Tx_n)\leqslant h\cdot[a\cdot L(Tx_n,x_n)+(1-a)\cdot N(Tx_n,x_n)], \tag{2.16}$$

where

$$\begin{aligned}L(Tx_n,x_n)=\max\{&d(fTx_n,fx_n),d(fTx_n,T^2x_n),d(fx_n,Tx_n),\\&d(fTx_n,Tx_n),d(fx_n,T^2x_n)\}\\\leqslant\max\{&d(fTx_n,fx_n),d(fTx_n,Tfx_n)+d(Tfx_n,T^2x_n),\\&d(fx_n,Tx_n),d(fTx_n,fx_n)+d(fx_n,Tx_n),d(fx_n,T^2x_n)\}\end{aligned}$$

and

$$N(Tx_n, x_n) \leqslant [\max\{d^2(fTx_n, fx_n), d(fTx_n, T^2x_n) \cdot d(fx_n, Tx_n), \\ d(fTx_n, Tx_n) \cdot d(fx_n, T^2x_n), d(fTx_n, T^2x_n) \cdot d(fx_n, T^2x_n), \\ d(fTx_n, Tx_n) \cdot d(fx_n, Tx_n)\}]^{1/2}, \\ \leqslant [\max\{d^2(fTx_n, fx_n), [d(fTx_n, Tfx_n) + d(Tfx_n, T^2x_n)] \\ \cdot d(fx_n, Tx_n), d(fTx_n, Tx_n) \cdot d(fx_n, T^2x_n), \\ [d(fTx_n, Tfx_n) + d(Tfx_n, T^2x_n)] \cdot d(fx_n, T^2x_n), \\ [d(fTx_n, Tfx_n) + d(Tfx_n, Tx_n)] \cdot d(fx_n, Tx_n)\}]^{1/2}.$$

Thus, we have

$$\lim_{n\to\infty} L(Tx_n, x_n) \leqslant \max\{d(Tz, z), d(Tz, z), 0, d(Tz, z), d(z, Tz)\} \\ = d(Tz, z) \tag{2.17}$$

and

$$\lim_{n\to\infty} N(Tx_n, x_n) \leqslant [\max\{d^2(Tz, z), 0, d^2(Tz, z), d^2(Tz, z), 0\}]^{1/2} \\ = d(Tz, z). \tag{2.18}$$

Hence from (2.16)∼(2.18) we obtain

$$d(Tz, z) \leqslant h \cdot [a \cdot d(Tz, z) + (1 - a) \cdot d(Tz, z)],$$

i.e., $d(Tz, z) \leqslant h \cdot d(Tz, z)$, which implies that $Tz = z$. Since $T(X) \subset f(X)$, there exists a point z' such that $z = Tz = Tz'$ and, using (2.11) again, we obtain

$$d(T^2x_n, Tz') \leqslant h \cdot \big[a \cdot \max\{d(fTx_n, z), d(fTx_n, T^2x_n), \\ d(z, Tz'), d(fTx_n, Tz'), d(z, T^2x_n)\} \\ + (1 - a) \cdot [\max\{d^2(fTx_n, z), d(fTx_n, T^2x_n) \cdot d(z, Tz'), \\ d(fTx_n, Tz') \cdot d(z, T^2x_n), d(fTx_n, T^2x_n) \cdot d(z, T^2x_n), \\ d(z, Tz') \cdot d(fTx_n, Tz')\}]^{1/2}\big]. \tag{2.19}$$

Passing to the limit in (2.19) as $n \to \infty$, we deduce that $d(z, Tz') \leqslant h \cdot d(z, Tz')$, i.e., $z = Tz' = fz'$ and, by Lemma 2.6, $fz = fTz' = Tfz' = Tz = z$.

Now, assume that f is continuous. Then $f^2x_n \to fz$ and $fTx_n \to fz$. By the f-weak compatibility of f and T and the continuity of f we have

$$\lim_{n\to\infty} d(fz, Tfx_n) \leqslant \lim_{n\to\infty} d(Tfx_n, z), \\ d(fz, z) \leqslant \lim_{n\to\infty} d(Tfx_n, z). \tag{2.20}$$

Using (2.14), (2.20) and the continuity of f, we have

$$\begin{aligned} d(Tfx_n, Tx_n) \leqslant h \cdot [a \cdot \max\{&d(f^2x_n, fx_n), d(f^2x_n, Tfx_n), d(fx_n, Tx_n), \\ &d(f^2x_n, Tx_n), d(fx_n, Tfx_n)\} + (1-a) \cdot \max[\{d^2(f^2x_n, fx_n), \\ &d(f^2x_n, Tfx_n) \cdot d(fx_n, Tx_n), d(f^2x_n, Tx_n) \cdot d(fx_n, Tfx_n), \\ &d(f^2x_n, Tfx_n) \cdot d(fx_n, Tfx_n), d(fx_n, Tx_n) \cdot d(f^2x_n, Tx_n)\}]^{1/2}], \end{aligned}$$

i.e., as $n \to \infty$,

$$\begin{aligned} d(fz, z) &\leqslant d(Tfx_n, z) \\ &\leqslant h \cdot [a \cdot \max\{d(fz, z), d(fz, Tfx_n), 0, d(fz, z), d(z, Tfx_n)\} \\ &\qquad + (1-a) \cdot [\max\{d^2(fz, z), 0, d(fz, z) \cdot d(z, Tfx_n), \\ &\qquad d(fz, Tfx_n) \cdot d(z, Tfx_n), 0\}]^{1/2}], \end{aligned}$$

i.e., as $n \to \infty$,

$$\begin{aligned} d(fz, z) &\leqslant d(Tfx_n, z) \\ &\leqslant h \cdot [a \cdot \max\{d(fz, z), d(Tfx_n, z), 0, d(Tfx_n, z), \\ &\qquad d(z, Tfx_n)\} + (1-a) \cdot [\max\{d^2(fz, z), 0, d^2(Tf_{x_n}, z), \\ &\qquad d^2(Tfx_n, z), 0\}]^{1/2}]. \end{aligned}$$

Thus, $Tfx_n \to z$ as $n \to \infty$ and so $fz = z$.

Again using (2.14) and (2.20), we obtain, as $n \to \infty$,

$$\begin{aligned} d(Tz, Tfx_n) \leqslant h \cdot [a \cdot \max\{&d(fz, f^2x_n), d(fz, Tz), d(f^2x_n, Tfx_n), \\ &d(fz, Tfx_n), d(f^2x_n, Tz)\} + (1-a) \cdot [\max\{d^2(fz, f^2x_n), \\ &d(fz, fz) \cdot d(f^2x_n, Tfx_n), d(fz, Tfx_n) \cdot d(f^2x_n, Tz), \\ &d(fz, fz)d(f^2x_n, Tz), d(f^2x_n, Tfx_n) \cdot d(fz, Tfx_n)\}]^{1/2}], \end{aligned}$$

i.e.,

$$\begin{aligned} d(Tz, z) \leqslant h \cdot [a \cdot \max\{0, d(z, Tz), 0, 0, d(z, Tz)\} \\ + (1-a) \cdot [\max\{0, 0, 0, d^2(z, Tz), 0\}]^{1/2}]. \end{aligned}$$

Thus, we have

$$d(Tz, z) \leqslant h \cdot d(z, Tz),$$

which is a contradiction. Therefore, z is a common fixed point of f and T. From (2.14), the uniqueness of z follows easily. This completes the proof. □

Next, we give an example to discuss the validity of Theorem 2.8.

Example 2.2. Let $X = [0, \infty)$ with the usual metric. Let f and $T\colon X \to X$ be mappings defined by $fx = \frac{1}{2}(x^2 + x)$ and $Tx = \frac{1}{3}(x^2 + x)$ for x in X, respectively. Then T and f are continuous and $T(X) = f(X) = X$. Since $fx = Tx$ if and only if $x \to 1$, we can and do choose a sequence $\{x_n\}$ in X such that $x_n \to 1$. Then we have

$$\lim_{n\to\infty} d(Tfx_n, fTx_n) = \lim_{n\to\infty} \left| \frac{1}{36}x_n^4 + \frac{1}{6}x_n^3 - \frac{11}{36}x_n^2 + \frac{1}{9} \right| = 0.$$

Therefore, f and T are compatible and so f and T are f-weak compatible. Since

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{3}|x^2 - y^2| \\ &= \frac{2(x+y)}{3(x+y+1)} \left\{ \tfrac{1}{2}|x^2 - y^2 + x - y| \right\} \\ &\leqslant \frac{2}{3} d(fx, fy), \end{aligned}$$

i.e.,

$$d(Tx, Ty) \leqslant h \cdot [a \cdot d(fx, fy) + (1 - a) \cdot [d^2(fx, fy)]^{1/2}],$$

i.e.,

$$d(Tx, Ty) \leqslant h \cdot [a \cdot L(x, y) + (1 - a) \cdot N(x, y)]$$

for all $x, y \in X$, all conditions of Theorem 2.8 are fulfilled with $h \in \left[\frac{2}{3}, 1\right)$, $0 \leqslant a \leqslant 1$ and the point 1 is the unique common fixed point of f and T.

3. Fixed point theorems for Kannan type multi-valued mappings

Kannan [11] established a fixed point theorem for a single-valued mapping T defined on a complete metric space (X, d) satisfying

$$d(Tx, Ty) \leqslant \alpha[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $0 < \alpha < \frac{1}{2}$. The range of α is crucial even to the existence part of this result in the setting of a complete metric space. However, in a more restrictive yet quite natural setting, elaborate fixed point theorems exist for the case $\alpha = \frac{1}{2}$. This wider class of mappings were studied by Kannan in [12]. Recently, Beg and Azam [1], [2], Shiu, Tan and Wong [22] and Wong [26] have also studied such mappings.

In this section, we consider a mapping $T\colon X \to CB(X)$ satisfying the following condition:

$$H^r(Tx, Ty) \leqslant \alpha_1(d(x, Tx))d^r(x, Tx) + \alpha_2(d(y, Ty))d^r(x, Tx) \tag{3.1}$$

for all $x, y \in X$, where $\alpha_i\colon R \to [0,1)$ $(i = 1,2)$ and r is a fixed positive real number.

If there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, then $\{x_n\}$ is said to be asymptotically T-regular. In fact, we establish the following:

Theorem 3.1. *Let (X,d) be a complete metric space and let $T\colon X \to CB(X)$ be a multi-valued mapping satisfying* (3.1) *for all $x, y \in X$, where $\alpha_i\colon R \to [0,1)$ $(i = 1,2)$ and r is a fixed positive real number. If there exists an asymptotically T-regular sequence $\{x_n\}$ in X, then T has a fixed point x^* in X. Moreover, $Tx_n \to Tx^*$ as $n \to \infty$.*

Proof. By hypothesis, we have

$$\begin{aligned}H^r(Tx_n, Tx_m) &\leqslant \alpha_1(d(x_n, Tx_n))d^r(x_n, Tx_n) + \alpha_2(d(x_m, Tx_m))d^r(x_m, Tx_m)\\ &\to 0 \quad \text{as } n, m \to \infty.\end{aligned}$$

This shows that $\{Tx_n\}$ is a Cauchy sequence in $(CB(X), H)$. Since $(CB(X), H)$ is complete, there exists $K^* \in CB(X)$ such that $H(Tx_n, K^*) \to 0$ as $n \to \infty$. Suppose $x^* \in K^*$. Then by (3.1) we have

$$\begin{aligned}d^r(x^*, Tx^*) &\leqslant H^r(K^*, Tx^*)\\ &\leqslant \lim_{n\to\infty} H^r(Tx_n, Tx^*)\\ &\leqslant \lim_{n\to\infty} [\alpha_1(d(x_n, Tx_n))d^r(x_n, Tx_n) + \alpha_2(d(x^*, Tx^*))d^r(x^*, Tx^*)]\\ &\leqslant \alpha_2(d(x^*, Tx^*))d^r(x^*, Tx^*),\end{aligned}$$

which implies that

$$(1 - \alpha_2(d(x^*, Tx^*)))d^r(x^*, Tx^*) \leqslant 0,$$

i.e., $d(x^*, Tx^*) = 0$, and so $x^* \in Tx^*$. Now

$$\begin{aligned}H^r(K^*, Tx^*) &= \lim_{n\to\infty} H^r(Tx_n, Tx^*)\\ &\leqslant \alpha_2(d(x^*, Tx^*))d^r(x^*, Tx^*)\\ &\leqslant d^r(x^*, Tx^*) = 0.\end{aligned}$$

Therefore, we obtain $Tx^* = K^* = \lim\limits_{n\to\infty} Tx_n$. This completes the proof. □

Theorem 3.2. *Let (X,d) be a complete metric space and let $T\colon X \to CB(X)$ be a multi-valued mapping satisfying* (3.1)*. If there exists an asymptotically T-regular sequence $\{x_n\}$ in X and Tx_n is compact for all $n \in \mathbb{N}$, then each cluster point of $\{x_n\}$ is a fixed point of T.*

P r o o f. Let $y_n \in Tx_n$ be such that $d(x_n, y_n) = d(x_n, Tx_n)$. It is obvious that a cluster point of $\{x_n\}$ is a cluster point of $\{y_n\}$. Suppose that y^* is such a cluster point of $\{x_n\}$ and $\{y_n\}$. Then as in Theorem 3.1, we have

$$\begin{aligned} d^r(x^*, Tx^*) &\leqslant H^r(Tx_n^*, Tx^*) \\ &\leqslant \alpha_1(d(x_n, Tx_n))d^r(x_n, Tx_n) + \alpha_2(d(x^*, Tx^*))d^r(x^*, Tx^*) \\ &\leqslant \alpha_1(d(x_n, Tx_n))d^r(x_n, Tx_n), \end{aligned}$$

which implies that $y^* \in Tx^*$. By (3.1) again,

$$\begin{aligned} d^r(y^*, Ty^*) &\leqslant H^r(Tx^*, Ty^*) \\ &\leqslant \alpha_1(d(x^*, Tx^*))d^r(x^*, Tx^*) + \alpha_2(d(y^*, Ty^*))d^r(y^*, Ty^*), \end{aligned}$$

i.e.,

$$(1 - \alpha_2(d(y^*, Ty^*)))d^r(y^*, Ty^*) \leqslant 0.$$

Therefore, we have $y^* \in Ty^*$. This completes the proof. □

Theorem 3.3. *Let (X, d) be a complete metric space and let $T\colon X \to CB(X)$ be a multi-valued mapping satisfying* (3.1) *with $\alpha_1(d(x, Tx)) + \alpha_2(d(y, Ty)) \leqslant 1$. If* $\inf\{d(x, Tx)\colon x \in X\} = 0$, *then T has a fixed point in X.*

P r o o f. In view of Theorem 3.2 it suffices to show that there exists an asymptotically T-regular sequence $\{x_n\}$ in X.

Pick x_0 in X and consider a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$. Then the inequality (3.1) implies

$$\begin{aligned} d^r(x_n, Tx_n) &\leqslant H^r(Tx_{n-1}, Tx_n) \\ &\leqslant \alpha_1(d(x_{n-1}, Tx_{n-1}))d^r(x_{n-1}, Tx_{n-1}) \\ &\quad + \alpha_2(d(x_n, Tx_n))d^r(x_n, Tx_n) \\ &\leqslant \frac{\alpha_1(d(x_{n-1}, Tx_{n-1}))}{1 - \alpha_2(d(x_n, Tx_n))} d^r(x_{n-1}, Tx_{n-1}) \\ &\leqslant d^r(x_{n-1}, Tx_{n-1}). \end{aligned} \tag{3.2}$$

It follows from (3.2) that the sequence $\{d(x_n, Tx_n)\}$ is decreasing. Therefore, we have

$$d(x_n, Tx_n) \to \inf\{d(x_n, Tx_n)\colon n \in \mathbb{N}\}$$

and so $d(x_n, Tx_n) \to 0$. Therefore, $\{x_n\}$ is asymptotically T-regular. This completes the proof. □

Theorems 3.1, 3.2 and 3.3 generalize the results of Shiau, Tan and Wong [22] and Beg and Azam [2]. In these theorems we have not only dropped the hypothesis of compactness of Tx (cf. Theorem 1 in [22]) but also emphasized to the fact that the mapping T belongs to a wider class of mappings.

Acknowledgement. The first author was supported in part by the University Grant Commission, New Delhi, India, and the second and third authors were supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1995, Project No. BSRI-95-1405.

References

[1] *I. Beg and A. Azam*: Fixed point theorems for Kannan mappings. Indian J. Pure and Appl. Math. *17(11)* (1986), 1270–1275.
[2] *I. Beg and A. Azam*: Fixed points of asymptotically regular multivalued mappings. J. Austral. Math. Soc. (Series A) *53* (1992), 313–326.
[3] *K. M. Das and K. V. Naik*: Common fixed point theorems for commuting maps on a metric space. Proc. Amer. Math. Soc. *77* (1979), 369–373.
[4] *B. Fisher*: Common fixed points of four mappings. Bull. Inst. Math. Acad. Sinica *11* (1983), 103–113.
[5] *G.Jungck*: Commuting mappings and fixed points. Amer. Math. Monthly *83* (1976), 261-163.
[6] *G.Jungck*: Compatible mappings and common fixed points. Internat. J. Math. & Math. Sci. *9* (1986), 771–779.
[7] *G.Jungck*: Common fixed points for commuting and compatible maps on compacta. Proc. Amer. Math. Soc. *103* (1988), 997–983.
[8] *H. Kaneko*: Single-valued and multi-valued f-contractions. Boll. Un. Mat. Ital. *4-A* (1985), 29–33.
[9] *H. Kaneko*: A common fixed point of weakly commuting multi-valued mappings. Math. Japonica *33(5)* (1988), 741–744.
[10] *H. Kaneko and S. Sessa*: Fixed point theorem for compatible multi-valued and single-valued mappings. Internat. J. Math. & Math. Sci. *12* (1989), 257–262.
[11] *R. Kannan*: Some results on fixed points. Bull. Calcutta. Math. Soc. *60* (1968), 71–76.
[12] *R. Kannan*: Fixed point theorem in reflexive Banach spaces. Proc. Amer. Math. Soc. *38* (1973), 111–118.
[13] *T. Kubiak*: Fixed point theorems for contractive type multi-valued mappings. Math. Japonica *30* (1985), 89–101.
[14] *H. K. Pathak*: On a fixed point theorem of Jungck. to appear in Proceedings of the First World Congress of Nonlinear Analyst, 1992.
[15] *H. K. Pathak*: On common fixed points of weak compatible mappings in metric and Banach spaces. to appear in Nonlinear Functional Analysis and its Applications, special volume (Ed. T. M. Rassias) (1993).
[16] *H. K. Pathak*: Fixed point theorem for weak compatible multi-valued and single-valued mappings. Acta. Math. Hungar. *67(1–2)* (1995), 69–78.
[17] *H. K. Pathak and V. Popa*: On common fixed points of weak compatible mappings in metric spaces. Studii Si Ćercetări Stiintifice, Seria: Mathematică *3* (1994), 89–100.
[18] *H. K. Pathak and V. Popa*: Common fixed points of weak compatible mappings. Studia Univ. Babes-Bolyai Mathematica *34(1)* (1994), 65–78.

[19] *S. Nadler*: Multi-valued contraction mappings. Pacific J. Math. *20* (1969), 475–488.
[20] *B. E. Rhoades, S. Sessa, M. S. Khan and M. Swaleh*: On fixed points of asymptotically regular mappings. J. Austral. Math. Soc. (Series A) *43* (1987), 328–346.
[21] *S. Sessa*: On a weak commutativity condition of mappings in fixed point considerations. Publ. Inst. Math. (Beograd) *32(46)* (1982), 146–153.
[22] *C. Shiau, K. K. Tan and C. S. Wong*: A class of quasi-nonexpansive multi-valued maps. Canad. Math. Bull. *18* (1975), 707–714.
[23] *S. L. Singh and S. P. Singh*: A fixed point theorem. Indian J. Pure and Appl. Math. *11* (1980), 1584–1586.
[24] *S. L. Singh, K. S. Ha and Y. J. Cho*: Coincidence and fixed points of nonlinear hybrid contractions. Internat. J. Math. & Math. Sci. *12(2)* (1989), 247–256.
[25] *R. E. Smithson*: Fixed points for contractive multi-functions. Proc. Amer. Math. Soc. *27* (1971), 192–194.
[26] *C. S. Wong*: On Kannan maps. Proc. Amer. Math. Soc. *47* (1975), 105–111.

Authors' addresses: H. K. Pathak, Department of Mathematics, Kalyan Mahavidyalaya, Bhilai Nagar (M.P.) 490 006, India; S. M. Kang and Y. J. Cho, Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea;.