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AFFINE COMPLETNESS OF PROJECTABLE LATTICE ORDERED GROUPS

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Affine completeness of algebraic systems was studied in [3], [5], [6], [8]–[13]. In the present paper we prove that a nonzero abelian linearly ordered group fails to be affine complete. Then by applying Proposition 2.2, [9] we obtain that an abelian projectable lattice ordered group G is affine complete if and only if $G = \{0\}$; this is a generalization of Theorem (A) from [9].

1. Preliminaries

For lattice ordered groups we apply the usual terminology and notation (cf., e.g., [1]).

Let A be a universal algebra. We denote by $\operatorname{Con} A$ the set of all congruences of A. Next, let $P(A)$ be the set of all polynomials of A.

Let N be the set of all positive integers and $n \in N$. A mapping $f\colon A^n \to A$ is said to be compatible with $\operatorname{Con} A$ if, whenever $\Theta \in \operatorname{Con} A$, $a_i, b_i \in A$ and $a_i \Theta b_i$ for $i = 1, 2, \ldots, n$, then $f(a_1, \ldots, a_n) \Theta f(b_1, \ldots, b_n)$.

The algebra A is called *affine complete* if each mapping $f\colon A^n \to A$ which is compatible with $\operatorname{Con} A$ belongs to $P(A)$.

1.1. Lemma. *Let G be an abelian lattice ordered group and let $p(x) \in P(G)$ be such that $p(x)$ fails to be a constant. There exist $a, x_0 \in G$ and an integer n such that, whenever $x_1 \in G$ and $x_1 \geqslant x_0$, then $p(x_1) = a + nx_1$.*

P r o o f. This is a consequence of Lemma 3 and Remark 3.1 in [9]. □

1.2. Proposition. ([9], Proposition 2.2.) *Let G be a projectable lattice ordered group. Assume that G is abelian and that it is not linearly ordered. Then G is not affine complete.*

2. The case of linearly ordered groups

If I is a linearly ordered set and for each $i \in I$, G_i is a linearly ordered group, then the lexicographic product of the indexed system $(G_i)_{i\in I}$ will be denoted by $\Gamma_{i\in I} G_i$ (cf., e.g., [4], Chap. II).

Let R be the additive group of all reals with the natural linear order. If $G_i = R$ for each $i \in I$, then we put

$$\Gamma_{i\in I} G_i = V(I).$$

2.1. Theorem. (Hahn [7].) *Let G be an abelian linearly ordered group. Then there exists a linearly ordered set I and an isomorphism φ of G into $V(I)$.*

For a more general result and a shorter proof cf. Conrad, Harvey and Holland [2].

If G, I and φ are as in 2.1, then for each $0 \neq x \in G$ there exists $i_0 \in I$ such that $\varphi(x)(i_0) \neq 0$, and $\varphi(x)(i) = 0$ whenever $i \in I$, $i < i_0$. We denote

$$i(x) = i_0.$$

Next, let I_1 be the set of all $i_1 \in I$ such that $i(x) = i_1$ for some $x \in G$. In what follows we suppose that $G \neq \{0\}$. Hence $I_1 \neq \emptyset$. Put

$$\varphi_1(x)(i) = \varphi(x)(i) \quad \text{for each } i \in I_1.$$

Then φ_1 is a homomorphism of G into $V(I_1)$.

Let $0 \neq x \in G$. We have $\varphi_1(x)(i_1) \neq 0$ for $i_1 = i(x)$, whence $\varphi_1(x) \neq 0$ and thus φ_1 is an isomorphism of G into $V(I_1)$.

Hence without loss of generality we can suppose that $I = I_1$.

Let I' be a subset of I such that either $I' = \emptyset$ or I' is an ideal of I. For $x, y \in G$ we put $x\Theta(I')y$ if for each $i' \in I'$ the relation

$$\varphi(x)(i') = \varphi(y)(i')$$

is valid. From the definition of $\Theta(I')$ we immediately obtain

2.2. Lemma. *$\Theta(I')$ is a congruence relation on G.*

For a congruence relation Θ on G and for $x \in G$ we denote by $x(\Theta)$ the class in Θ containing x (i.e., $x(\Theta) = \{y \in G \colon y\Theta x\}$).

2.3. Lemma. *Let $\Theta \in \operatorname{Con} G$ such that Θ is not the greatest element of $\operatorname{Con} G$. Then there is an ideal I' of I such that $\Theta = \Theta(I')$.*

P r o o f. We denote by I' the set of all $i' \in I$ having the property that there exists $x \in G$ with $x \notin 0(\Theta)$ such that $i(x) = i'$. From the fact that Θ is not the greatest element of $\operatorname{Con} G$ we obtain that $I' \neq \emptyset$.

Let $i' \in I'$, $i_1 \in I$ and $i_1 < i'$. There exists $y \in G$ with $i(y) = i_1$. If $y \in 0(\Theta)$, then $i(|y|) = i(y)$, $|y| \in 0(\Theta)$ and

$$-|y| < x < |y|,$$

whence $x \in 0(\Theta)$, which is a contradiction. Thus $y \in 0(\Theta)$ and hence $y_1 \in I'$. Therefore I' is an ideal in I.

Now let $0 \neq x \in 0(\Theta)$, $x(i) = i_1$. Assume that $i_1 \in I'$. Hence there is $z \in G$ such that $z(i) = i_1$ and $z \notin 0(\Theta)$. But then there is a positive integer n with

$$-n|x| < z < n|x|,$$

implying that $z \in 0(\Theta)$, which is a contradiction. Thus $i_1 \in I'$. This yields that

$$x\Theta(I')0.$$

Hence $\Theta \leqslant \Theta(I')$.

Next, let $0 \neq z \in 0(\Theta(I'))$, $i_1 = i(z)$. In other words, $z\Theta(I')0$, and hence $i_1 \notin I'$. Suppose that $z \notin 0(\Theta)$; then $i_1 \in I'$, which is a contradiction. Thus $z \in 0(\Theta)$ and therefore $\Theta(I') \leqslant \Theta$.

Summarizing we obtain that $\Theta = \Theta(I')$.

It is clear that if Θ is the greatest element of $\operatorname{Con} G$, then $\Theta = \Theta(I')$, where $I' = I$. □

Using the relation $I' = I$ we conclude that for each $i_1 \in I$ there exists $t \in G$ such that $i(t) = i_1$. Hence by applying the Axiom of Choice we obtain that there exists a mapping $\psi\colon I \to G$ having the property that whenever $i_1 \in I$, then $\psi(i_1) = t$ is an element of G with

$$i(\psi(i_1)) = i_1.$$

For each $i_1 \in I$ we denote $\psi(i_1) = x^{i_1}$.

We define a mapping $f\colon G \to G$ as follows. We put $f(0) = 0$. Let $x \in G$, $x \neq 0$. Denote $i(x) = i_1$; we set

$$f(x) = \begin{cases} x^{i_1} & \text{if} \quad \varphi(x)(i_1) = \varphi(kx^{i_1})(i_1) \quad \text{and } k \text{ is an odd integer,} \\ 2x^{i_1} & \text{otherwise.} \end{cases}$$

2.4. Lemma. *$f(x)$ does not belong to $P(G)$.*

P r o o f. By way of contradiction, assume that $f(x)$ belongs to $P(G)$. Then there exist a, x_0 and n with the properties as in 1.1. Next, there exist $i_1 \in I$ and a positive integer m_0 such that

$$m_0 x^{i_1} > x_0.$$

Let m_1 be a positive integer, $m_1 > m_0$. In view of the definition of f,

$$f(2m_0x^{i_1}) = f(2m_1x^{i_1}) = 2x^{i_1}.$$

On the other hand, 1.1 yields

$$f(2m_0x^{i_1}) = a + n.2m_0x^{i_1},$$
$$f(2m_1x^{i_1}) = a + n.2m_1x^{i_1},$$

whence

$$2n(m_1 - m_0)x^{i_1} = 0.$$

Since $m_1 - m_0 > 0$ we obtain that $n = 0$, thus $f(x) = a$ for $x > x_0$. We have $2m_0x^{i_1} > x_0$, $(2m_0 + 1)x^{i_1} > x_0$ and

$$f(2m_0x^{i_1}) \neq f((2m_0 + 1)x^{i_1}),$$

which is a contradiction. □

2.5. Lemma. *The mapping f is compatible with* Con G.

P r o o f. Let $x, y \in G$ and $\Theta \in \operatorname{Con} G$. Suppose that $x\Theta y$ is valid. In view of 2.3 there exists $I_1 \subseteq I$ such that either $I_1 = \emptyset$ or I_1 is an ideal of I, and $\Theta = \Theta(I_1)$. Hence

(1) $$\varphi(x)(i) = \varphi(y)(i) \quad \text{for each } i \in I_1.$$

We have to verify whether the relation

$$\varphi(f(x))(i) = \varphi(f(y)(i))$$

holds for each $i \in I$.

The case $x = y$ is trivial. Suppose that $x \neq y$.

First let $x = 0$. Put $i(y) = i_2$. In view of (1) we have $i_2 \notin I_1$ and $f(y) \in \{x^{i_2}, 2x^{i_2}\}$. Thus $f(y)(i) = 0$ for each $i \in I_1$.

Next, let $x \neq 0 \neq y$ and let i_2 be as above. Put $i(x) = i_1$. If $i_1, i_2 \in I \setminus I_1$, then $\varphi(f(x))(i) = 0 = \varphi(f(y))(i)$ for each $i \in I_1$.

Suppose that $i_1 \in I$. Then in view of (1) we have $i_2 = i_1$ and, at the same time, $f(x) = f(y)$. This completes the proof. □

2.6. Theorem. *Let G be a nonzero abelian linearly ordered group. Then G is not affine complete.*

Proof. This is a consequence of 2.4 and 2.5. □

Now we proceed to the case of projectable lattice ordered groups.

2.7. Theorem. *Let H be an abelian projectable lattice ordered group. Then the following conditions are equivalent:*

(i) *H is affine complete.*

(ii) $H = \{0\}$.

Proof. The implication (ii)⇒(i) is trivial. From 2.6 and 1.2 we infer that (i)⇒(ii) holds. □

Since each complete lattice ordered group is abelian and projectable, the above theorem generalizes Theorem (A) from [9].

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