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DRL-SEMIGROUPS AND MV-ALGEBRAS

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The notion of a DRl-semigroup was introduced by K.L.N. Swamy in [7] as a common generalization of Brouwerian algebras and abelian lattice ordered groups (l-groups). A $DRl-semigroup$ is an algebra $A=(A,+,0,\vee,\wedge,-)$ of type $\langle 2,0,2,2,2\rangle$ such that

(1) $(A,+,0)$ is a commutative monoid,

(2) $(A,\vee,\wedge)$ is a lattice,

(3) $(A,+,\vee,\wedge)$ is a lattice ordered semigroup (l-semigroup), i.e. A satisfies the identities

$$x+(y\vee z)=(x+y)\vee(x+z),$$
$$x+(y\wedge z)=(x+y)\wedge(x+z).$$

(4) If "$\leqslant$" denotes the order on A induced by the lattice $(A,\vee,\wedge)$ then for each $x,y\in A$, $x-y$ is the smallest $z\in A$ such that $y+z\geqslant x$.

(5) A satisfies the identities

$$((x-y)\vee 0)+y\leqslant x\vee y,$$
$$x-x\geqslant 0.$$

By [7], Theorem 1, DRl-semigroups form a variety of algebras of type $\langle 2,0,2,2,2\rangle$, because condition (4) can be equivalently replaced by the identities

(4i) $x+(y-x)\geqslant y$,

(4ii) $x-y\leqslant(x\vee z)-y$,

(4iii) $(x+y)-y\leqslant x$.

The notion of an MV-algebra was introduced by C.C. Chang in [2], [3] as an algebraic counterpart of the Lukasiewicz infinite valued propositional logic.

An MV-*algebra* is an algebra $A = (A, \oplus, \neg, 0)$ of type $\langle 2, 1, 0\rangle$ satisfying the following identities. (See e.g. [4].)

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(MV2) $x \oplus y = y \oplus x$;

(MV3) $x \oplus 0 = x$;

(MV4) $\neg\neg x = x$;

(MV5) $x \oplus \neg 0 = \neg 0$;

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

D. Gluschankof in [5] studied some connections between cyclic ordered groups and MV-algebras. In this paper we deal with the connections between DRl-semigroups and MV-algebras.

Let $G = (G, +, 0, -(.), \vee, \wedge)$ be an abelian l-group and $0 \leqslant u \in G$. For any $x, y \in [0, u] = \{x \in G;\ 0 \leqslant x \leqslant u\}$, set $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. Then $\Gamma(G, u) = ([0, u], \oplus, \neg, 0)$ is an MV-algebra and for any MV-algebra A there exist an abelian l-group G and $0 < u \in G$ such that A is isomorphic to $\Gamma(G, u)$. Recently, these connections were studied by J. Jakubík in [6] also for complete MV-algebras and complete l-groups.

If $A = (A, \oplus, \neg, 0)$ is an MV-algebra and if we set $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$, then $(A, \vee, \wedge, 0, \neg 0)$ is a bounded distributive lattice. (See e.g. [4], [5].)

Theorem 1. *If $G = (G, +, 0, -(.), \vee, \wedge)$ is an abelian l-group, $0 < u \in G$, $A = [0, u]$, and if we set for any $x, y \in A$*

$$x \oplus y = (x + y) \wedge u,$$
$$x \ominus y = ((x - y) \vee 0) \wedge u,$$

then $(A, \oplus, 0, \vee, \wedge, \ominus)$ is a bounded DRl-semigroup with the least element 0 and the greatest element u satisfying the properties

(i) *$\forall x \in A$; $u \ominus (u \ominus x) = x$,*

(ii) *$\forall x, y \in A$; $x \oplus (y \ominus x) = y \oplus (x \ominus y)$,*

in which $u \oplus u = u$ and $u \ominus x = u - x$ for any $x \in A$.

P r o o f. We will show that $(A, \oplus, \vee, \wedge, \ominus)$ is a DRl-semigroup.

a) $\Gamma(G,u)$ is an MV-algebra, hence $(A,\oplus,0)$ is a commutative monoid. If $x,y,z\in A$ then

$$\begin{aligned}
x\oplus(y\vee z) &= (x+(y\vee z))\wedge u = ((x+y)\vee(x+z))\wedge u\\
&= ((x+y)\wedge u)\vee((x+z)\wedge u) = (x\oplus y)\vee(x\oplus z),\\
x\oplus(y\wedge z) &= (x+(y\wedge z))\wedge u = (x+y)\wedge(x+z)\wedge u\\
&= ((x+y)\wedge u)\wedge((x+z)\wedge u) = (x\oplus y)\wedge(x\oplus z),
\end{aligned}$$

therefore $(A,\oplus,\vee,\wedge)$ is an l-semigroup.

b) For any $x,y\in A$, we have

$$\begin{aligned}
y\oplus((x-y)\vee 0)\wedge u) &= (y+(((x-y)\vee 0)\wedge u))\wedge u\\
&= (y+((x-y)\vee 0))\wedge(y+u)\wedge u\\
&= ((y+(x-y))\vee y)\wedge u = (x\vee y)\wedge u\\
&= x\vee y \geqslant x.
\end{aligned}$$

Let $r\in A$, $y\oplus r\geqslant x$, i.e. $(y+r)\wedge u\geqslant x$. Since $y+r\geqslant x$, $r\geqslant((x-y)\vee 0)\wedge u$. Consequently, $x\ominus y$ is the smallest element in A satisfying $y\oplus z\geqslant x$.

c) If $x,y\in A$ then by b)

$$((x\ominus y)\vee 0)\oplus y = (x\ominus y)\oplus y = x\vee y.$$

d) For each $x\in A$,

$$x\ominus x = ((x-x)\vee 0)\wedge u = 0.$$

Hence $(A,\oplus,0,\vee,\wedge,\ominus)$ is a DRl-semigroup and, moreover,

$$\begin{aligned}
u\oplus u &= (u+u)\wedge u = u,\\
u\ominus x &= ((u-x)\vee 0)\wedge u = (u-x)\wedge u = u-x
\end{aligned}$$

for each $x\in A$.

We will verify the validity of conditions (i) and (ii).

(i): $u\ominus(u\ominus x) = u-(u-x) = x$.

(ii): By b),

$$\begin{aligned}
x\oplus(y\ominus x) &= (x+(((y-x)\vee 0)\wedge u))\wedge u\\
&= ((x+((y-x)\vee 0)\wedge(x+u)\wedge u\\
&= ((x+(y-x))\vee x)\wedge u = (x\vee y)\wedge u\\
&= x\vee y = y\oplus(x\ominus y).
\end{aligned}$$

□

Corollary 2. *Let $A = (A, \oplus, \neg, 0)$ be an MV-algebra. For any $x, y \in A$, set*

$$x \leqslant y \Leftrightarrow \neg(\neg x \oplus y) \oplus y = y. \tag{1}$$

Then "$\leqslant$" is a lattice order on A (with the lattice operations $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$), for any $r, s \in A$ there exists the least element $r \ominus s$ with the property $s \oplus (r \ominus s) \geqslant r$, and $(A, \oplus, 0, \vee, \wedge, \ominus)$ is a DRl-semigroup with the smallest element 0 and the greatest element $\neg 0$.

Proof. Let $G = (G, +, 0, -(.), \vee, \wedge)$ be an abelian l-group, $0 < u \in G$, and let $A \cong \Gamma(G, u)$. We have to verify that the order on $\Gamma(G, u)$ obtained by (1) is the same as that induced on $[0, u]$ by the order of the l-group G.

Let $x, y \in [0, u]$. Suppose that $x \leqslant y$ in G. Then

$$\begin{aligned}\neg(\neg x \oplus y) \oplus y &= (u - (((u - x) + y) \wedge u)) \oplus y \\ &= ((x - y) \vee 0) \oplus y = 0 \oplus y = y.\end{aligned}$$

Conversely,

$$\begin{gathered}\neg(\neg x \oplus y) \oplus y = y \Longrightarrow \\ (((x - y) \vee 0) + y) \wedge u = y \Longrightarrow (x \vee y) \wedge u = y \Longrightarrow \\ x \vee y = y \Longrightarrow x \leqslant y.\end{gathered}$$

This implies the assertion. □

Theorem 3. *Let $(A, +, 0, \vee, \wedge, -)$ be a bounded DRl-semigroup with the smallest element 0 and the greatest element 1 satisfying the conditions*

(i) $\forall x \in A$; $1 - (1 - x) = x$,

(ii) $\forall x, y \in A$; $x + (y - x) = y + (x - y)$.

Set $\neg x = 1 - x$ for any $x \in A$. Then $(A, +, \neg, 0)$ is an MV-algebra.

Proof. Let us show that conditions (MV1)–(MV6) are satisfied.

(MV1)–(MV3) are contained directly in the definition of a DRl-semigroup.

(MV4): If $x \in A$ then, by (i), $\neg\neg x = 1 - (1 - x) = x$.

(MV5): It is clear (by [7], Lemma 1) that $\neg 0 = 1$ (and $1 + 1 = 1$). If $x \in A$, then $0 \leqslant x$ implies $1 \leqslant x + 1$, hence $x + 1 = 1$. Thus $x + \neg 0 = \neg 0$.

(MV6): Let $x, y \in A$. Then by [7], Lemma 6, and by (i) and (ii), $\neg(\neg x + y) + y = (1 - ((1 - x) + y)) + y = ((1 - (1 - x)) - y) + y = (x - y) + y = (y - x) + x = \neg(\neg y + x) + x$. □

Let $A = (A, \oplus, \neg, 0)$ be an MV-algebra and $\emptyset \neq I \subseteq A$. Then I will be called an *ideal* of A if

(a) $\forall a, b \in I;\ a \oplus b \in I$,

(b) $\forall a \in I,\ x \in A;\ \neg(\neg(a \oplus \neg x) \oplus \neg x) \in I$.

Recall that if $B = (B, +, 0, \vee, \wedge, -)$ is a DRl-semigroup and $c, d \in B$, then by the *symmetric difference* of c and d we mean $c * d = (c - d) \vee (d - c)$. (Hence "$*$" is a metric operation on A.) A non-void subset $J \subseteq B$ is called an *ideal* of B if

(c) $\forall a, b \in J;\ a + b \in J$,

(d) $\forall a \in J, x \in B;\ x * 0 \leqslant a * 0 \Longrightarrow x \in J$.

Under conditions (c) and (d), if $x \in B$, $0 \leqslant x$, then $x * 0 = x$. Hence in any DRl-semigroup induced by an MV-algebra, condition (d) can be replaced by

(d′) $\forall a \in J,\ x \in B;\ x \leqslant a \Longrightarrow x \in J$.

Then it is obvious that in MV-algebras the ideals in the sense of MV-algebras and those in the sense of DRl-semigroups coincide. (Orders on MV-algebras will be always introduced by (1) from Corollary 2.)

In [8], Theorem 1.2, it is proved that the ideals and the congruences of DRl-semigroups are in a one-to-one correspondence. We will show an analogous correspondence also for the ideals and the congruences of MV-algebras.

Proposition 4. *If I is an ideal of an MV-algebra $A = (A, \oplus, \neg, 0)$ then the relation $\equiv_I$ on A such that*

$$\forall x, y \in A;\ \ x \equiv_I y \Leftrightarrow x * y \in I,$$

is a congruence on the MV-algebra A.

P r o o f. Suppose that $A = \Gamma(G, u)$, where G is an abelian l-group and $0 < u \in G$. By [8], Theorem 1.2, $\equiv_I$ is an equivalence such that

$$\forall x, y, u, v \in A;\ \ x \equiv_I y,\ \ u \equiv_I v \Longrightarrow (x \oplus u) \equiv_I (y \oplus v).$$

Let $x, y \in A$, $x \equiv_I y$, i.e. $x * y \in I$. Then

$$\begin{aligned}
\neg x * \neg y &= (u - x) * (u - y) \\
&= ((u - x) \ominus (u - y)) \vee ((u - y) \ominus (u - x)) \\
&= ((((u - x) - (u - y)) \vee 0) \wedge u) \vee ((((u - y) - (u - x)) \vee 0) \wedge u) \\
&= (((y - x) \vee 0) \wedge u) \vee (((x - y) \vee 0) \wedge u) \\
&= (y \ominus x) \vee (x \ominus y) = x * y \in I,
\end{aligned}$$

hence $\neg x \equiv_I \neg y$. Therefore "$\equiv_I$" is a congruence on $(A, \oplus, \neg, 0)$. $\square$

Proposition 5. *If "$\sim$" is a congruence on an MV-algebra $A = (A, \oplus, \neg, 0)$ then $I_\sim = \{x \in A;\ x \sim 0\}$ is an ideal of A.*

Proof. The lattice operations on A are defined by

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y),$$

hence "$\sim$" is a congruence also on the induced lattice $(A, \vee, \wedge)$.

If $a, b \in I_\sim$, i.e. $a \sim 0$, $b \sim 0$, then $(a \oplus b) \sim 0$, and so $a \oplus b \in I_\sim$.

Let $a \in I_\sim$, $x \in A$ and $x \leqslant a$. Then $x \vee a \in I_\sim$, thus $(x \vee a) \sim 0$, hence also $(x \wedge (x \vee a)) \sim (x \wedge 0)$, that is $x \sim 0$, and therefore $x \in I_\sim$. □

Theorem 6. *The ideals and the congruences of any MV-algebra are in a one-to-one correspondence.*

Proof. If A is an MV-algebra then the ideals on A coincide with the ideals of the induced DRl-semigroup. By [8], Theorem 1.2 and its proof, the ideals of any DRl-semigroup correspond one-to-one to its congruences and this correspondence is expressed by the same formulas as in Propositions 4 and 5. □

In [9], some results concerning the lattices of ideals of semiregular normal autometrized lattice ordered algebras are obtained. The DRl-semigroups are special cases of these algebras, hence the following theorem is an immediate consequence of [9], Theorem 6.

Theorem 7. *The ideals of any MV-algebra A form (under ordering by set inclusion) a complete algebraic Brouwerian lattice $\mathcal{I}(A)$.*

Theorem 8. *The lattice* **MV** *of all varieties of MV-algebras is a complete dually algebraic dually Brouwerian lattice.*

Proof. It is well-known that the lattice of subvarieties of any variety of algebras $\mathcal{M}$ is dually isomorphic to the lattice of fully characteristic congruences of the free algebra with countable rank in $\mathcal{M}$, and hence, by Theorem 6, the lattice **MV** is dually isomorphic to the lattice $\mathcal{I}_c(F)$ of fully characteristic (i.e. closed under all endomorphisms) ideals of the free MV-algebra F with a countable set of free generators. Obviously, $\mathcal{I}_c(F)$ is a complete sublattice of the lattice $\mathcal{I}(F)$, and thus it is Brouwerian. Moreover, $\mathcal{C}_c(F)$, the lattice of fully characteristic congruences, is algebraic, because the fully characteristic congruences corresponding to the finite sets of identities are its compact elements. (If $p = q$ is an identity, then its corresponding congruence is the least fully characteristic congruence θ such that $p(u_0, u_1, \ldots)\theta q(u_0, u_1, \ldots)$ for free generators $u_0, u_1, \ldots$.) □

Proposition 9. *If A is an MV-algebra and $I \in \mathcal{I}(A)$, then the pseudocomplement of I in $\mathcal{I}(A)$ is*

$$I^{\perp} = \{x \in A;\ \neg(\neg(a \oplus \neg x) \oplus \neg x) = 0,\ \textit{for each}\ a \in I\}.$$

Proof. If A is a DRl-semigroup and $I \in \mathcal{I}(A)$, then, by [9], Lemma 7, the pseudocomplement of I in $\mathcal{I}(A)$ is $I^* = \{x \in A;\ x \wedge a = 0,\ \text{for each}\ a \in I\}$. This implies the assertion. □

The ideal $I^{\perp}$ from Proposition 9 will be called the *polar* of $I \in \mathcal{I}(A)$. If $J \in \mathcal{I}(A)$, then J is called a *polar in* A, if there is some $I \in \mathcal{I}(A)$ such that J is its polar. Denote the set of all polars in an MV-algebra A by $\mathcal{P}(A)$. It is obvious that if $I \in \mathcal{I}(A)$, then $I \in \mathcal{P}(A)$ if and only if $(I^{\perp})^{\perp} = I$. From Glivenko's theorem (see e.g. [1]) we have:

Theorem 10. *If A is an MV-algebra then the set of its polars $\mathcal{P}(A)$ ordered by set inclusion is a complete Boolean algebra.*

Finally, we will show some connections between homomorphisms of MV-algebras and DRl-semigroups. (Recall that if G and H are abelian l-groups, $0 < u \in G$ and $\overline{f}\colon G \longrightarrow H$ is an l-group homomorphism, then f, the restriction of $\overline{f}$ to $[0, u]$, is an MV-algebra homomorphism of $\Gamma(G, u)$ into $\Gamma(H, \overline{f}(u))$. See e.g. [4].)

Proposition 11. *Let G and H be abelian l-groups, $0 < u \in G$, $0 < v \in H$, and $A = \Gamma(G, u)$, $B = \Gamma(H, v)$. Suppose that $f\colon A \longrightarrow B$ is a homomorphism of MV-algebras which is a restriction of an l-group homomorphism $\overline{f}\colon G \longrightarrow H$. Then f is a homomorphism of the DRl-semigroup $(A, \oplus, \vee, \wedge, \ominus)$ into the DRl-semigroup $(B, \oplus, \vee, \wedge, \ominus)$.*

Proof. We have

$$f(u) = f(\neg 0) = \neg f(0) = v,$$

hence also $\overline{f}(u) = v$.

Let $x, y \in A$. Then

$$f(x \ominus y) = f(((x - y) \vee 0) \wedge u) = ((f(x) - f(y)) \vee 0) \wedge v = f(x) \ominus f(y).$$

□

Proposition 12. *Let $(A, +, 0, \vee, \wedge, -)$ and $(B, +, 0', \vee, \wedge, -)$ be DRl-semigroups with the least elements 0 and $0'$, and the greatest elements 1 and $1'$, respectively, satisfying conditions* (i) *and* (ii) *from Theorem* 3*, and let $g\colon A \longrightarrow B$ be a homomorphism of DRl-semigroups such that $g(1) = 1'$. Then g is a homomorphism of induced MV-algebras.*

P r o o f. If $x \in A$, then

$$g(\neg x) = g(1 - x) = g(1) - g(x) = 1' - g(x) = \neg g(x).$$

□

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