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INTERPOLATION THEOREM FOR A CONTINUOUS FUNCTION ON ORIENTATIONS OF A SIMPLE GRAPH

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Abstract. Let G be a simple graph. A function f from the set of orientations of G to the set of non-negative integers is called a continuous function on orientations of G if, for any two orientations O_1 and O_2 of G, $|f(O_1) - f(O_2)| \leqslant 1$ whenever O_1 and O_2 differ in the orientation of exactly one edge of G.

We show that any continuous function on orientations of a simple graph G has the interpolation property as follows:

If there are two orientations O_1 and O_2 of G with $f(O_1) = p$ and $f(O_2) = q$, where $p < q$, then for any integer k such that $p < k < q$, there are at least m orientations O of G satisfying $f(O) = k$, where m equals the number of edges of G.

It follows that some useful invariants of digraphs including the connectivity, the arc-connectivity and the absorption number, etc., have the above interpolation property on the set of all orientations of G.

1. Introduction

A variety of research has been devoted to the orientations of a graph. For example, it is well known that every graph without self loops admits an acyclic orientation; Stanley [22] studied the set of acyclic orientations of a simple graph G and counted the number of acyclic orientations of G by using the chromatic polynomial of G; Robbins [16] proved that a nontrivial graph G admits a strongly connected orientation if and only if G is 2 edge-connected; Chvátal and Thomassen [5] further showed that every 2 edge-connected graph of radius r admits an orientation of radius at most $r^2 + r$; Gerards [7] established an orientation theorem characterizing the class of graphs in which the edges can be oriented in such a way that going along any circuit in the graph, the number of forward edges minus the number of backward edges is equal

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to 0, -1 or 1. Roberts and Xu [17–20] considered optimizing the orientations of grid graphs with respect to various measures; Donald and Elwin [6] investigated the structure of the set of strongly connected orientations of a graph G and showed that any two strongly connected orientations of G can be connected by a sequence of operations called simple transformations.

The above indicates one half of the background for our present work. The other half comes from a number of research related to the interpolation property for some invariants of spanning subgraphs of a given graph. In 1980, at the fourth International Conference on Graph Theory and Applications held in Kalamazoo, G. Chartrand asked [see [3], p. 610]: If a graph G contains spanning trees having n and m end-vertices, with $m < n$, does G contain a spanning tree with k end-vertices for every integer k with $m < k < n$? This problem piqued the interest of many graph theorists. It was first affirmatively settled by Schuster [21] in 1983. In 1984 and 1985, Cai [2] and Lin [13] gave different proofs (Lin's is the shortest). Several different generalizations also appeared in Schuster [21], Liu [14], Barefoot [1], and Zhang and Chen [23]. Zhang and Guo [24] further considered similar problem for directed graphs and got the corresponding interpolation theorems. Harary et al. [8, 10, 11] and Lewinter [12] obtained interpolation theorems for more invariants of spanning trees. Recently, Harary and Plantholt [9] classified many known interpolation theorems for spanning trees in [8, 10, 11, 12, 21], obtained interpolation results for new invariants and generalized to other families of spanning subgraphs. More recently, S. Zhou [25] used the same idea as in Lin [13] to give a short proof of interpolation theorems for many invariants on spanning subgraphs with equal size.

In this paper, we shall consider the set of all orientations of a simple graph G and establish a general interpolation theorem. It follows that some useful invariants of digraphs (including the connectivity, the arc-connectivity, the absorption number and some other invariants introduced in this paper) have the interpolation property on the set of all orientations of a simple graph G.

Throughout the paper, $G = (V(G), E(G))$ is always assumed to be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. An orientation of G is the digraph obtained from G by assigning a direction to each edge of G. The essential concepts in this paper are introduced in the following two definitions.

Definition 1.1 (Graph of orientations of G). For any two distinct orientations O_1 and O_2 of G, we say that O_1 and O_2 are adjacent if they differ in the orientation of exactly one edge of the underlying graph G. This adjacency relation determines a simple graph $\hat{G}$ with the vertex set $V(\hat{G})$ representing all the orientations of G.We call $\hat{G}$ the graph of orientations of G.

Definition 1.2 (Continuous functions on G). Let f be a function from the vertex set $V(G)$ to the set of non-negative integers. We say that f is a continuous function on G if, $|f(u) - f(v)| \leqslant 1$ for any two adjacent vertices u, v, of G. This definition was motivated by Lovácz [15].

For simplicity, a continuous function on the graph of orientations of G will be called a continuous functio on orientations of G. For other general graph theoretical terminology, the reader is refered to the book of Chartrand and Lesniak [4].

2. Main results

Theorem. *Any continuous function f on orientations of a simple graph G has the following interpolation property:*

If there are two orientations O_1 and O_2 of G with $f(O_1) = p$ and $f(O_2) = q$, where $p < q$, then for any integer k such that $p < k < q$, there are at least m orientations O of G satisfying $f(O) = k$, where m equals the number of edges of G.

Proof. Let $\hat{G}$ be the graph of orientations of G. We first show that $\hat{G}$ is isomorphic to the m-cube I^m where $m = |E(G)|$. (Recall that the m-cube I^m is the graph whose vertices are the m-dimensional vectors of 0's and 1's, two vertices being adjacent if and only if they differ in exactly one coordinate.) In fact, for any edge of G, it can be assigned exactly two distinct directions. We may correspond them to 0 and 1, respectively, Then a vertex of $\hat{G}$ (i.e., an orientation of G) corresponds to an m-dimensional vector of 0's and 1's. It is easily seen that this is a one-to-one correspondence from $V(\hat{G})$ to $V(I^m)$ and preserves adjacency relation. Therefore it gives an isomorphism between the graphs $\hat{G}$ and I^m.

Notice that I^m is m-regular. So its connectivity $k(I^m) \leqslant m$. On the other hand, since I^m is the product of m paths of length 1, we may use Menger's Theorem to show $k(I^m) \geqslant m$ by induction on m. Thus we have $k(I^m) = m$ and so $k(\hat{G}) = m$. Therefore, there are m internally disjoint paths P between O_1 and O_2 in $\hat{G}$. Since f is continuous, there must exist at least one O with $f(O) = k$ on every such path P, and the theorem follows. □

In order to apply the theorem, we recall and introduce some invariants for a digraph $D = (V(D), A(D))$ with the vertex set $V(D)$ and the arc set $A(D)$. (Note that the counterparts of these invariants for undirected graphs are familiar and have been extensively studied.)

Definition 2.1. The connectivity $k_1(D)$ of D is defined to be the minimum number of vertices whose removal from D leaves the remaining digraph not strongly connected or reduces D to a single vertex.

Definition 2.2. The arc-connectivity $k_2(D)$ of D is defined to be the minimum number of arcs whose removal from D leaves the remaining digraph not strongly connected or reduces D to a single vertex.

Definition 2.3. The directed arboricity $k_3(D)$ of D is the minimum number of subsets into which $A(D)$ can be partitioned so that each subset induces a directed forest. (A directed forest is a digraph of which every component is a rooted ditree, where a rooted ditree is a digraph T in which there is a vertex, called the root of T, being able to reach any other vertex of T by a directed path and the underlying undirected graph of T is a tree.)

Definition 2.4. The directed vertex arboricity $k_4(D)$ of D is the minimum number of subsets into which $V(D)$ can be partitioned so that each subset induces a directed forest.

Definition 2.5. The directed linear arboricity $k_5(D)$ of D is the minimum number of subsets into which $A(D)$ can be partitioned so that each subset induces a directed linear forest. (A directed linear forest is a directed forest of which each component is a directed path.)

Definition 2.6. The directed linear vertex arboricity $k_6(D)$ of D is the minimum number of subsets into which $V(D)$ can be partitioned so that each subset induces a directed linear forest.

Definition 2.7. The absorption number $k_7(D)$ is the minimum of the cardinalities $|S|$ over all such subsets S of $V(D)$ of which each S satisfies the following: for any $v \in V(D) - S$, there is an arc in D from v to a vertex of S.

Now we give the following result on the above invariants.

Corollary. *For any simple graph G, each of the invariants k_i $(i = 1, 2, \ldots, 7)$ has the interpolation property on the orientations of G. That is, if there are two orientations O_1 and O_2 of G with $k_i(O_1) = p$ and $k_i(O_2) = q$, where $p < q$, then for any integer k such that $p < k < q$, there are orientations O of G satisfying $k_i(O) = k$. And the number of such O's is not less than the number of edges of G.*

Proof. For any given $i = 1, 2, \ldots, 7$, the function defined by $f(O) = k_i(O)$ for each $O \in V(\hat{G})$ is easily seen to be a continuous function on $\hat{G}$. Then the result immediately follows from the Theorem. □

Remark. There are other invariants, such as maximum (in-, out-)degree, minimum (in-, out-)degree, and the number of disjoint directed cycles, etc., which can also be included in the corollary.

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