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BEST SIMULTANEOUS L_p APPROXIMATIONS

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Abstract. In this paper we study simultaneous approximation of n real-valued functions in $L_p[a, b]$ and give a generalization of some related results.

1. BEST SIMULTANEOUS L_p APPROXIMATIONS

The problem of simultaneous approximation to two or more real-valued functions belonging to $L_p[a, b]$ by elements of a subset S of $L_p[a, b]$ has been studied by several authors. Phillips and Sahney [3] gave results for the L_1 and L_2 norms. The problem of the best simultaneous approximation to an arbitrary number of functions discussed by Holland and Sahney [1], who generalized the results in [3] for the L_2 norms. Ling [2] considered simultaneous Chebyshev approximation in the Sum norm. Holland, McCabe, Phillips and Sahney [4] considered the best simultaneous L_1 approximations and studied the relation between the best simultaneous approximations and the best L_1 approximations to the arithmetics mean of n functions. The problem of simultaneous L_p approximation to two real valued functions f_1 and f_2 when p is an odd natural number was discussed by Karakuş in [5] and when p is non-integer real number by Karakuş-Atacik in [6].

In this paper we study the best simultaneous L_p approximation to n functions.

Definition 1. Let $p \geq 1$ be real number and $S \subset L_p[a, b]$ a non-empty set of real-valued functions. Let us assume that real-valued functions f_1, f_2, \dots, f_n and all $s \in S$ are L_p integrable. If there exists an element $s^* \in S$ such that

$$(1) \quad \inf_{s \in S} \sum_{i=1}^n \|f_i - s\|_p^p = \sum_{i=1}^n \|f_i - s^*\|_p^p,$$

then s^* is said to be a *best simultaneous approximation to the functions* f_1, f_2, \dots, f_n *in the* L_p *norm.*

Theorem 1. *Let* $f_i, i = 1, 2, \dots, n$ ($n \geq 2$) *and* s *be as defined above.*

a) *If* p *is an even natural number, then*

$$(2) \quad \inf_{s \in S} \sum_{i=1}^n \|f_i - s\|_p^p \\ = \inf_{s \in S} \left\{ \frac{2}{n-1} \sum_{k=0}^{p/2} \binom{p}{2k} \sum_{i < j} \int_a^b \left[\frac{f_i(x) + f_j(x)}{2} - s(x) \right]^{p-2k} \right. \\ \left. \times \left[\frac{f_i(x) - f_j(x)}{2} \right]^{2k} dx \right\}$$

b) *If* p *is an odd natural number, the*

$$(3) \quad \inf_{s \in S} \sum_{i=1}^n \|f_i - s\|_p^p = \inf_{s \in S} \left\{ \frac{2}{n-1} \sum_{k=0}^{(p-1)/2} \binom{p}{2k} \right. \\ \left. \times \sum_{i < j} \int_a^b \left[\max \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\} \right]^{p-2k} \right. \\ \left. \times \left[\min \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\} \right]^{2k} dx \right\}.$$

We first prove the following lemma.

Lemma 1. *Let* $n \geq 2$ *be a natural number and let* $1 \leq i < j \leq n$. *For arbitrary real numbers* a_i, a_j *and* $p \geq 1$ *let*

$$(4) \quad A_{ij} = \binom{p}{2k} \left(\frac{a_i + a_j}{2} \right)^{p-2k} \left(\frac{a_i - a_j}{2} \right)^{2k}.$$

a) *If* p *is an even natural number, then*

$$(5) \quad \sum_{i=1}^n a_i^p = \frac{2}{n-1} \sum_{k=0}^{p/2} \sum_{i < j} A_{ij}.$$

b) *If* p *is an odd natural number, then*

$$(6) \quad \sum_{i=1}^n a_i^p = \frac{2}{n-1} \sum_{k=0}^{(p-1)/2} \sum_{i < j} A_{ij}.$$

c) If p is a non-integer real number, $a_i + a_j = 1$ and $-1 \leq a_i - a_j \leq 1$, then

$$(7) \quad \sum_{i=1}^n a_i^p = \frac{2}{n-1} \sum_{k=0}^{\infty} \sum_{i < j} A_{ij}.$$

Proof. a) To prove Lemma 1(a), we use the identity

$$(8) \quad (a+b)^p + (a-b)^p = 2 \sum_{k=0}^{p/2} \binom{p}{2k} a^{p-2k} b^{2k}$$

where p is an even natural number and a, b are arbitrary real numbers. Let us choose $a+b = a_i$, $a-b = a_j$. Then we have

$$(9) \quad a_i^p + a_j^p = 2 \sum_{k=0}^{p/2} A_{ij}.$$

By using (9), we obtain

$$(10) \quad \begin{aligned} \frac{2}{n-1} \sum_{k=0}^{p/2} \sum_{i < j} A_{ij} &= \frac{2}{n-1} \sum_{k=0}^{p/2} \{ (A_{12} + A_{13} + \dots + A_{1n}) \\ &\quad + (A_{23} + \dots + A_{2n}) + \dots + (A_{(n-1)n}) \} \\ &= \frac{2}{n-1} \left\{ \left(\frac{a_1^p + a_2^p}{2} + \frac{a_1^p + a_3^p}{2} + \dots + \frac{a_1^p + a_n^p}{2} \right) \right. \\ &\quad \left. + \left(\frac{a_2^p + a_3^p}{2} + \dots + \frac{a_2^p + a_n^p}{2} \right) + \dots + \left(\frac{a_{n-1}^p + a_n^p}{2} \right) \right\} \\ &= a_1^p + a_2^p + \dots + a_n^p. \end{aligned}$$

b) In this case, we have

$$(11) \quad a_i^p + a_j^p = 2 \sum_{k=0}^{(p-1)/2} A_{ij}.$$

The proof of part (b) is similar to part (a).

c) By using the series

$$(12) \quad (1+y)^p + (1-y)^p = 2 \sum_{k=0}^{\infty} \binom{p}{2k} y^{2k}, \quad -1 \leq y \leq 1$$

and writing $a_i + a_j = 1$, $a_i - a_j = y$ we have

$$(13) \quad a_i^p + a_j^p = 2 \sum_{k=0}^{\infty} A_{ij}.$$

Using this result, we obtain (7). □

Proof of Theorem 1. We first show the existence of the right hand side of (2) in the sense of the L_p norm. From the Hölder inequality

$$\int_a^b |g(x)h(x)| dx \leq \left[\int_a^b |g(x)|^r dx \right]^{1/r} \left[\int_a^b |h(x)|^t dx \right]^{1/t}$$

where $1/r + 1/t = 1$, $g \in L_r$ and $h \in L_t$. If any $s \in S$, $1/r = (p - 2k)/p$ and $1/t = 2k/p$ we have

$$\begin{aligned} & \frac{2}{n-1} \sum_{k=0}^{p/2} \binom{p}{2k} \sum_{i < j} \int_a^b \left[\frac{f_i(x) + f_j(x)}{2} - s(x) \right]^{p-2k} \left[\frac{f_i(x) - f_j(x)}{2} \right]^{2k} dx \\ & \leq \frac{2}{n-1} \sum_{k=0}^{p/2} \binom{p}{2k} \sum_{i < j} \left\{ \int_a^b \left[\frac{f_i(x) + f_j(x)}{2} - s(x) \right]^p dx \right\}^{(p-2k)/p} \\ & \quad \times \left\{ \int_a^b \left[\frac{f_i(x) - f_j(x)}{2} \right]^p dx \right\}^{2k/p} \\ & = \frac{2}{n-1} \sum_{k=0}^{p/2} \binom{p}{2k} \sum_{i < j} \left\| \frac{f_i + f_j}{2} - s \right\|_p^{p-2k} \left\| \frac{f_i - f_j}{2} \right\|_p^{2k}, \end{aligned}$$

which implies the existence of the right hand side of (2). On the other hand, for any $s \in S$ and a pair (i, j) we define

$$\begin{aligned} g_s(x) &= \max \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\}, \\ h_s(x) &= \min \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\}. \end{aligned}$$

We have $g_s, h_s \in L_p$. The existence of the right hand side of (3) in the sense of the L_p norm is shown as in the proof of (2).

a) Let $s \in S$ and $a_i = f_i - s$. Then by Lemma 1(a)

$$(14) \quad \sum_{i=1}^n [f_i(x) - s(x)]^p = \frac{2}{n-1} \sum_{k=0}^{p/2} \binom{p}{2k} \times \sum_{i < j} \left[\frac{f_i(x) + f_j(x)}{2} - s(x) \right]^{p-2k} \left[\frac{f_i(x) - f_j(x)}{2} \right]^{2k}.$$

Integrating each side from a to b and then taking the infimum over all $s \in S$, we obtain (2).

b) Let $s \in S$ and $a_i = |f_i - s|$. Then by Lemma 1(b)

$$\begin{aligned}
 (15) \quad & \sum_{i=1}^n |f_i(x) - s(x)|^p \\
 &= \frac{2}{n-1} \sum_{k=0}^{(p-1)/2} \binom{p}{2k} \sum_{i < j} \left[\frac{|f_i(x) - s(x)| + |f_j(x) - s(x)|}{2} \right]^{p-2k} \\
 & \quad \left[\frac{|f_i(x) - s(x)| - |f_j(x) - s(x)|}{2} \right]^{2k}.
 \end{aligned}$$

For arbitrary real numbers m and n we have identities

$$\begin{aligned}
 (16) \quad & |m+n| + |m-n| = 2 \max\{|m|, |n|\}, \\
 & ||m+n| - |m-n|| = 2 \min\{|m|, |n|\}.
 \end{aligned}$$

If we replace $m+n$ and $m-n$ in (16) by $f_i(x) - s(x)$ and $f_j(x) - s(x)$, respectively, and use (15), we obtain

$$\begin{aligned}
 & \sum_{i=1}^n |f_i(x) - s(x)|^p \\
 &= \frac{2}{n-1} \sum_{k=0}^{(p-1)/2} \binom{p}{2k} \sum_{i < j} \left[\max \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\} \right]^{p-2k} \\
 & \quad \times \left[\min \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\} \right]^{2k}.
 \end{aligned}$$

Integrating both sides of this equality from a to b and taking the infimum over all $s \in S$, we have the result of Theorem 1(b).

Remark 1. In Theorem 1(a):

a) If we take $p = 2$ we see that Theorem 1 in [1] is a special case of Theorem 1(a). On the other hand, if we take $n = 2$, we obtain Theorem 3 in [2].

b) If p is an even natural number, then for $n = 2$ Theorem II in [1] is a special case of Theorem 1(a).

Remark 2. In Theorem 1(b):

a) If we replace $[a, b]$ by $[0, 1]$ put $p = 1$ and $n = 2$, then we obtain Theorem 2 in [3].

b) If we take $p = 1$, we see that Theorem 5 in [4] is a special case of Theorem 1(b). Really, $\text{sgn}(f_i(x) - s(x))$ is always positive or negative according to the hypothesis of Theorem 5 in [4]. Hence

$$\max \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\} = \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|.$$

Then (3) implies

$$\begin{aligned} \inf_{s \in S} \sum_{i=1}^n \|f_i - s\|_1 &= \inf_{s \in S} \left\{ \frac{2}{n-1} \int_a^b \sum_{i < j} \left(\frac{f_i(x) + f_j(x)}{2} - s(x) \right) dx \right\} \\ &= \inf_{s \in S} n \left\| \frac{1}{n} \sum_{i=1}^n f_i - s \right\|_1. \end{aligned}$$

c) If p is an odd natural number, then for $n = 2$ Theorem 1 in [5] is a special case of Theorem 1(b).

Theorem 2. Let f_1, f_2, \dots, f_n and s be as in Definition 1 and let $p > 1$ be a non-integer real number. If $f_i(x) - s(x) \neq 0$, then

$$\begin{aligned} &\inf_{s \in S} \sum_{i=1}^n \|f_i - s\|_p^p \\ &= \inf_{s \in S} \left\{ \frac{2}{n-1} \sum_{k=0}^{\infty} \binom{p}{2k} \sum_{i < j} \int_a^b \left[\max \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\} \right]^{p-2k} \right. \\ &\quad \left. \times \left[\min \left\{ \left| \frac{f_i(x) + f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x) - f_j(x)}{2} \right| \right\} \right]^{2k} dx \right\}. \end{aligned}$$

Proof. The existence of the right hand side of the equality in the sense of the L_p -norm is shown as in the proof of Theorem 1(b) and using the absolute convergence of the series on the right hand side under the given hypothesis. To prove Theorem 2 it is sufficient to take $a_i = |f_i(x) - s(x)|$ in Lemma 1(c). \square

Remark 3. In Theorem 2, if we take $n = 2$, we see that Theorem in [6] is a special case of Theorem 2.

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