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ALMOST HYPER-HERMITIAN STRUCTURES IN BUNDLE SPACES
OVER MANIFOLDS WITH ALMOST CONTACT 3-STRUCTURE

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Abstract. We consider almost hyper-Hermitian structures on principal fibre bundles with one-dimensional fiber over manifolds with almost contact 3-structure and study relations between the respective structures on the total space and the base. This construction suggests the definition of a new class of almost contact 3-structure, which we called trans-Sasakian, closely connected with locally conformal quaternionic Kähler manifolds. Finally we give a family of examples of hypercomplex manifolds which are not quaternionic Kähler.

1. INTRODUCTION

An almost hyper-Hermitian (quaternion-Hermitian) manifold is a Riemannian $4n$ -manifold which admits a reduction of its frame bundle to the subgroup $\mathrm{Sp}(n)$ ($\mathrm{Sp}(n)\mathrm{Sp}(1)$) of $\mathrm{SO}(4n)$. These two types of manifolds are of special interest because $\mathrm{Sp}(n)$ and $\mathrm{Sp}(n)\mathrm{Sp}(1)$ are included in the list of Berger ([1]) of the possible holonomy groups of locally irreducible Riemannian manifolds that are not locally symmetric. An almost hyper-Hermitian (quaternion-Hermitian) manifold is said to be hyper-Kähler (quaternionic Kähler), if its reduced holonomy group is a subgroup of $\mathrm{Sp}(n)$, $n \geq 1$ ($\mathrm{Sp}(n)\mathrm{Sp}(1)$, $n > 1$). The terms “quaternionic Kähler” and “hyper-Kähler” were introduced by Calabi and Ishihara in 1973. A few years before, Kuo ([13]) defined a new type of geometric structure closely related to both quaternion-Hermitian and almost hyper-Hermitian structures, the almost contact 3-structure. A particular and interesting class of almost contact 3-structure is the Sasakian 3-structure. Riemannian manifolds with Sasakian 3-structure are called 3-Sasakian manifolds. They are Einstein and $(4n + 3)$ -dimensional and have many links with quaternionic Kähler and hyper-Kähler manifolds. In fact, if the distribution formed by the three Killing vector fields of a Sasakian 3-structure is regular then the space

of leaves is quaternionic Kähler, which was shown by Ishihara in 1973 ([8]). Later in 1975, Konishi ([11]) proved the existence of a Sasakian 3-structure on a certain principal $SO(3)$ bundle over any quaternionic Kähler manifold of positive scalar curvature. Recently, Boyer, Galicki and Mann ([3]) have shown that for any quaternionic Kähler manifold M of positive scalar curvature there exists a commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{U} & & \\
 & \swarrow & \downarrow & \searrow & \\
 \mathbb{C}^*/\mathbb{Z}_2 & & & & \mathcal{S} \\
 & \searrow & \downarrow & \swarrow & \\
 \mathcal{L} & & \mathbb{H}^*/\mathbb{Z}_2 & & \\
 & \searrow & \downarrow & \swarrow & \\
 \mathbb{C}P^1 & & M & & \mathbb{R}P^3
 \end{array}$$

where \mathcal{U} is hyper-Kähler (the Swann bundle associated to M [19]), \mathcal{L} is Kähler-Einstein (the twistor space associated to M [18]) and \mathcal{S} is 3-Sasakian (the Konishi bundle associated to M [11]). The map $\iota: \mathcal{S} \rightarrow \mathcal{U}$ is the inclusion of a level set of a natural real valued function while the other maps are fibrations where each map is denoted by its associated fiber.

In this paper we consider principal fibre bundles with one-dimensional structure group over manifolds with almost contact metric 3-structures. On the total bundle space we construct an almost hyper-Hermitian structure defined from an arbitrary connection form and the almost contact metric 3-structure of the base. In this context, we find relations among classes of the almost hyper-Hermitian structure, classes of the almost contact metric 3-structure and the curvature of the connection form. These relations lead us to consider a new class of almost contact 3-structure, called trans-Sasakian, which is closely connected with locally conformal quaternionic Kähler structures. Finally, the mentioned relations have suggested us a construction of a family of hypercomplex manifolds which are not quaternionic semi-Kähler.

2. QUATERNION-HERMITIAN STRUCTURES

Quaternion-Hermitian manifolds have been broadly treated by diverse authors (see [2], [8], [18], and [19]). In this section we review some basic definitions, known facts and prove some new results.

A $4n$ -dimensional manifold M ($n > 1$) is said to be *quaternion-Hermitian* if M is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ and a rank-three subbundle \mathcal{J} of the endomorphism bundle $\text{End } TM$ such that locally \mathcal{J} has an *adapted basis* J_1, J_2, J_3 with $J_i^2 = -1$, $J_1 J_2 = J_3 = -J_2 J_1$ and $\langle J_i X, J_i Y \rangle = \langle X, Y \rangle$, for $i = 1, 2, 3$. This is equivalent to saying that M has a reduction of its structure group to $\text{Sp}(n) \text{Sp}(1)$.

At each point of a $4n$ -dimensional quaternion-Hermitian manifold there is a local orthonormal frame field, called *adapted frame*, given in the following way:

$$\{E_1, \dots, E_n, J_1E_1, \dots, J_1E_n, J_2E_1, \dots, J_2E_n, J_3E_1, \dots, J_3E_n\}.$$

From the three local two-forms $F^i(X, Y) = \langle X, J_iY \rangle$, one may define a global four-form Ω by the local formula

$$(2.1) \quad \Omega = F^1 \wedge F^1 + F^2 \wedge F^2 + F^3 \wedge F^3.$$

The following lemma will be useful later.

Lemma 2.1. *Let M be a quaternion-Hermitian $4n$ -manifold ($n > 1$) and α a skew-symmetric p -form on M ($p \leq 2$). Then $\alpha \wedge \Omega = 0$ if and only if $\alpha = 0$.*

P r o o f. Throughout the proof (i, j, k) is always a cyclic permutation of $(1, 2, 3)$, $r, s = 1, \dots, n$ with $r \neq s$ and we consider an adapted local frame of M ordered as in (2.1). First, from

$$\begin{aligned} \alpha \wedge \Omega(E_s, J_iE_s, E_r, J_1E_r, J_2E_r, J_3E_r) &= 0, \\ \alpha \wedge \Omega(J_jE_s, J_kE_s, E_r, J_1E_r, J_2E_r, J_3E_r) &= 0, \\ \alpha \wedge \Omega(E_r, J_iE_r, E_s, J_1E_s, J_2E_s, J_3E_s) &= 0, \\ \alpha \wedge \Omega(J_jE_r, J_kE_r, E_s, J_1E_s, J_2E_s, J_3E_s) &= 0, \end{aligned}$$

we have

$$\begin{aligned} 3\alpha(E_s, J_iE_s) + \alpha(E_r, J_iE_r) + \alpha(J_jE_r, J_kE_r) &= 0, \\ 3\alpha(J_jE_s, J_kE_s) + \alpha(E_r, J_iE_r) + \alpha(J_jE_r, J_iE_r) &= 0, \\ \alpha(E_s, J_iE_s) + \alpha(J_jE_s, J_kE_s) + 3\alpha(E_r, J_iE_r) &= 0, \\ \alpha(E_s, J_iE_s) + \alpha(J_jE_s, J_kE_s) + 3\alpha(J_jE_r, J_kE_r) &= 0. \end{aligned}$$

From these equations $\alpha(E_r, J_iE_r) = \alpha(J_jE_r, J_kE_r) = \alpha(E_s, J_iE_s) = \alpha(J_jE_s, J_kE_s) = 0$. Secondly, we consider

$$\begin{aligned} \alpha \wedge \Omega(E_s, J_1E_s, J_2E_s, E_r, J_1E_r, J_2E_r) &= 0, \\ \alpha \wedge \Omega(E_s, J_3E_s, J_1E_s, E_r, J_3E_r, J_1E_r) &= 0, \\ \alpha \wedge \Omega(E_s, J_2E_s, J_3E_s, E_r, J_2E_r, J_3E_r) &= 0, \\ \alpha \wedge \Omega(J_1E_s, J_2E_s, J_3E_s, J_1E_r, J_2E_r, J_3E_r) &= 0, \end{aligned}$$

then we have

$$\begin{aligned}\alpha(E_s, E_r) + \alpha(J_1 E_s, J_1 E_r) + \alpha(J_2 E_s, J_2 E_r) &= 0, \\ \alpha(E_s, E_r) + \alpha(J_1 E_s, J_1 E_r) + \alpha(J_3 E_1, J_3 E_r) &= 0, \\ \alpha(E_s, E_r) + \alpha(J_2 E_s, J_2 E_r) + \alpha(J_3 E_1, J_3 E_r) &= 0, \\ \alpha(J_1 E_s, J_1 E_r) + \alpha(J_2 E_s, J_2 E_r) + \alpha(J_3 E_1, J_3 E_r) &= 0.\end{aligned}$$

Hence, $\alpha(E_s, E_r) = \alpha(J_i E_1, J_i E_r) = 0$. At this point, we can conclude $\alpha = 0$. \square

Remark 2.2. In [12] it is shown that $\alpha \wedge \Omega = 0$ implies $\alpha = 0$, when α is a p -form such that $p + 4 \leq n + 1$.

If Ω is parallel with respect to the Levi-Civita connection ∇ of $\langle \cdot, \cdot \rangle$, then the holonomy group of M is a subgroup of $\text{Sp}(n)\text{Sp}(1)$ ($n > 1$) and M is said to be *quaternionic Kähler*. The quaternionic Kähler condition is equivalent to the existence of three local one-forms $\alpha^1, \alpha^2, \alpha^3$ such that

$$(2.2) \quad \nabla J_i = \alpha^i \otimes J_j - \alpha^k \otimes J_k$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$ ([8]). If the exterior derivative $d\Omega$ vanishes, M is said to be *quaternionic almost-Kähler*. In [19] it is shown that every quaternionic almost-Kähler manifold of dimension ≥ 12 is quaternionic Kähler. The dimension eight is included in the following result.

Proposition 2.3. *Let M be a quaternion-Hermitian $4n$ -manifold ($n > 1$). Then the following statements are equivalent:*

- i) M is quaternionic Kähler.
- ii) $d\Omega = 0$ and $dF^i = \alpha^i \wedge F^i + b^i \wedge F^j + c^i \wedge F^k$.
- iii) There exist three local one-forms $\alpha^1, \alpha^2, \alpha^3$ such that $dF^i = \alpha^i \wedge F^j - \alpha^k \wedge F^k$ where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Proof. The equivalence of the first two statements was established in [19]. Taking (2.2) into account, it is easy to see that the third statement follows from the first. Finally, it is a straightforward computation that the second statement follows from the third. \square

The quaternionic nearly-Kähler condition, i.e., $d\Omega = 5\nabla\Omega$, is equivalent to the quaternionic Kähler condition ([20]). If the coderivative $\delta\Omega$ vanishes, M is said to be *quaternionic semi-Kähler*. In [2] it is shown that $\delta\Omega = - * dk\Omega^{n-1}$, where k is constant and $*$ denotes Hodge's star operator. Then every quaternionic almost-Kähler manifold is quaternionic semi-Kähler. The converse is also true for dimension eight.

A quaternion-Hermitian $4n$ -manifold M ($n > 1$) is said to be *locally conformal quaternionic Kähler*, if $dF^i = \alpha \wedge F^i + \alpha^i \wedge F^j - \alpha^k \wedge F^k$ for all cyclic permutations (i, j, k) of $(1, 2, 3)$ and some one-forms $\alpha, \alpha^1, \alpha^2, \alpha^3$. In this case $d\Omega = 2\alpha \wedge \Omega$ and using Lemma 2.1 we have $d\alpha = 0$, then locally $\alpha = df$. If we consider the metric $e^{-f}\langle \cdot, \cdot \rangle$, the structure considered on a neighborhood of a point is also quaternion-Hermitian and satisfies the third statement of Proposition 2.3. Moreover, if we have $d\Omega = 2\alpha_U \wedge \Omega = 2\alpha_V \wedge \Omega$ for all points of $U \cap V$, U, V open sets of M , by Lemma 2.1 $\alpha_U = \alpha_V$ on $U \cap V$, then the one-form α is global.

An *almost hyper-Hermitian structure* on M is a quaternion-Hermitian structure such that the subbundle \mathcal{J} has an adapted basis J_1, J_2, J_3 of global tensor fields. In this case, M has a reduction of its structure group to $\text{Sp}(n)$. If M has an almost hyper-Hermitian structure such that F^1, F^2, F^3 are closed, M is said to be *hyper-Kähler*. Hitchin [6] showed that this implies that J_1, J_2, J_3 are integrable and hence the holonomy group is contained in $\text{Sp}(n)$. An alternative condition to impose on an almost hyper-Hermitian structure is that J_1, J_2, J_3 all be integrable. In this case the manifold M is said to be *hypercomplex (hyper-Hermitian)*. A manifold M is said to be *locally conformal hyper-Kähler*, if M has an almost hyper-Hermitian structure such that $dF^i = \alpha \wedge F^i$ for some one-form α . In this case α is closed and we can do a local conformal change of the metric such that the almost hyper-Hermitian structure considered on a neighborhood of the point is hyper-Kähler for the new metric.

3. ALMOST CONTACT 3-STRUCTURES

In this section we show, together with some definitions and known facts (see [13], [14], some new results about almost contact 3-structures which will be used later. An *almost contact structure* (φ, ξ, η) on a differentiable manifold is an aggregate consisting of a tensor field φ of type $(1, 1)$, a vector field ξ and a one-form η which satisfy $\eta(\xi) = 1, \varphi^2 = -I + \xi \otimes \eta$, where \otimes means the tensor product and I is the identity tensor.

A $(4n + 3)$ -manifold M ($n \geq 1$) possesses an *almost contact metric 3-structure*, if M has a Riemannian metric $\langle \cdot, \cdot \rangle$ and three almost contact structures, $(\varphi_i, \xi_i, \eta^i)$, $i = 1, 2, 3$, satisfying

$$\begin{aligned} \eta^i(\xi_j) &= \delta_j^i, & \varphi_i(\xi_j) &= -\varphi_j(\xi_i) = \xi_k, & \eta^i \circ \varphi_j &= -\eta^j \circ \varphi_i = \eta_k, \\ \varphi^i \circ \varphi_j - \eta^j \otimes \xi_i &= -\varphi_j \circ \varphi_i + \eta^i \otimes \xi_j = \varphi_k, & \langle \varphi_i X, \varphi_i Y \rangle &= \langle X, Y \rangle, \end{aligned}$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$ and any X, Y vector fields on M . In this case, the structure group of M admits a reduction to $\text{Sp}(n) \times I_3$. At each point

of a $(4n + 3)$ -manifold with an almost contact metric 3-structure there is a local orthonormal frame field, called *adapted frame*, given in the following way:

$$(3.1) \quad \{E_1, \dots, E_n, \varphi_1 E_1, \dots, \varphi_1 E_n, \varphi_2 E_1, \dots, \varphi_2 E_n, \varphi_3 E_1, \dots, \varphi_3 E_n, \xi_1, \xi_2, \xi_3\}.$$

Let F^i be the two-forms given by $F^i(X, Y) = \langle X, \varphi_i Y \rangle$. Associated to an almost contact metric 3-structure there is a four-form given by

$$(3.2) \quad \Omega = F^1 \wedge F^1 + F^2 \wedge F^2 + F^3 \wedge F^3.$$

For almost contact 3-structures it is also needed to consider the three-form

$$(3.3) \quad \Psi = \eta^1 \wedge F^1 + \eta^2 \wedge F^2 + \eta^3 \wedge F^3.$$

We will make use of the following lemma in the sequel.

Lemma 3.1. *Let M be a $(4n + 3)$ -manifold ($n \geq 1$) with an almost contact 3-structure and α a skew-symmetric two-form on M . Then*

- i) $\alpha \wedge \Psi = 0$ if and only if $\alpha = 0$.
- ii) $C_{12} \circ C_{13}(\alpha \otimes \Omega) = 0$ if and only if $\alpha = 0$, where C denotes the metric contraction.

Proof. Throughout the proof (i, j, k) is always a cyclic permutation of $(1, 2, 3)$, $r, s = 1, \dots, n$ with $r \neq s$ and we consider an adapted local frame of M ordered as in (3.1). i) First, we develop $\alpha \wedge \Psi(E_r, E_s, \xi_1, \xi_2, \xi_3) = 0$, then we get $3\alpha(E_r, E_s) = 0$. Secondly, we consider $\alpha \wedge \Psi(\xi_i, E_r, \varphi_1 E_r, \varphi_2 E_r, \varphi_3 E_r) = 0$, $\alpha \wedge \Psi(E_r, \varphi_i E_r, \xi_1, \xi_2, \xi_3) = 0$ and $\alpha \wedge \Psi(\varphi_j E_r, \varphi_k E_r, \xi_1, \xi_2, \xi_3) = 0$, then we have

$$\begin{aligned} \alpha(E_r, \varphi_i E_r) + \alpha(\varphi_j E_r, \varphi_k E_r) &= 0, \\ \alpha(\xi_j, \xi_k) + 3\alpha(E_r, \varphi_i E_r) &= 0, \\ \alpha(\xi_j, \xi_k) + 3\alpha(\varphi_j E_r, \varphi_k E_r) &= 0. \end{aligned}$$

From these equations, $\alpha(\xi_j, \xi_k) = 0$, $\alpha(E_r, \varphi_i E_r) = 0$ and $\alpha(\varphi_j E_r, \varphi_k E_r) = 0$. Finally, we consider $\alpha \wedge \Psi(\xi_i, \xi_j, E_r, \varphi_i E_r, \varphi_j E_r) = 0$ and get $\alpha(\xi_i, \varphi_i E_r) + \alpha(\xi_j, \varphi_j E_r) = 0$. Hence, $\alpha(\xi_1, \varphi_1 E_r) = -\alpha(\xi_2, \varphi_2 E_r) = \alpha(\xi_3, \varphi_3 E_r) = -\alpha(\xi_1, \varphi_1 E_r) = 0$. In a similar way, we can get $\alpha(\xi_i, E_r) = 0$, $\alpha(\xi_i, \varphi_j E_r) = 0$ and $\alpha(\xi_i, \varphi_k E_r) = 0$. So we conclude $\alpha = 0$.

- ii) From $C_{12} \circ C_{13}(\alpha \otimes \Omega)(\xi_i, \xi_j) = 0$, we get

$$(3.4) \quad \sum_{l=1}^n \alpha(E_l, \varphi_k E_l) + \sum_{l=1}^n \alpha(\varphi_i E_l, \varphi_k E_l) = 0.$$

Now, from $C_{12} \circ C_{13}(\alpha \otimes \Omega)(E_r, \varphi_i E_r) = 0$, $C_{12} \circ C_{13}(\alpha \otimes \Omega)(E_r, \varphi_j E_r) = 0$ and $C_{12} \circ C_{13}(\alpha \otimes \Omega)(E_r, \varphi_k E_r) = 0$, taking (3.4) into account, we get

$$\begin{aligned}\alpha(\xi_i, \xi_j) - \alpha(E_r, \varphi_k E_r) + 2\alpha(\varphi_i E_r, \varphi_j E_r) &= 0, \\ \alpha(\xi_i, \xi_j) - \alpha(\varphi_i E_r, \varphi_j E_r) + 2\alpha(E_r, \varphi_k E_r) &= 0.\end{aligned}$$

From these equations we have

$$(3.5) \quad -\alpha(\xi_i, \xi_j) = \alpha(\varphi_i E_r, \varphi_j E_r) = \alpha(E_r, \varphi_k E_r).$$

Now using (3.5) in (3.4) we obtain $-2n\alpha(\xi_i, \xi_j) = 0$. Hence

$$(3.6) \quad 0 = \alpha(\xi_i, \xi_j) = \alpha(\varphi_i E_r, \varphi_j E_r) = \alpha(E_r, \varphi_k E_r).$$

Let us compute successively $C_{12} \circ C_{13}(\alpha \otimes \Omega)(\xi_1, \varphi_1 E_r)$, $C_{12} \circ C_{13}(\alpha \otimes \Omega)(\xi_2, \varphi_2 E_r)$ and $C_{12} \circ C_{13}(\alpha \otimes \Omega)(\xi_3, \varphi_3 E_r)$, obtaining

$$\begin{aligned}\alpha(\xi_2, \varphi_2 E_r) + \alpha(\xi_3, \varphi_3 E_r) &= 0, \\ \alpha(\xi_1, \varphi_1 E_r) + \alpha(\xi_3, \varphi_3 E_r) &= 0, \\ \alpha(\xi_1, \varphi_1 E_r) + \alpha(\xi_2, \varphi_2 E_r) &= 0.\end{aligned}$$

From these equations we get

$$(3.7) \quad \alpha(\xi_1, \varphi_1 E_r) = \alpha(\xi_2, \varphi_2 E_r) = \alpha(\xi_3, \varphi_3 E_r) = 0.$$

In a similar way we can obtain

$$(3.8) \quad \alpha(\xi_i, E_r) = \alpha(\xi_j, \varphi_k E_r) = \alpha(\xi_k, \varphi_j E_r) = 0.$$

If the dimension of M is seven, the proof is already concluded. Let us complete the proof for dimension higher than seven. From $C_{12} \circ C_{13}(\alpha \otimes \Omega)(E_r, E_s) = 0$ and $C_{12} \circ C_{13}(\alpha \otimes \Omega)(\varphi_i E_r, \varphi_i E_s) = 0$ we have

$$\begin{aligned}\alpha(\varphi_1 E_r, \varphi_1 E_s) + \alpha(\varphi_2 E_r, \varphi_2 E_s) + \alpha(\varphi_3 E_r, \varphi_3 E_s) &= 0, \\ \alpha(E_r, E_s) + \alpha(\varphi_j E_r, \varphi_j E_s) + \alpha(\varphi_k E_r, \varphi_k E_s) &= 0.\end{aligned}$$

These equations yield

$$(3.9) \quad \alpha(E_r, E_s) = \alpha(\varphi_1 E_r, \varphi_1 E_s) = \alpha(\varphi_2 E_r, \varphi_2 E_s) = \alpha(\varphi_3 E_r, \varphi_3 E_s) = 0.$$

In a similar way we can get

$$(3.10) \quad \alpha(E_r, \varphi_i E_s) = \alpha(\varphi_i E_r, E_s) = \alpha(\varphi_j E_r, \varphi_k E_s) = \alpha(\varphi_k E_r, \varphi_j E_s) = 0.$$

From (3.6), (3.7), (3.8), (3.9) and (3.10) we conclude $\alpha = 0$. □

If Ω and Ψ are parallel with respect to the Levi-Civita connection, the almost contact 3-structure is said to be *cosymplectic*. If Ω and Ψ are closed, we say that M has an *almost-cosymplectic 3-structure*. If the forms Ω and Ψ are coclosed, i.e., $\delta\Omega = \delta\Psi = 0$, we say that M has a *semi-cosymplectic 3-structure*. An almost contact metric 3-structure is said to be *hypernormal*, if the three almost contact structures are normal, i.e., $N_{\varphi_i} + 2d\eta^i \otimes \xi_i = 0$, where N_{φ_i} is the Nijenhuis tensor of φ_i , i.e.,

$$N_{\varphi_i} = \varphi_i^2[X, Y] + [\varphi_i X, \varphi_i Y] - \varphi_i[\varphi_i X, Y] - \varphi_i[X, \varphi_i Y].$$

If we suppose that the two-forms F^1, F^2, F^3 and the one-forms η^1, η^2, η^3 are closed, we say that M has a *hypercospymplectic 3-structure*. One can use Hitchin's argument to deduce that in this case the three almost contact structures are normal. Therefore, the three almost contact structures are cosymplectic, i.e., $\nabla F^i = 0$ and $\nabla \eta^i = 0$, $i = 1, 2, 3$ ([5]).

Definition 3.2. An almost contact metric 3-structure is said to be *a-Sasakian* ($a \in \mathbb{R}$, $a \neq 0$), if it is hypernormal and $d\eta^i = aF^i$. When $a = 1$, the almost contact 3-structure is said to be *Sasakian*. A hypercospymplectic structure can be considered a 0-Sasakian structure.

Definition 3.3. An almost contact metric 3-structure is said to be *trans-Sasakian*, if

$$\begin{aligned} dF^i &= \alpha \wedge F^i + \alpha^i \wedge F^j - \alpha^k \wedge F^k, \\ d\eta^i &= aF^i + r_i F^j - r_k F^k + \alpha \wedge \eta^i + \alpha^i \wedge \eta^j - \alpha^k \wedge \eta^k \end{aligned}$$

for some a, r_1, r_2, r_3 differentiable local functions, $\alpha, \alpha^1, \alpha^2, \alpha^3$ local one-forms on M and for all (i, j, k) cyclic permutations of $(1, 2, 3)$. In this case we have $d\Omega = 2\alpha \wedge \Omega$ and $d\Psi = 2\alpha \wedge \Psi + a\Omega$.

Lemma 3.4. Let M be a $(4n + 3)$ -manifold ($n \geq 1$) with a *trans-Sasakian structure*. Then the local functions a, r_1, r_2, r_3 and the local forms $\alpha, \alpha^1, \alpha^2, \alpha^3$ given in Definition 3.3 are global.

Proof. Let us suppose $a_U, r_{1U}, r_{2U}, r_{3U}, \alpha_U, \alpha_U^1, \alpha_U^2, \alpha_U^3$ defined on U and $a_V, r_{1V}, r_{2V}, r_{3V}, \alpha_V, \alpha_V^1, \alpha_V^2, \alpha_V^3$ defined on V , where U, V are no disjoint open sets of M . On $U \cap V$ we have $a_U = a_V = d\eta^i(E, \varphi_i E)$, $r_{iU} = r_{iV} = d\eta^i(E, \varphi_j E)$, where E is a unitary vector orthogonal to ξ_1, ξ_2, ξ_3 . Therefore a, r_1, r_2, r_3 are global differentiable functions on M . Now, from $d\Psi = 2\alpha \wedge \Psi + a\Omega$ we have $(\alpha_U - \alpha_V) \wedge \Psi = 0$. By Lemma 3.1, $\alpha_U = \alpha_V$. Therefore α is a global one-form. Finally, from

Definition 3.3 we have

$$\begin{aligned} 0 &= (\alpha_U^i - \alpha_V^i) \wedge F^j - (\alpha_U^k - \alpha_V^k) \wedge F^k, \\ 0 &= (\alpha_U^i - \alpha_V^i) \wedge \eta^j - (\alpha_U^k - \alpha_V^k) \alpha^k \wedge \eta^k. \end{aligned}$$

If E is a unitary vector orthogonal to ξ_1, ξ_2 and ξ_3 we have

$$0 = ((\alpha_U^i - \alpha_V^i) \wedge F^j - (\alpha_U^k - \alpha_V^k) \wedge F^k)(\xi_r, E, \varphi_j E) = -\alpha_U^i(\xi_r) + \alpha_V^i(\xi_r),$$

where $r = 1, 2, 3$. Moreover,

$$0 = ((\alpha_U^i - \alpha_V^i) \wedge \eta^j - (\alpha_U^k - \alpha_V^k) \alpha^k \wedge \eta^k)(E, \xi_j) = \alpha_U^i(E) - \alpha_V^i(E).$$

Hence, $\alpha_U^i = \alpha_V^i$. Then the forms α^1, α^2 and α^3 are global. \square

4. ALMOST HYPER-HERMITIAN STRUCTURES IN PRINCIPAL FIBRE BUNDLES OVER MANIFOLDS WITH ALMOST CONTACT 3-STRUCTURE

From now on, M will be a $(4n + 3)$ -manifold ($n \geq 1$) with an almost contact metric 3-structure $(\varphi_i, \xi_i, \eta^i, \langle \cdot, \cdot \rangle)$, $i = 1, 2, 3$ and $\mathfrak{X}(M)$ will denote the Lie algebra of C^∞ vector fields on M . Let ω be an arbitrary connection form on \overline{M} , where $\overline{M} = \overline{M}(M, G, \pi)$ denotes a principal fibre bundle with a one-dimensional connected structure group G and projection π . We use X^H and A^* to denote the horizontal lift of $X \in \mathfrak{X}(M)$ and the fundamental vector field with respect to $A \in \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of G . Then we have ([10])

$$(4.1) \quad \begin{aligned} [A^*, X^H] &= 0, \quad [A^*, B^*] = 0, \\ \omega([X^H, Y^H]) &= -2\hat{\Omega}(X^H, Y^H), \quad \mathfrak{h}[X^H, Y^H]_p = [X, Y]_p^H \end{aligned}$$

for $A, B \in \mathfrak{g}$, $X, Y \in \mathfrak{X}(M)$, where $\hat{\Omega}$ denotes the curvature form of ω and \mathfrak{h} is the horizontal component of a vector in $T_p\overline{M}$.

Making use of the connection form ω and the almost contact 3-structure on M , $(\varphi_i, \xi_i, \eta^i, \langle \cdot, \cdot \rangle)$, one can define three almost complex structures J_1, J_2, J_3 on \overline{M} in the following way ([21]):

$$(4.2) \quad (J_i)_p = -\omega_p(X_p)(\xi_i^H)_p + (\varphi_{i\pi(p)}\pi_{*p}X_p)^H + \eta_{\pi(p)}^i(\pi_{*p}X_p)\hat{\xi}_p^*, \quad i = 1, 2, 3,$$

where p is a point of \overline{M} , $\hat{\xi} \in \mathfrak{g}$ with $\hat{\xi} \neq 0$ and $\hat{\eta}$ is the dual form of $\hat{\xi}$.

Let $\langle \rangle_0$ be the tensor metric field on \overline{M} given by $\langle \rangle_0 = \pi^*\langle \rangle + \hat{\eta}\omega \otimes \hat{\eta}\omega$. In [15] it is shown that each J_i is almost Hermitian with respect to $\langle \rangle_0$. By a straightforward computation one can check that $J_1 J_2 = J_3 = -J_2 J_1$. Then we have:

Proposition 4.1. $(J_1, J_2, J_3, \langle \rangle_0)$ is an almost hyper-Hermitian structure on \overline{M} .

Let us denote by \overline{F}^i and F^i the respective two-forms defined from the i -th almost complex structure on \overline{M} and the i -th almost contact structure on M . Analogously, $\overline{\Omega}$ and Ω represent the respective four-forms on \overline{M} and M . The three-form on M is denoted by Ψ as in Section 2. The relations among all these forms is given in the following lemma.

Lemma 4.2. We have

- i) $\overline{F}^i = \pi^* F^i + \hat{\eta}\omega \wedge \pi^* \eta^i, \quad i = 1, 2, 3;$
- ii) $\overline{\Omega} = \pi^* \Omega + 2\hat{\eta}\omega \wedge \pi^* \Psi.$

Proof. Using (4.2) we get

$$\overline{F}^i(X^H, Y^H) = F^i(X, Y) \circ \pi, \quad \overline{F}^i(X^H, \hat{\xi}^*) = -\eta^i(X) \circ \pi, \quad \overline{F}^i(A^*, Y^*) = 0$$

for $X, Y \in \mathfrak{X}(M)$ and $A, B \in \mathfrak{g}$. Now it is immediate that $\pi^* F^i + \hat{\eta}\omega \wedge \pi^* \eta^i$ coincides with \overline{F}^i . We can deduce ii) using i), (2.1), (3.2) and (3.3). \square

From now on $\{E_1, E_2, \dots, E_{4n+3}\}$ will be an adapted frame of M ordered as in (3.1) and we will write $\omega, \hat{\Omega}$ instead of $\hat{\eta}\omega, \hat{\eta}\hat{\Omega}$. The curvature form $\hat{\Omega}$ is tensorial of $(\text{Ad}, \mathfrak{g})$ type, where Ad is the adjoint representation. But here, the Lie group G is abelian, hence we have $\hat{\Omega}_{pg}(X_{pg}^H, Y_{pg}^H) = \hat{\Omega}_p(X_p^H, Y_p^H)$ for all $p \in \overline{M}, g \in G$ and $X, Y \in \mathfrak{X}(M)$. Thus we can define a two-form on M , denoted also by $\hat{\Omega}$, given by $\hat{\Omega}(X, Y) = \hat{\Omega}(X^H, Y^H)$.

Now we consider $\overline{\nabla}$ and ∇ , the respective Levi-Civita connections of $\langle \rangle_0$ and $\langle \rangle$. From the Koszul formula ([10]) using (4.1) we obtain the following lemma.

Lemma 4.3. For $A, B \in \mathfrak{g}$ and $X, Y \in \mathfrak{X}(M)$, we have

- i) $\overline{\nabla}_{A^*} B^* = 0,$
- ii) $\overline{\nabla}_{X^H} A^* = \overline{\nabla}_{A^*} X^H = \frac{1}{2} \hat{\eta}(A) \sum_{i=1}^{4n+3} \hat{\Omega}(X^H, E_i^H) E_i^H,$
- iii) $\overline{\nabla}_{X^H} Y^H = -\frac{1}{2} (\hat{\Omega}(X^H, Y^H))^* + (\nabla_X Y)^H.$

The covariant derivative of \overline{F}^i in terms of F^i, η^i and ω is given in the next lemma.

Lemma 4.4. For $X, Y, Z \in \mathfrak{X}(M)$ we have

- i) $\bar{\nabla}_{\hat{\xi}^*}(\bar{F}^i)(\hat{\xi}^*, X^H) = \frac{1}{2}\hat{\Omega}(\xi_i^H, X^H),$
 ii) $\bar{\nabla}_{\hat{\xi}^*}(\bar{F}^i)(X^H, Y^H) = -\frac{1}{2}\{\hat{\Omega}(X^H, (\varphi_i Y)^H) + \hat{\Omega}((\varphi_i X)^H, Y^H)\},$
 iii) $\bar{\nabla}_{X^H}(\bar{F}^i)(\hat{\xi}^*, Y^H) = \nabla_X(\eta^i)(Y) \circ \pi - \frac{1}{2}\hat{\Omega}(X^H, (\varphi_i Y)^H),$
 iv) $\bar{\nabla}_{X^H}(\bar{F}^i)(Y^H, Z^H) = \nabla_X(F^i)(Y, Z) \circ \pi + \frac{1}{2}(X \diamond \hat{\Omega} \wedge \eta^i)(Y, Z) \circ \pi,$
 where \diamond denotes the interior product.

P r o o f. We have

$$\bar{\nabla}_{\hat{\xi}^*}(\bar{F}^i)(\hat{\xi}^*, X^H) = \hat{\xi}^* \bar{F}^i(\hat{\xi}^*, X^H) - \bar{F}^i(\bar{\nabla}_{\hat{\xi}^*} \hat{\xi}^*, X^H) - \bar{F}^i(\hat{\xi}^*, \bar{\nabla}_{\hat{\xi}^*} X^H).$$

By Lemma 4.2 i) and Lemma 4.3 i) the first and second summands vanish. Now using again Lemma 4.3 ii) and Lemma 4.2 i) we have

$$\bar{\nabla}_{\hat{\xi}^*}(\bar{F}^i)(\hat{\xi}^*, X^H) = -\frac{1}{2} \sum_{j=1}^{4n+3} \hat{\Omega}(X^H, E_j^H) \eta^i(E_j) \circ \pi.$$

But $\eta^i(E_j) = 0$ if $E_j \neq \xi_i$ and $\eta^i(\xi_i) = 1$, hence we have i). Part ii) is deduced in a similar way using Lemma 4.3 ii) and taking $\hat{\xi}^* \bar{F}^i(X^H, Y^H) = \hat{\xi}^*(F^i(X, Y) \circ \pi) = 0$ into account.

To show iii) we use Lemma 4.2 and Lemma 4.3 ii) and iii) to reach

$$\bar{\nabla}_{X^H}(\bar{F}^i)(\hat{\xi}^*, Y^H) = X \eta^i(Y) \circ \pi - \frac{1}{2} \sum_{j=1}^{4n+3} \hat{\Omega}(X^H, E_j^H) F^i(E_j, Y) \circ \pi - \eta^i(\nabla_X Y) \circ \pi.$$

From this equality iii) is immediate. Part iv) can be proved in a similar way, taking Lemma 4.2 and Lemma 4.3 into account. \square

In the following results we relate the almost hyper-Hermitian structure of \bar{M} with the almost contact 3-structure on M . First, from Lemma 4.4 one easily gets the following result.

Theorem 4.5. *Two of the following conditions imply the remaining one:*

- (a) *The almost hyper-Hermitian structure on \bar{M} is hyper-Kähler.*
- (b) *The almost contact metric 3-structure of M is hypercosymplectic.*
- (c) *The curvature form $\hat{\Omega}$ of the connection one-form ω vanishes.*

As a direct consequence of the fact proved in [21, Proposition 3.1, p. 178] the following theorem can be obtained.

Theorem 4.6. *Two of the following conditions imply the remaining one:*

- (a) The almost hyper-Hermitian structure of \overline{M} is hypercomplex.
- (b) The almost contact metric 3-structure of M is hypernormal.
- (c) The curvature form $\hat{\Omega}$ of the connection one-form ω is the pullback by π of a two-form on M belonging to $\mathfrak{sp}(n)$, the Lie algebra of $\mathrm{Sp}(n) \times I_3$, i.e.,

$$\hat{\Omega}(\varphi_i X, \varphi_i Y) = \hat{\Omega}(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$ and $i = 1, 2, 3$.

Theorem 4.7. *Two of the following conditions imply the remaining one:*

- (a) The almost hyper-Hermitian structure of \overline{M} is locally conformal hyper-Kähler.
- (b) The almost contact metric 3-structure of M is α -Sasakian.
- (c) The curvature form $\hat{\Omega}$ of the connection one-form ω vanishes.

Proof. First, let us suppose that M has an α -Sasakian 3-structure and the curvature form $\hat{\Omega} = 0$. From Lemma 4.2 we have $d\overline{F}^i = -a\omega \wedge \overline{F}^i$. Hence we have that \overline{M} is locally conformal hyper-Kähler.

Now, we suppose a) and c), i.e., $\hat{\Omega} = 0$ and $d\overline{F}^i = \overline{\alpha} \wedge \overline{F}^i$, $i = 1, 2, 3$. From the argument given in [6], it is immediate that \overline{M} is hypercomplex. By Theorem 4.6 it follows that M has a hypernormal 3-structure. On the other hand, for all $p \in \overline{M}$ there is a local section through p , $\sigma: U \rightarrow \pi^{-1}(U)$ and $T_p\overline{M} = T_p\sigma(U) \oplus T_p\pi^{-1}(x)$, where $x = \pi(p)$. On $\sigma(U)$, the one-form $\overline{\alpha}$ can be expressed as $\overline{\alpha} = \alpha - f\omega$, where α is $\overline{\alpha}$ restricted to $T_q\sigma(U)$ and $f(q) = -\overline{\alpha}_q(\hat{\xi}_q^*)$, for all $q \in \sigma(U)$. From Lemma 4.2, we have

$$\alpha \wedge \pi^* F^i - \omega \wedge (\alpha \wedge \pi^* \eta^i + f\pi^* F^i) = \pi^* dF^i - \omega \wedge \pi^* d\eta^i.$$

Therefore, $dF^i = \sigma^* \alpha \wedge F^i$, $d\eta^i = \sigma^* \alpha \wedge \eta^i + \sigma^* f F^i$. In [16, Theorem 2.4, p. 192] it is proved that for a normal almost contact structure satisfying these equations we have $\sigma^* \alpha = \delta \eta^i \eta^i$, $i = 1, 2, 3$. Hence $\sigma^* \alpha = 0$ and $\sigma^* f$ is locally constant. In conclusion, M has an α -Sasakian 3-structure.

Finally, from a) and b), taking Lemma 4.2 into account, we have

$$\alpha \wedge \pi^* F^i - \omega \wedge (\alpha \wedge \pi^* \eta^i + f\pi^* F^i) = \hat{\Omega} \wedge \pi^* \eta^i - a\omega \wedge \pi^* F^i,$$

where $\overline{\alpha} = \alpha - f\omega$ on $\sigma(U)$ as before. Then

$$\sigma^* \alpha \wedge \pi^* \eta^i + (a - \sigma^* f)\pi^* F^i = 0, \quad \sigma^* \alpha \wedge F^i - \hat{\Omega} \wedge \pi^* \eta^i = 0.$$

Hence, $\sigma^* \alpha = 0$, $a = \sigma^* f$ and $\hat{\Omega} \wedge \eta^i = 0$. If X_p and Y_p are vectors orthogonal to ξ_{ip} , then $0 = \hat{\Omega}_p \wedge \eta_p^i(X_p, Y_p, \xi_{ip}) = \hat{\Omega}_p(X_p, Y_p)$. On the other hand, from Theorem 4.6 we have $\hat{\Omega}(\varphi_i X, \varphi_i Y) = \hat{\Omega}(X, Y)$. Then $\hat{\Omega}(\xi_i, X) = 0$. Therefore $\hat{\Omega}(X, Y) = \hat{\Omega}(X - \eta^i(X)\xi_i, \varphi_i Y - \eta^i(Y)\xi_i) = 0$, noting that $X - \eta^i(X)\xi_i$ is orthogonal to ξ_i . Hence $\hat{\Omega}$ vanishes. \square

Now the covariant derivative of $\bar{\Omega}$ is expressed in terms of Ω , Ψ and ω .

Lemma 4.8. For $X, Y, Z, U, V \in \mathfrak{X}(M)$ we have

$$(4.3) \quad \bar{\nabla}_{\hat{\xi}^*}(\bar{\Omega})(\hat{\xi}^*, X^H, Y^H, Z^H) = \mathcal{A}(C_{13} \hat{\Omega} \otimes \Psi)(X, Y, Z) \circ \pi,$$

$$(4.4) \quad \bar{\nabla}_{\hat{\xi}^*}(\bar{\Omega})(X^H, Y^H, Z^H, U^H) = \frac{1}{2} \mathcal{A}(C_{13} \hat{\Omega} \otimes \Omega)(X, Y, Z, U) \circ \pi,$$

$$(4.5) \quad \bar{\nabla}_{X^H}(\bar{\Omega})(\hat{\xi}^*, Y^H, Z^H, U^H) = 2\nabla_X(\Psi)(Y, Z, U) \circ \pi \\ + \frac{1}{2} C_{13} \hat{\Omega} \otimes \Omega(X, Y, Z, U) \circ \pi,$$

$$(4.6) \quad \bar{\nabla}_{X^H}(\bar{\Omega})(Y^H, Z^H, U^H, V^H) = \nabla_X(\Omega)(Y, Z, U, V) \circ \pi \\ + (X \diamond \hat{\Omega}) \wedge \Psi(Y, Z, U) \circ \pi,$$

where \mathcal{A} denotes the skew-symmetrization of a tensor, C the metric contraction, $\hat{\Omega}$ the curvature form of ω and \diamond the interior product.

P r o o f. It follows by a straightforward computation, taking Lemma 4.2 and Lemma 4.3 into account. \square

For the exterior derivative and the coderivative of $\bar{\Omega}$ we have the following expressions.

Lemma 4.9. For all $X, Y, Z \in \mathfrak{X}(M)$ we have

- i) $d\bar{\Omega} = \pi^* d\Omega + 2\hat{\Omega} \wedge \pi^* \Psi - 2\omega \wedge \pi^* d\Psi$;
- ii) $\delta\bar{\Omega}(X^H, Y^H, Z^H) = \delta\Omega(X, Y, Z) \circ \pi$;
- iii) $\delta\bar{\Omega}(\hat{\xi}^*, X^H, Y^H) = -2\delta\Psi(X, Y) \circ \pi + \frac{1}{2} C_{12} \circ C_{13}(\hat{\Omega} \otimes \Omega)(X, Y) \circ \pi$, where C denotes the metric contraction.

P r o o f. The expression i) is a direct consequence of Lemma 4.2. Let us prove ii): if $\{E_1, E_2, \dots, E_{4n+3}\}$ is an adapted frame of M , then $\{E_1^H, E_2^H, \dots, E_{4n+3}^H, \hat{\xi}^*\}$ is an adapted frame of \bar{M} , hence we have

$$\delta\bar{\Omega}(X^H, Y^H, Z^H) = -\bar{\nabla}_{\hat{\xi}^*} \bar{\Omega}(\hat{\xi}^*, X^H, Y^H, Z^H) - \sum_{j=1}^{4n+3} \bar{\nabla}_{E_j^H} \bar{\Omega}(E_j^H, X^H, Y^H, Z^H).$$

From (4.3) and (4.6) we get ii). To show iii) we have

$$\delta\bar{\Omega}(\hat{\xi}^*, X^H, Y^H) = -\bar{\nabla}_{\hat{\xi}^*} \bar{\Omega}(\hat{\xi}^*, \hat{\xi}^*, X^H, Y^H) - \sum_{j=1}^{4n+3} \bar{\nabla}_{E_j^H} \bar{\Omega}(E_j^H, \hat{\xi}^*, X^H, Y^H),$$

and using (4.6) we get

$$\begin{aligned} \delta\bar{\Omega}(\hat{\xi}^*, X^H, Y^H) &= 2 \sum_{j=1}^{4n+3} \nabla_{E_j}(\Psi)(E_j, X, Y) \\ &+ \frac{1}{2} \sum_{j,k=1}^{4n+3} \hat{\Omega}(E_j^H, E_k^H)\Omega(E_j, E_k, X, Y) \circ \pi, \end{aligned}$$

then iii) follows. □

Next, we give a relation between a quaternionic semi-Kähler structure and a semi-cosymplectic 3-structure.

Theorem 4.10. *Two of the following conditions imply the remaining one:*

- (a) *The almost hyper-Hermitian structure on \bar{M} is quaternionic semi-Kähler.*
- (b) *The almost contact metric 3-structure on M is semi-cosymplectic.*
- (c) *The curvature form $\hat{\Omega}$ of ω vanishes.*

Proof. It follows directly from Lemma 4.9 and Lemma 3.1. □

Theorem 4.11. *Two of the following conditions imply the remaining one:*

- (a) *The almost hyper-Hermitian structure on \bar{M} is quaternionic Kähler.*
- (b) *The almost contact metric 3-structure on M is cosymplectic.*
- (c) *The curvature form $\hat{\Omega}$ of ω vanishes.*

Proof. It follows from (4.3), (4.4), (4.5), (4.6) and Theorem 4.10. □

To study the quaternionic almost Kähler case, we need before to prove the following statement.

Proposition 4.12. *Let M be a $(4n + 3)$ -manifold ($n \geq 1$). Then every almost cosymplectic 3-structure on M is semi-cosymplectic.*

Proof. We consider the product manifold $M \times \mathbb{R}$ where \mathbb{R} is the set of real numbers. We take the projection map ω on the second factor as the connection form. We consider the almost hyper-Hermitian structure on $M \times \mathbb{R}$ defined, as in (4.2), from ω and the almost contact 3-structure on M . By Lemma 4.9, the almost hyper-Hermitian structure on $M \times \mathbb{R}$ is quaternionic almost-Kähler. By the argument given in [2], $M \times \mathbb{R}$ is quaternionic semi-Kähler. Using now Theorem 4.10 we deduce that M has a semi-cosymplectic 3-structure. □

Theorem 4.13. *Two of the following conditions imply the remaining one:*

- (a) *The almost hyper-Hermitian structure on \overline{M} is quaternionic almost Kähler.*
- (b) *The almost contact metric 3-structure on M is almost cosymplectic.*
- (c) *The curvature form $\hat{\Omega}$ of ω vanishes.*

Proof. It follows from Lemma 4.9, Theorem 4.10 and Proposition 4.12. \square

Theorem 4.14. *Two of the following conditions imply the remaining one:*

- (a) *The almost hyper-Hermitian structure of \overline{M} is locally conformal quaternionic Kähler.*
- (b) *The almost contact metric 3-structure of M is trans-Sasakian.*
- (c) *The curvature form $\hat{\Omega}$ of ω vanishes.*

Proof. First, let us suppose that M has a trans-Sasakian 3-structure and the curvature form $\hat{\Omega} = 0$. From Lemma 4.2 we have

$$d\overline{F}^i = (\pi^*\alpha - \pi^*a\omega) \wedge \overline{F}^i + (\pi^*\alpha^i - \pi^*r_i\omega) \wedge \overline{F}^j - (\pi^*\alpha^k - \pi^*r_k\omega) \wedge \overline{F}^k.$$

Hence we have that \overline{M} is locally conformal quaternionic Kähler.

Now, we suppose a) and c), i.e., $\hat{\Omega} = 0$ and $d\overline{F}^i = \overline{\alpha} \wedge \overline{F}^i + \overline{\alpha}^i \wedge \overline{F}^j - \overline{\alpha}^k \wedge \overline{F}^k$. For all $p \in \overline{M}$ there is a local section through p , $\sigma: U \rightarrow \pi^{-1}U$ and $T_p\overline{M} = T_p\sigma(U) \oplus T_p\pi^{-1}(x)$, where $x = \pi(p)$. On $\sigma(U)$, the one-forms $\overline{\alpha}$, $\overline{\alpha}^i$ can be expressed as $\overline{\alpha} = \alpha - f\omega$, $\overline{\alpha}^i = \alpha^i - r_i\omega$ where α and α^i are $\overline{\alpha}$ and $\overline{\alpha}^i$ restricted to $T_p\sigma(U)$ and $f(q) = -\overline{\alpha}_q(\hat{\xi}_q^*)$, $r_i(q) = -\overline{\alpha}_q^i(\hat{\xi}_q^*)$ for all $q \in \sigma(U)$. From Lemma 4.2, we have

$$\begin{aligned} \pi^*dF^i - \omega \wedge d\eta^i &= \alpha \wedge \pi^*F^i + \alpha^i \wedge \pi^*F^j - \alpha^k \wedge \pi^*F^k \\ &\quad - f\omega \wedge \pi^*F^i - r_i\omega \wedge \pi^*F^j - r_k\omega \wedge \pi^*F^k \\ &\quad - \omega \wedge \alpha \wedge \pi^*\eta^i - \omega \wedge \alpha^i \wedge \pi^*\eta^j - \omega \wedge \alpha^k \wedge \pi^*\eta^k. \end{aligned}$$

Therefore

$$\begin{aligned} dF^i &= \sigma^*\alpha \wedge F^i + \sigma^*\alpha^i \wedge F^j - \sigma^*\alpha^k \wedge F^k, \\ d\eta^i &= \sigma^*f F^i + \sigma^*r_i F^j - \sigma^*r_k F^k + \sigma^*\alpha \wedge \eta^i + \sigma^*\alpha^i \wedge \eta^j - \sigma^*\alpha^k \wedge \eta^k. \end{aligned}$$

Hence M has a trans-Sasakian 3-structure.

Finally, from a) and b), taking Lemma 4.2 into account, we have

$$\alpha \wedge \pi^*\Omega - f\omega \wedge \pi^*\Omega - 2\omega \wedge \alpha \wedge \pi^*\Psi = \pi^*\beta \wedge \pi^*\Omega + \hat{\Omega} \wedge \pi^*\Psi - 2\omega \wedge \pi^*\beta \wedge \pi^*\Psi - a\omega \wedge \pi^*\Omega$$

where $\overline{\alpha} = \alpha - f\omega$ on $\sigma(U)$ as before and β , a are the one-form and the function given in the definition of a trans-Sasakian 3-structure. Then

$$(\beta - \sigma^*\alpha) \wedge \Omega + \hat{\Omega} \wedge \Psi = 0, \quad (\beta - \sigma^*\alpha) \wedge \Psi + (a - \sigma^*f)\Omega = 0.$$

If $a - \sigma^* f = 0$, then taking Lemma 3.1 into account, we have $\beta - \sigma^* \alpha = 0$. Then $\hat{\Omega} \wedge \Psi = 0$ and using again Lemma 3.1 we have $\hat{\Omega} = 0$.

If $a - \sigma^* f \neq 0$, then

$$0 = -\frac{1}{a - \sigma^* f} (\beta - \sigma^* \alpha) \wedge (\beta - \sigma^* \alpha) \wedge \Psi + \hat{\Omega} \wedge \Psi = \hat{\Omega} \wedge \Psi.$$

Now, taking Lemma 3.1 into account, we have $\hat{\Omega} = 0$. □

Corollary 4.15. *Let M be a connected $(4n + 3)$ -manifold ($n \geq 1$) with a trans-Sasakian 3-structure, i.e.,*

$$\begin{aligned} dF^i &= \alpha \wedge F^i + \alpha^i \wedge F^j - \alpha^k \wedge F^k, \\ d\eta^i &= aF^i + r_i F^j - r^k F^k + \alpha \wedge \eta^i + \alpha^i \wedge \eta^j - \alpha^k \wedge \eta^k \end{aligned}$$

for all (i, j, k) cyclic permutations of $(1, 2, 3)$. Then α is closed and a is constant.

Proof. We consider $M \times \mathbb{R}$ with the projection map ω on the second factor as a connection form. On $M \times \mathbb{R}$ we have a quaternion-Hermitian structure defined as in (4.2). By Theorem 4.14, $M \times \mathbb{R}$ is locally conformal quaternionic Kähler and $d\bar{\Omega} = 2\bar{\alpha} \wedge \bar{\Omega}$, where $\bar{\alpha} = \pi^* \alpha - a\omega$. Then $d\alpha = 0$ and $da = 0$. □

5. EXAMPLES

I. Trivial principal fibre bundles over 3-Sasakian manifolds

In [4] it is shown that any 3-Sasakian homogeneous space is one of the following homogeneous spaces:

$$\begin{aligned} \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)} \cong S^{4n-1}, \quad \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1) \times \mathbb{Z}_2} \cong \mathbb{R}P^{4n-1}, \quad \frac{\mathrm{SU}(m)}{\mathrm{S}(\mathrm{U}(m-2) \times \mathrm{U}(1))}, \\ \frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4) \times \mathrm{Sp}(1)}, \quad \frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)}, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)}, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)}, \quad \frac{\mathrm{E}_8}{\mathrm{E}(7)}, \end{aligned}$$

where $n \geq 1$, $\mathrm{Sp}(0)$ is the identity group, $m \geq 3$ and $k \geq 7$.

By Theorem 4.7, we have the following locally conformal hyper-Kähler manifolds:

$$\begin{aligned} S^{4n-1} \times S^1, \quad \mathbb{R}P^{4n-1} \times S^1, \quad \frac{\mathrm{SU}(m)}{\mathrm{S}(\mathrm{U}(m-2) \times \mathrm{U}(1))} \times S^1, \quad \frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4) \times \mathrm{Sp}(1)} \times S^1, \\ \frac{\mathrm{G}_2}{\mathrm{Sp}(1)} \times S^1, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)} \times S^1, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)} \times S^1, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)} \times S^1, \quad \frac{\mathrm{E}_8}{\mathrm{E}(7)} \times S^1, \end{aligned}$$

where $n > 1$, $m > 3$ and $k \geq 7$. This is an alternative way of obtaining these examples given in [17].

II. Nontrivial principal fibre bundles over a $(4n + 3)$ -dimensional torus

Let us recall the following well known theorem about classification of principal circle bundles.

Theorem 5.1. ([9, p. 35]) *There is a one-to-one correspondence between equivalence classes of principal circle bundles over a manifold M and the cohomology group $H^2(M, \mathbb{Z})$. Furthermore, given an integral closed two-form $\hat{\Omega}$ on M , there is a principal circle bundle $\pi: \overline{M} \rightarrow M$ with a connection form ω such that $\hat{\Omega}$ is the curvature of ω ($\pi^*(\hat{\Omega}) = d\omega$).*

We consider a $(4n + 3)$ -dimensional torus \mathbb{T}^{4n+3} ($n \geq 1$). Let $\{\alpha_1, \alpha_2, \dots, \alpha_{4n+3}\}$ a basis for one-forms such that each α_i is integral and closed. On \mathbb{T}^{4n+3} we consider the metric tensor field given by $\langle \rangle = \sum_{l=1}^{4n+3} \alpha_l \otimes \alpha_l$ and the almost contact metric 3-structure consisting of

- the $(1, 1)$ tensor fields

$$(5.1) \quad \varphi_i = \sum_{l=1}^n \{ E_{in+l} \otimes \alpha_l - E_l \otimes \alpha_{i+l} + E_{kn+l} \otimes \alpha_{jn+l} - E_{jn+l} \otimes \alpha_{kn+l} + E_{4n+k} \otimes \alpha_{4n+j} - E_{4n+j} \otimes \alpha_{4n+k} \},$$

where $\{E_1, \dots, E_{4n+3}\}$ is the orthonormal frame dual of $\{\alpha_1, \dots, \alpha_{4n+3}\}$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$;

- the one-forms $\eta^1 = \alpha_{4n+1}$, $\eta^2 = \alpha_{4n+2}$ and $\eta^3 = \alpha_{4n+3}$;
- the vector fields $\xi_1 = E_{4n+1}$, $\xi_2 = E_{4n+2}$ and $\xi_3 = E_{4n+3}$.

Since each α_i is closed, it can be checked that $(\varphi_i, \eta^i, \xi_i, \langle \rangle)$ is a hypercosymplectic 3-structure. Hence we can also claim that \mathbb{T}^{4n+3} has a hypernormal 3-structure.

By Theorem 5.1 we have a nontrivial principal circle bundle $\pi: \overline{M} \rightarrow \mathbb{T}^{4n+3}$ corresponding to $[\hat{\Omega}] \in H^2(\mathbb{T}^{4n+3}, \mathbb{Z})$, where

$$(5.2) \quad \hat{\Omega} = \mathfrak{S}_{ijk} \sum_{l=1}^n \{ \alpha_l \wedge \alpha_{in+l} - \alpha_{jn+l} \wedge \alpha_{kn+l} \}$$

and \mathfrak{S} denotes the cyclic sum. There is a connection one-form ω on \overline{M} with curvature $d\omega = \pi^*(\hat{\Omega})$. We will also denote $\pi^*(\hat{\Omega})$ by $\hat{\Omega}$.

We consider on \overline{M} the almost hyper-Hermitian structure $(J_1, J_2, J_3, \langle \rangle_0)$ defined as in (4.2) from the connection form ω and the almost contact 3-structure of \mathbb{T}^{4n+3} .

Theorem 5.2. *On the $(4n + 4)$ -dimensional manifold \overline{M} ($n \geq 1$) there is a hypercomplex structure which is not quaternionic semi-Kähler.*

Proof. Since we have a hypernormal 3-structure on \mathbb{T}^{4n+3} , we only need to check condition c) of Theorem 4.6, i.e.,

$$(5.3) \quad \hat{\Omega}(\varphi_i X, \varphi_i Y) = \hat{\Omega}(X, Y)$$

for $X, Y \in \mathfrak{X}(\mathbb{T}^{4n+3})$ and $i = 1, 2, 3$. Note that conditions (5.3) are bilinear, so we only have to check those conditions for any pair (E_r, E_s) of the adapted frame $\{E_1, \dots, E_{4n+3}\}$.

From the expression (5.2) of $\hat{\Omega}$, taking (5.1) into account, we have

$$(5.4) \quad 0 = \hat{\Omega}(E_{4n+j}, E_r) = \hat{\Omega}(\varphi_i E_{4n+j}, \varphi_i E_r),$$

where $i, j = 1, 2, 3$ and $r = 1, 2, \dots, 4n + 3$.

From now on $r, s = 1, 2, \dots, n, r \neq s$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$. From (5.2), taking (5.1) into account, we get

$$(5.5) \quad \hat{\Omega}(E_r, E_s) = \hat{\Omega}(\varphi_i E_r, \varphi_i E_s) = \hat{\Omega}(E_{in+r}, E_{in+s}) = 0.$$

Similarly we have

$$(5.6) \quad \begin{aligned} \hat{\Omega}(E_r, E_{jn+s}) &= \hat{\Omega}(\varphi_i E_r, \varphi_i E_{jn+s}) = \hat{\Omega}(E_{jn+r}, E_{kn+s}) \\ &= \hat{\Omega}(\varphi_i E_{jn+r}, \varphi_i E_{kn+s}) = 0. \end{aligned}$$

Now using again expression (5.2) of $\hat{\Omega}$ and taking (5.1) into account, we have

$$(5.7) \quad \hat{\Omega}(E_r, E_{in+r}) = \hat{\Omega}(\varphi_i E_r, \varphi_i E_{in+r}) = 1.$$

In a similar way we have

$$(5.8) \quad \hat{\Omega}(E_r, E_{jn+r}) = \hat{\Omega}(\varphi_i E_r, \varphi_i E_{jn+r}) = -1,$$

$$(5.9) \quad \hat{\Omega}(E_r, E_{kn+r}) = \hat{\Omega}(\varphi_i E_r, \varphi_i E_{kn+r}) = 1,$$

$$(5.10) \quad \hat{\Omega}(E_{jn+r}^H, E_{kn+r}) = \hat{\Omega}(\varphi_i E_{jn+r}, \varphi_i E_{kn+r}) = -1.$$

From (5.4), (5.5), (5.6), (5.7), (5.8), (5.9) and (5.10) we can claim that conditions (5.3) are satisfied. Then by Theorem 4.6 the almost hyper-Hermitian structure on \overline{M} is hypercomplex. If \overline{M} were an quaternionic semi-Kähler manifold then by Theorem 4.10, $\hat{\Omega}$ would vanish, which is a contradiction. \square

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