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THE KURZWEIL CONSTRUCTION OF AN INTEGRAL IN ORDERED SPACES

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Abstract. This paper generalizes the results of papers which deal with the Kurzweil-Henstock construction of an integral in ordered spaces. The definition is given and some limit theorems for the integral of ordered group valued functions defined on a Hausdorff compact topological space T with respect to an ordered group valued measure are proved in this paper.

Keywords: lattice ordered group valued function and measure, Kurzweil-Henstock construction of an integral, limit theorems

MSC 2000: 28B15

Introduction

Let us recall the definition of the Kurzweil integral of a real function.

A function $f\colon \langle a,b\rangle \to \mathbb{R}$ is integrable in the Kurzweil sense if there is $c \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a function $\delta\colon \langle a,b\rangle \to (0,\infty)$ such that

$$\left|\sum_{i=1}^{n} f(t_i)m(E_i) - c\right| < \varepsilon$$

for every partition $D = \{(E_i,t_i),\ i = 1,2,\dots,n\}$, where $E_1, E_2, \dots, E_n$ are nonoverlapping closed intervals with $\bigcup\limits_{i=1}^{n} E_i = \langle a,b\rangle$ and $t_i \in E_i$, $E_i \subset (t_i - \delta(t_i),\ t_i + \delta(t_i))$ for $i = 1,2,\dots,n$.

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We say that δ is a gauge on $\langle a,b\rangle$ and the partition D is δ-fine. The set of all δ-fine partitions we denote by $\mathscr{A}(\delta)$.

When the range X of the function f is only partially ordered, the ε-technique is replaced by the double sequence technique working in the weak σ-distributive vector lattices.

A conditionally σ-complete vector lattice (lattice ordered group) X (that is, every bounded sequence $(a_i)_i \subset X$ has the supremum $\bigvee\limits_i a_i$) is called weakly σ-distributive, if for every bounded double sequence $(a_{ij})_{i,j} \subset X$ such that $a_{ij} \downarrow 0$ ($j \to \infty$, $i = 1,2,\ldots$) we have

$$\bigwedge_{\varphi\in N^N} \bigvee_i a_{i\varphi(i)} = 0.$$

The equality $|x| = x \vee 0 + (-x) \vee 0$ holds for x in a lattice ordered group X.

The definition of the Kurzweil integral of a function $f\colon \langle a,b\rangle \to X$ was introduced and some properties of the integral were proved by Riečan in [7]. A limit theorem for uniformly convergent sequences of Kurzweil integrable functions is proved in [8] and the limit theorem for monotone and with a common regulating sequence convergent sequences is obtained in [11].

The Kurzweil integral of a function $f\colon T \to \mathbb{R}$, where T is a Hausdorff compact topological space was defined in [6]. Now, the gauge is a function $\delta\colon T \to 2^T$, where $\delta(t)$ is a neighbourhood of t. A partition $D = \{(E_i,t_i),\ i = 1,2,\ldots,n\}$ is a δ-fine $\mathscr{P}$-partition of T, if E_i and E_j have no common interior points for $i \neq j$, $\bigcup\limits_{i=1}^{n} E_i = T$, $E_i \subset \delta(t_i)$, $t_i \in \overline{E_i}$ and $E_i \in \mathscr{P}$ for $i = 1,2,\ldots,n$, where $\mathscr{P}$ is a family of Borel subsets of T.

If $\mathscr{U}(T)$ is the set of all neighbourhood gauges, $\mathscr{P}$ is the σ-algebra generated by the family of all compact subsets of T and $\mathscr{A}(\delta/E,\mathscr{P})$ is the set of all δ-fine $\mathscr{P}$-partitions D of $E \in \mathscr{P}$, then $\mathscr{A}(\delta/E,\mathscr{P}) \neq \emptyset$ for every $\delta \in \mathscr{U}(T)$ or $\delta \in \mathscr{U}(\overline{E})$ and every $E \in \mathscr{P}$ (see [9], Lemma 1 and Remark 2). In general we do not need all neighbourhood gauges and all Borel subsets. (Haluška has written about it in [3].)

In the case when $f\colon T \to X$ and $L(X,Y)$ is the set of all linear continuous or regular operators from X to Y (X,Y are some vector lattices), the Kurzweil integral of f with respect to $L(X,Y)$-valued measure can be found in [9] and [3].

The Kurzweil integral of $f\colon T \to X$ with respect to a Y-valued measure was applied by Száz in [10]. Száz supposes that X,Y and Z are normed spaces which are equipped with a bilinear map $(x,y) \mapsto xy$ from $X \times Y$ into Z. Now, the Riesz space will take the place of the normed spaces. We will define the Kurzweil type integral of lattice ordered group valued functions defined on T with respect to a lattice ordered group valued measure.

First we shall list assumptions concerning the range spaces X, Y, Z, the domain T and a given measure $\mu\colon \mathscr{S} \to Y$.

Assumptions 1. X, Y, Z are assumed to be Abelian lattice ordered groups, moreover Z being conditionally σ-complete and weakly σ-distributive. Further, a mapping $b\colon X \times Y \to Z$ is given satisfying the following conditions:

(i) $b(x_1 + x_2, y) = b(x_1, y) + b(x_2, y)$ for every $x_1, x_2 \in X$, $y \in Y$.

(ii) $b(x, y_1 + y_2) = b(x, y_1) + b(x, y_2)$ for every $x \in X$, $y_1, y_2 \in Y$.

(iii) If $x \in X$, $y \in Y$, $x \geqslant 0$, $y \geqslant 0$, then $b(x, y) \geqslant 0$.

(iv) If $x_n \in X$ $(n = 1, 2, \ldots)$, $y \in Y$, $y \geqslant 0$ and $x_n \downarrow 0$, then $b(x_n, y) \downarrow 0$.

(v) If $x_n \in X$, $y_n \in Y$, $x_n \geqslant 0$, $y_n \geqslant 0$ $(n = 1, 2, \ldots)$ and $\bigvee\limits_{n=1}^{\infty} x_n$, $\bigvee\limits_{n=1}^{\infty} y_n$ exist, then

$$\bigvee_n b(x_n, y_1) = b\Big(\bigvee_n x_n, y_1\Big), \bigvee_n b(x_1, y_n) = b\Big(x_1, \bigvee_n y_n\Big).$$

In the sequel we will write $x \cdot y$ or xy instead of $b(x, y)$.

Examples. 1. Let X, Z be Riesz spaces, $Y = L(X, Z)$ the space of linear positive mappings from X to Z. Then the mapping $b\colon X \times Y \to Z$ defined by $b(x, y) = y(x)$ is a biadditive map.

2. Let X be a Riesz space, $Y = \mathbb{R}$, $Z = X$, $b(x, y) = x \cdot y$ (scalar multiplication).

3. Let Y be a Riesz space, $X = \mathbb{R}$, $Z = Y$, $b(x, y) = x \cdot y$ (scalar multiplication).

Assumptions 2. We consider a Hausdorff compact topological space T, a subfamily $\mathscr{P}$ of Borel subsets of T and a subfamily $\mathscr{U}(T)$ of neighbourhood gauges η on T such that $\mathscr{A}(\eta/E, \mathscr{P}) \neq \emptyset$ for every $\eta \in \mathscr{U}(T)$ and every $E \in \mathscr{P}$. Finally a measure $\mu\colon \mathscr{S} \to Y$ is given, i.e., such a mapping that the following conditions are satisfied:

(i) $\mathscr{S}$ is the σ-algebra of Borel subsets of T, i.e., the σ-algebra generated by the family of all compact subsets of T.

(ii) $\mu(E) \geqslant 0$ for every $E \in \mathscr{S}$.

(iii) $\mu\Big(\bigcup\limits_{n=1}^{k} E_n\Big) = \sum\limits_{n=1}^{k} \mu(E_n)$ whenever $E_1, \ldots, E_k \in \mathscr{S}$, E_i and E_j have no common interior points $(i \neq j)$.

(iv) μ is regular in the following sense: For every $E \in \mathscr{S}$ there exists a bounded sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \downarrow 0$ $(k \to \infty, n = 1, 2, \ldots)$ such that for every $\varphi\colon N \to N$ there exist a compact set F and an open set U such that $F \subset E \subset U$ and

$$\mu(U \backslash F) < \bigvee_i a_{i\varphi(i)}.$$

Definition 3. Let $f\colon T \to X$ be any mapping, $\mu\colon \mathscr{S} \to Y$ a regular measure, $D = \{(E_1, t_1) \dots, (E_n, t_n)\}$ a partition, $E_1, \dots, E_n \in \mathscr{P}$. Then we define

$$S(f, D) = \sum_{i=1}^{n} f(t_i)\mu(E_i).$$

The function f is integrable (with respect to μ), if there exists $z \in Z$ and a bounded double sequence $(a_{nk})_{n,k} \subset Z$, $a_{nk} \downarrow 0$ $(k \to \infty, n = 1, 2, \dots)$ such that for every $\varphi\colon N \to N$ there exists $\eta \in \mathscr{U}(T)$ such that

$$|S(f, D) - z| < \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

for any $D \in \mathscr{A}(\eta)$ $(= \mathscr{A}(\eta, \mathscr{P}))$.

The element z from Definition 3 is determined uniquely (for the proof see [9], Lemma 6) and z will be denoted by $\int f \, \mathrm{d}\mu$. It is no problem to prove the following elementary properties of the integral (see [9], Theorem 7, Theorem 8):

(i) If $f, g\colon T \to X$ are integrable, then $f + g$, $f - g$ are integrable and

$$\int (f + g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu, \int (f - g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu - \int g \, \mathrm{d}\mu.$$

(ii) If $f\colon T \to X$ is integrable and $f(t) \geqslant 0$ for every $t \in T$, then $\int f \, \mathrm{d}\mu \geqslant 0$.

Definition 4. A mapping $f\colon T \to X$ is integrable on a set $E \in \mathscr{P}$, if there exist $z \in Z$ and a bounded sequence $a_{nk} \downarrow 0$ $(k \to \infty,\ n = 1, 2, \dots)$ and for every $\varphi\colon N \to N$ there exists $\eta \in \mathscr{U}(T)$ such that

$$|S_E(f, D) - z| < \bigvee_{i} a_{i\varphi(i)}$$

whenever $D \in \mathscr{A}(\eta/E)$, where $S_E(f, D) = \sum\limits_{i=1}^{n} f(t_i)\mu(E_i)$. The element z will be denoted by $\int_E f \, \mathrm{d}\mu$.

The proofs of the following propositions are the same as the proof of Lemma 11, Theorem 12 and Theorem 13 of [9]. From now on the space Z is assumed to be conditionally complete (i.e., every bounded subset of Z has the supremum).

Proposition 5. (Cauchy-Bolzano condition.) *A mapping $f\colon T \to X$ is integrable on $E \in \mathscr{P}$ if and only if the following condition is satisfied:*

There exists a bounded sequence $(a_{nk})_{n,k} \subset Z$, $a_{nk} \downarrow 0$ $(k \to \infty, n = 1, 2, \ldots)$ *and for every* $\varphi\colon N \to N$ *there is* $\eta \in \mathscr{U}(T)$ *such that*

$$|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for all $D_1, D_2 \in \mathscr{A}(\eta/E)$.

Proposition 6. *If* $E, F, G \in \mathscr{P}$, $E = F \cup G$, F *and* G *have no common interior points and* $f\colon T \to X$ *is integrable on* E, *then* f *is integrable on both* F *and* G, *and*

$$\int_E f \,\mathrm{d}\mu = \int_F f \,\mathrm{d}\mu + \int_G f \,\mathrm{d}\mu.$$

Proposition 7. *If* $f\colon T \to X$ *is a simple measurable function* $f = \sum\limits_{i=1}^{n} \chi_{E_i} x_i$, $E_i \in \mathscr{S}$ $(i = 1, 2, \ldots, n)$, $E_i \cap E_j = \emptyset$ $(i \neq j)$, *then* f *is integrable and*

$$\int f \,\mathrm{d}\mu = \sum_{i=1}^{n} x_i \mu(E_i).$$

Limit theorems

Theorem 8. (Henstock lemma.) *Let* $g\colon T \to X$ *be an integrable function. Let* $(a_{ij})_{i,j}$ *be such a bounded sequence with* $a_{ij} \downarrow 0$ $(j \to \infty,\ i = 1, 2, \ldots)$ *that for every* $\varphi\colon N \to N$ *there exists* $\eta \in \mathscr{U}(T)$ *such that*

$$\left| \int g \,\mathrm{d}\mu - S(g, D) \right| < \bigvee_i a_{i\varphi(i)}$$

for every $D \in \mathscr{A}(\eta)$. *Then for every* $D \in \mathscr{A}(\eta)$, $D = \{(E_i, t_i), i = 1, 2, \ldots, n\}$ *and every* $\alpha \neq \emptyset$, $\alpha \subset \{1, 2, \ldots, n\}$ *we have*

$$\left| \sum_{i \in \alpha} \int_{E_i} g \,\mathrm{d}\mu - \sum_{i \in \alpha} g(t_i)\mu(E_i) \right| \leqslant \bigvee_i a_{i\varphi(i)}.$$

P r o o f. It is the same as the proof of Lemma 2 of [11]. □

Definition 9. We say that $f_n \to f$ converges with a common regulating sequence (w.c.r.s.), if there exists a bounded $(a_{ij})_{i,j}$ with $a_{ij} \downarrow 0 (j \to \infty, i = 1, 2, \ldots)$ such that for every $\varphi\colon N \to N$ and every $t \in T$ there exists $p = p(t)$ such that

$$|f_n(t) - f(t)| < \bigvee_i a_{i\varphi(i)}$$

for any $n \geqslant p$.

Theorem 10. *Let $(f_n)_n$ be a sequence of integrable functions. Let one of the following assumptions (A or B) be satisfied:*

(A) *The sequence $(f_n)_n$ has uniformly regulated integrals, i.e., there exists a triple sequence (a_{nij}) satisfying the following properties:*

(i) *$(a_{nij})_{i,j}$ is bounded for every n and $a_{nij} \downarrow 0$ $(j \to \infty)$.*

(ii) *$\sum\limits_{n=1}^{m} \bigvee\limits_{i=1}^{\infty} a_{ni\varphi(i+n+1)}$ is bounded for every $\varphi\colon N \to N$.*

(iii) *For every $\varphi\colon N \to N$ and every n there is $\eta_n \in \mathscr{U}(T)$ such that*

$$\left|\int f_n \,\mathrm{d}\mu - S(f_n, D)\right| < \bigvee_i a_{ni\varphi(i+n+1)}$$

for every $D \in \mathscr{A}(\eta_n)$.

(B) *There is $a \in Z$ such that $|S(f_k, D) - \int f_k \,\mathrm{d}\mu| \leqslant a$ for every $k \in N$ and every partition D.*

If $f_n \to f$ converges with a common regulating sequence, then there is a bounded sequence $(b_{ij})_{i,j}$ with $b_{ij} \downarrow 0$ such that for every $\varphi\colon N \to N$ there is $\eta \in \mathscr{U}(T)$ such that

$$\left|\int f_n \,\mathrm{d}\mu - S(f_n, D)\right| < \bigvee_i b_{i\varphi(i)} + \sum_{m=l}^{n-1} \left|\int_{F_m} (f_m - f_n) \,\mathrm{d}\mu\right|$$

for every $D = \{(E_k, t_k),\ k = 1, 2, \ldots, n\} \in \mathscr{A}(\eta)$, every $l \in N$ and every $n \in N$, $n > l$, where $F_m = \bigcup\limits_{p(t_k)=m} E_k$.

P r o o f. By the w.c.r.s. convergence there is a bounded sequence $(a_{ij})_{i,j}$ with $a_{ij} \downarrow 0$ $(j \to \infty,\ i = 1, 2, \ldots)$ such that for every $\varphi\colon N \to N$ and every $t \in T$ there is $p(t) \in N$ such that

$$|f_n(t) - f_m(t)| < \bigvee_i a_{i\varphi(i)}$$

for every $n, m \geqslant p(t)$. Since f_n is integrable, there is $a_{nij} \downarrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ such that for every $\varphi\colon N \to N$ there is $\eta_n \in \mathscr{U}(T)$ such that

$$\left|\int f_n \,\mathrm{d}\mu - S(f_n, D)\right| < \bigvee_i a_{ni\varphi(i+n+1)}$$

for every $D \in \mathscr{A}(\eta_n)$. Put

$$\eta(t) = \eta_1(t) \cap \ldots \cap \eta_{p(t)}(t).$$

Then $\eta \in \mathscr{U}(T)$. Let $D \in \mathscr{A}(\eta)$, $D = \{(E_1, t_1), \ldots, (E_s, t_s)\}$. For an arbitrary $l \in N$, fix $n > l$. By the Henstock lemma (Theorem 8)

$$(*) \qquad \Big| \sum_{p(t_k) \geqslant n} f_n(t_k)\mu(E_k) - \sum_{p(t_k) \geqslant n} \int_{E_k} f_n \, \mathrm{d}\mu \Big| \leqslant \bigvee_i a_{ni\varphi(i+n+1)}.$$

By the same lemma

$$\Big| \sum_{p(t_k) = m} f_m(t_k)\mu(E_k) - \sum_{p(t_k) = m} \int_{E_k} f_m \, \mathrm{d}\mu \Big| \leqslant \bigvee_i a_{mi\varphi(i+m+1)}.$$

Therefore

$$\begin{aligned}
&\Big| \sum_{p(t_k) < n} f_n(t_k)\mu(E_k) - \sum_{p(t_k) < n} \int_{E_k} f_n \, \mathrm{d}\mu \Big| \\
&\quad \leqslant \Big| \sum_{p(t_k) < n} f_n(t_k)\mu(E_k) - \sum_{p(t_k) < n} f_{p(t_k)}(t_k)\mu(E_k) \Big| \\
&\qquad + \sum_{m=l}^{n-1} \Big| \sum_{p(t_k) = m} f_m(t_k)\mu(E_k) - \sum_{p(t_k) = m} \int_{E_k} f_m \, \mathrm{d}\mu \Big| + \sum_{m=l}^{n-1} \Big| \int_{F_m} (f_m - f_n) \, \mathrm{d}\mu \Big| \\
&\quad \leqslant \sum_{p(t_k) < n} |f_n(t_k) - f_{p(t_k)}(t_k)| \mu(E_k) + \sum_{m=1}^{n-1} \bigvee_i a_{mi\varphi(i+m+1)} \\
&\qquad + \sum_{m=l}^{n-1} \Big| \int_{F_m} (f_m - f_n) \, \mathrm{d}\mu \Big| \\
(**) &\quad \leqslant \bigvee_i a_{i\varphi(i)}\mu(T) + \sum_{m=1}^{n-1} \bigvee_i a_{mi\varphi(i+m+1)} + \sum_{m=l}^{n-1} \Big| \int_{F_m} (f_m - f_n) \, \mathrm{d}\mu \Big|.
\end{aligned}$$

Put $b_{1ij} = a_{ij}\mu(T)$, $b_{nij} = a_{n-1ij}$ $(n = 2, 3, \ldots)$. By $(*)$ and $(**)$ we obtain

$$\Big| S(f_n, D) - \int f_n \, \mathrm{d}\mu \Big| \leqslant \sum_{m=l}^{n} \bigvee_i b_{mi\varphi(i+m+1)} + \sum_{m=l}^{n-1} \Big| \int_{F_m} (f_m - f_n) \, \mathrm{d}\mu \Big|.$$

Moreover, by the assumptions (A) or (B) there is $c \in Z$ such that

$$\Big| S(f_n, D) - \int f_n \, \mathrm{d}\mu \Big| \leqslant c$$

for every $n \in N$ and every $D \in \mathscr{A}(\eta)$. Now by the Fremlin lemma ([11], Lemma 1), there is a bounded sequence $(b_{ij})_{ij}$ with $b_{ij} \downarrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) such that

$$c \wedge \sum_{m=1}^{\infty} \bigvee_i b_{mi\varphi(i+m+1)} \leqslant \bigvee_i b_{i\varphi(i)}.$$

□

Theorem 11. *Let $(f_n)_n$ be a sequence of integrable functions. Let $(f_n)_n$ have uniformly approximable integrals, i.e. there is a bounded (b_{ij}) with $b_{ij} \downarrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) such that for every $\varphi\colon N \to N$ there is $\eta \in \mathscr{U}(T)$ such that $|\int f_n \,\mathrm{d}\mu - S(f_n, D)| < \bigvee_i b_{i\varphi(i)}$ for every $D \in \mathscr{A}(\eta)$ and $n \in N$. Let $f_n \to f$ with a common regulating sequence. Then f is integrable and $\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$.*

Proof. The proof is similar to the proof of Lemma 3 of [11]. It is proved there that $\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$ with respect to a double sequence, but this convergence implies the o-convergence in weakly σ-distributive groups (see [2], Proposition 1).

□

Theorem 12. (Levi.) *Let $(f_n)_n$ be a sequence of integrable functions, let $(\int f_n \,\mathrm{d}\mu)_n$ be bounded, $f_n \leqslant f_{n+1}$ ($n = 1, 2, \ldots$), $f_n \to f$ with a common regulating sequence. Let $(f_n)_n$ have uniformly regulated integrals (condition* A *in Theorem* 10*). Then f is integrable and*

$$\int f \,\mathrm{d}\mu = \bigvee_{n=1}^{\infty} \int f_n \,\mathrm{d}\mu.$$

Proof. By Theorem 10

$$\left|\int f_n \,\mathrm{d}\mu - S(f_n D)\right| \leqslant \bigvee_i b_{i\varphi(i)} + \left|\int_{F_m} (f_m - f_n) \,\mathrm{d}\mu\right|$$

for $n \geqslant l$, $l \in N$. Since $f_l \leqslant f_m \leqslant f_n$, we obtain

$$\begin{aligned}\left|\sum_{m=l}^{n-1} \int_{F_m} (f_m - f_n) \,\mathrm{d}\mu\right| &\leqslant \sum_{m=l}^{n-1} \int_{F_m} (f_n - f_l) \,\mathrm{d}\mu \leqslant \int (f_n - f_l) \,\mathrm{d}\mu \\ &= \left|\int f_n \,\mathrm{d}\mu - \int f_l \,\mathrm{d}\mu\right|,\end{aligned}$$

where $F_m = \bigcup\limits_{p(t_k)=m} E_k$.

Since $(\int f_n \,\mathrm{d}\mu)_n$ is bounded and increasing and Z is σ-complete (evenly complete), $\bigvee\limits_{n=1}^{\infty} \int f_n \,\mathrm{d}\mu$ exists. Therefore $\int f_n \,\mathrm{d}\mu - \int f_l \,\mathrm{d}\mu \to 0$ as $n, l \to \infty$. It follows that Theorem 11 is applicable. □

Theorem 13. (Levi). *Let $(f_n)_n$ be a sequence of integrable functions, $f_n \leqslant f_{n+1}$ $(n = 1, 2, \ldots)$, $f_n \to f$ with a common regulating sequence. Let f and f_1 be bounded. Then f is integrable and*

$$\int f \,\mathrm{d}\mu = \bigvee_{n=1}^{\infty} \int f_n \,\mathrm{d}\mu.$$

P r o o f. The same as in Theorem 12, only the assumption B in Theorem 11 must be used instead of the assumption A. □

Theorem 14. (Lebesgue.) *Let $(f_n)_n$ be a sequence of integrable functions, h a bounded integrable function such that $|f_n| \leqslant h$ for all n. Let $f_n \to f$ with a common regulating sequence. Then f is integrable and $\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$.*

P r o o f. Again we use Theorem 11. Put (for $j \leqslant k$)$g_{j,k} = \bigvee_{j \leqslant m \leqslant n \leqslant k} |f_n - f_m|$. Then $g_{j,k} \uparrow g_j$ $(k \to \infty)$. By Theorem 13, g_j is integrable and $\int g_j \,\mathrm{d}\mu = \bigvee_k \int g_{j,k} \,\mathrm{d}\mu$. Since $g_j \downarrow 0$, using again Theorem 13 we obtain $\int g_j \,\mathrm{d}\mu \downarrow 0$. Therefore

$$\sum_{m=l}^{n-1} \int_{F_m} (f_m - f_n) \,\mathrm{d}\mu \leqslant \left| \sum_{m=l}^{n-1} \int_{F_m} g_l \,\mathrm{d}\mu \right| \leqslant \int g_l \,\mathrm{d}\mu.$$

Again Theorems 10 and 11 are applicable. □

Theorem 15. (Uniform convergence.) *Let $(f_n)_n$ be a sequence of integrable functions converging uniformly to f, i.e., there exists a sequence $(a_n)_n \subset Z$, $a_n \downarrow 0$ such that $|f_n(t) - f(t)| \leqslant a_n$ for all $n \in N$ and all $t \in T$. Let f be bounded. Then f is integrable and*

$$\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu.$$

P r o o f. Let c be an upper bound of $|f|$. Then $|f_n(t)| \leqslant |f_n(t) - f(t)| + |f(t)| \leqslant a_1 + c$. Moreover,

$$|S(f_n, D)| \leqslant \sum_k |f(t_k)| \mu(E_k) \leqslant c\mu(T).$$

Therefore Theorems 10 and 11 are applicable. □

References

[1] *Boccuto, A.*: Riesz spaces, integration and sandwich theorems. Tatra Mountains Math. Publ. *3* (1993), 213–230.
[2] *Duchoň, M. – Riečan, B.*: On the Kurzweil-Stieltjes integral in ordered spaces. Tatra Mountains Math. Publ *8* (1996), 133–141.
[3] *Haluška, J.*: On integration in complete vector lattices. Tatra Mountains Math. Publ. *3* (1993), 201–212.
[4] *Henstock, R.*: The General Theory of Integration. Oxford, 1991.
[5] *Kurzweil, J.*: Nicht absolut konvergente Integrale. Teubner Leipzig, 1980.
[6] *Riečan, B.*: On the Kurzweil integral in compact topological spaces. Rad. Mat. *2* (1986), 151–163.
[7] *Riečan, B.*: On the Kurzweil integral for functions with values in ordered spaces I. Acta Math. Univ. Comeniana *56–57* (1990), 75–83.
[8] *Riečan, B. – Vrábelová, M.*: On the Kurzweil integral for functions with values in ordered spaces II. Math. Slov. *43* (1993), 471–475.
[9] *Riečan, B. – Vrábelová, M.*: On integration with respect to operator valued measures in Riesz spaces. Tatra Mountains Math. Publ. *2* (1993), 149–165.
[10] *Száz, A.*: The fundamental theorem of calculus in an abstract setting. Tatra Mountains Math. Publ. *2* (1993), 167–174.
[11] *Vrábelová, M. – Riečan, B.*: On the Kurzweil integral for functions with values in ordered spaces III. Tatra Mountains Math. Publ *8* (1996), 93–100.

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