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ON THE DEFECT SPECTRUM OF AN EXTENSION OF A BANACH SPACE OPERATOR

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Abstract. Let T be an operator acting on a Banach space X. We show that between extensions of T to some Banach space $Y \supset X$ which do not increase the defect spectrum (or the spectrum) it is possible to find an extension with the minimal possible defect spectrum.

Let X be a Banach space. Denote by $B(X)$ the algebra of all bounded operators in X. For $T \in B(X)$ denote by $R(T)$ and $N(T)$ its range and kernel, respectively. Denote by $\sigma_\pi(T) = \{\lambda \in \mathbb{C}\colon\ T-\lambda$ is not bounded below$\}$ and $\sigma_\delta(T) = \{\lambda \in \mathbb{C}\colon\ T-\lambda$ is not onto$\}$ its approximate point spectrum and its defect spectrum, respectively. Both $\sigma_\pi(T)$ and $\sigma_\delta(T)$ are compact subsets of the complex plane. The approximate point spectrum as well as the defect spectrum contain the boundary of the spectrum. If we consider any extension S of T to a larger Banach space $Y \supset X$ then the spectrum of S contains the approximate point spectrum of T. It was shown in [3] and [5] that the spectrum of an extension can be made the smallest possible, i.e. there exist a Banach space Y and an operator $S \in B(Y)$ such that $T = S|X$ and $\sigma(S) = \sigma_\pi(T)$.

First we study how the defect spectrum of an extension of a Banach space operator can be reduced. The situation is different than in the previous case because for any Banach space operator there is some extension which is onto. But it is natural to consider extensions S of T with the defect spectrum contained in the defect spectrum of T. We describe which subsets of the complex plane are the defect spectrum of some extension S of T under the assumption $\sigma_\delta(S) \subset \sigma_\delta(T)$. We also solve the same problem under another condition $\sigma(S) \subset \sigma(T)$.

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Then we consider a general case. We show for which pairs F_1, F_2 of subsets of the complex plane there is an extension S of T such that $\sigma_\pi(S) = F_1$ and $\sigma_\delta(S) = F_2$.

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Let K be a compact subset of $\mathbb{C}$. We shall denote by $\widehat{K} = \{\lambda \in \mathbb{C}\colon |p(\lambda)| \leqslant \sup\limits_{z\in K} |p(z)|$ for every polynomial $p\}$ the polynomial convex hull of K. It is well known that the polynomial convex hull of K is equal to the complement of the unbounded component of $\mathbb{C} \setminus K$. Thus $\partial\widehat{K} \subset \partial K$.

Lemma 1. *Let K, L be compact subsets of the complex plane $\mathbb{C}$. If $\widehat{K} = \widehat{L}$ then $\partial\widehat{K} \subset L$ and $\partial\widehat{L} \subset K$.*

P r o o f. Assume that there is $\lambda \in \partial\widehat{K} - L$. Find $\varepsilon > 0$ such that the open disc $D(\lambda, \varepsilon)$ is disjoint with L. As $\lambda \in \partial\widehat{K}$, we have that $(\mathbb{C} - \widehat{K}) \cap D(\lambda, \varepsilon) = (\mathbb{C} - \widehat{L}) \cap D(\lambda, \varepsilon)$ is non-empty. Thus $(\mathbb{C} - \widehat{L}) \cup D(\lambda, \varepsilon)$ is open, connected, unbounded set disjoint with L and therefore contained $\mathbb{C} - \widehat{L}$. Hence $\lambda \in \mathbb{C} - \widehat{L} = \mathbb{C} - \widehat{K}$ and this is a contradiction. The second inclusion follows from symmetry. $\square$

Lemma 2. *Let X be a Banach space, let $A \in \mathcal{L}(X)$. Then*

$$\widehat{\sigma}(A) = \widehat{\sigma}_\pi(A) = \widehat{\sigma}_\delta(A).$$

P r o o f. Follows from the fact that $\partial\sigma(A) \subset \sigma_\pi(A) \subset \sigma(A)$ and $\partial\sigma(A) \subset \sigma_\delta(A) \subset \sigma(A)$. $\square$

Lemma 3. *Let Y be a Banach space, let X be a closed subspace of Y. Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, $S|X = T$. Then*
(1) *$\sigma_\delta(S) \subset \widehat{\sigma}_\delta(T)$ implies $\partial\widehat{\sigma}(T) \subset \sigma_\delta(S)$,*
(2) *$\sigma(S) \subset \sigma(T)$ implies $\partial\sigma(T) \subset \sigma_\delta(S)$.*

P r o o f. As T is the restriction of S to X, we have $\sigma_\pi(T) \subset \sigma_\pi(S)$. Using Lemma 2 and the assumption $\sigma_\delta(S) \subset \widehat{\sigma}_\delta(T)$ we obtain

$$\widehat{\sigma}(T) = \widehat{\sigma}_\pi(T) \subset \widehat{\sigma}_\pi(S) = \widehat{\sigma}_\delta(S) \subset \widehat{\sigma}_\delta(T) = \widehat{\sigma}(T).$$

Thus we have shown that $\widehat{\sigma}(T) = \widehat{\sigma}_\delta(S)$ and we can apply Lemma 1.

(2) Let $\lambda \in \partial\sigma(T)$. The boundary of the spectrum of T is contained in the approximate point spectrum, so that $\lambda \in \sigma_\pi(T) \subset \sigma_\pi(S) \subset \sigma(S)$. There exist complex numbers λ_n $(n \in \mathbb{N})$ converging to λ and $\lambda_n \in \mathbb{C} - \sigma(T) \subset \mathbb{C} - \sigma(S)$. Thus $\lambda \in \partial\sigma(S) \subset \sigma_\delta(S)$.

We shall need the following lemma (cf. [1]). $\square$

Lemma 4. *Let Ω be a bounded, non-empty, open subset of $\mathbb{C}$. Then there exist a Hilbert space H and an operator $S_\Omega \in \mathcal{L}(H)$ such that*

(1) $\sigma(S_\Omega) = \overline{\Omega}$;

(2) $\partial\overline{\Omega} \subset \sigma_\delta \subset \partial\Omega$;

(3) *there exists $h_0 \in H$ such that $(g, h_0) \neq 0$ for any $\lambda \in \Omega$ and any non-zero $g \in N(S_\Omega - \lambda)$. Moreover $N(S_\Omega - \lambda) \neq \{0\}$.*

Proof. Put $\Omega^* = \{z \in \mathbb{C}\colon\ \bar{z} \in \Omega\}$. Denote by H the Bergman space of all square-integrable functions in Ω^* with respect to the planar Lebesgue measure. For $f, g \in H$ the scalar product is defined by

$$(f, g) = \frac{1}{\lambda(\Omega^*)} \int_{\Omega^*} f(z)\overline{g(z)}\,\mathrm{d}z\,\mathrm{d}\bar{z},$$

where $\lambda(\Omega^*)$ is the Lebesgue measure of the set Ω^*. This scalar product makes H into a Hilbert space. Define the operator M on the Hilbert space H by

$$(Mf)(z) = zf(z).$$

As for $f \in H$

$$\|Mf\|^2 = \frac{1}{\lambda(\Omega^*)} \int_{\Omega^*} |zf(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z} \leqslant \Big(\sup_{z\in\Omega^*} |z|\Big)^2 \|f\|^2,$$

we have $M \in \mathcal{L}(H)$. Fix $\lambda \in \Omega^*$. There exists $t > 0$ such that $D(\lambda, t) \subset \Omega^*$, where $D(\lambda, t)$ is an open disc with the centre in λ and radius t. Let $f \in H$. Consider the expression of f as a power series converging in the disc $D(\lambda, t)$, i.e. $f(z) = \sum\limits_{i=0}^{\infty} a_n(z-\lambda)^n$. Let $0 < s < t$. Then

$$\begin{aligned}
m(s) &= \int_{D(\lambda,s)} |f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z} = \int_{D(\lambda,s)} \Big|\sum_{n=0}^{\infty} a_n(z-\lambda)^n\Big|^2\,\mathrm{d}z\,\mathrm{d}\bar{z} \\
&= \int_0^s \int_0^{2\pi} \Big|\sum_{n=0}^{\infty} a_n r^n e^{in\varphi}\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\varphi = 2\pi \int_0^s \sum_0^{\infty} |a_n|^2 r^{2n+1}\,\mathrm{d}r \\
&= 2\pi \sum_{n=0}^{\infty} |a_n|^2 \frac{s^{2n+2}}{2n+2}.
\end{aligned}$$

Thus for $0 < 2\varepsilon < t$ we have

$$\frac{\int_{D(\lambda,\varepsilon)} |f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}}{\int_{\Omega^*} |f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}} \leqslant \frac{m(\varepsilon)}{m(2\varepsilon)} = \frac{\sum\limits_{n=0}^{\infty} |a_n|^2 \frac{\varepsilon^{2n+2}}{2n+2}}{\sum\limits_{n=0}^{\infty} |a_n|^2 \frac{(2\varepsilon)^{2n+2}}{2n+2}} \leqslant \frac{1}{4}$$

and therefore

$$\frac{3}{4}\int_{\Omega^*}|f(z)|^2\,\mathrm{d}z d\bar{z}\leqslant\int_{\Omega^*\setminus D(\lambda,\varepsilon)}|f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}\leqslant\int_{\Omega^*}|f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}.$$

Hence

$$\begin{aligned}\|(M-\lambda)f\|^2&=\frac{1}{\lambda(\Omega^*)}\int_{\Omega^*}|(z-\lambda)f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}\\&\geqslant\frac{1}{\lambda(\Omega^*)}\int_{\Omega^*\setminus D(\lambda,\varepsilon)}|(z-\lambda)f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}\\&\geqslant\frac{\varepsilon^2}{\lambda(\Omega^*)}\int_{\Omega^*\setminus D(\lambda,\varepsilon)}|f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}\\&\geqslant\frac{3\varepsilon^2}{4\lambda(\Omega^*)}\int_{\Omega^*}|f(z)|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}=\frac{3\varepsilon^2}{4}\|f\|^2.\end{aligned}$$

Set $S_\Omega=M^*$. We have shown that $M-\lambda$ is bounded below for $\lambda\in\Omega^*$, so that for $\mu\in\Omega$ the operator $S_\Omega-\mu=(M-\bar{\mu})^*$ is onto. For $\lambda\notin\overline{\Omega^*}$ it is clear that $[(M-\lambda)^{-1}f](z)=(z-\lambda)^{-1}f(z)$ $(f\in H)$. Thus $(S_\Omega-\bar{\lambda})=(M-\lambda)^*$ is invertible for $\lambda\notin\overline{\Omega^*}$.

We have proved that $\sigma(S_\Omega)\subset\overline{\Omega}$ and $\sigma_\delta(S_\Omega)\subset\partial\Omega$.

Denote by h_0 the function equal identically to 1 on Ω^*. Let $\lambda\in\Omega$ and $g\in N(S_\Omega-\lambda)=N((M-\bar{\lambda})^*)$, $g\neq 0$. Any function $f\in H$ can be written in the form

$$f(z)=\frac{f(z)-f(\bar{\lambda})}{z-\bar{\lambda}}(z-\bar{\lambda})+f(\bar{\lambda}).$$

As the analytic function $(f(z)-f(\bar{\lambda}))(z-\bar{\lambda})^{-1}$ is also in H, the first term of the latter sum is in $R(M-\bar{\lambda})=(N(S_\Omega-\lambda))^\perp$. Thus we obtain $(g,f)=(g,f(\bar{\lambda})h_0)=\overline{f(\bar{\lambda})}(g,h_0)$ and $(g,h_0)\neq 0$, otherwise $g\in H^\perp=\{0\}$. Further, if $f\in\sum\limits_{n=0}^{\infty}a_n(z-\bar{\lambda})^n$ in $D(\bar{\lambda},t)$ and $0<s<t$ then

$$|f(\bar{\lambda})|^2=|a_0|^2\leqslant\frac{m(s)}{\pi s^2}\leqslant\frac{\lambda(\Omega^*)}{\pi s^2}\|f\|^2.$$

Thus $f\to f(\bar{\lambda})$ is a non-zero bounded linear functional vanishing on $R(M-\bar{\lambda})=(N(S_\Omega-\lambda))^\perp$ and consequently $N(S_\Omega-\lambda)$ is non-trivial. Hence $\Omega\subset\sigma(S_\Omega)\subset\overline{\Omega}$, so that $\sigma(S_\Omega)=\overline{\Omega}$ and $\partial\overline{\Omega}\subset\sigma_\delta(S_\Omega)$. This completes the proof of (1), (2) and (3). □

Lemma 5. *Let Z be a Banach space, $D\in B(Z)$, $K>0$. Let M be a subspace of $R(D)$ such that M is dense in Z and for any $y\in M$ there exists $x\in Z$ satisfying $Dx=y$ and $\|x\|\leqslant K\|y\|$. Then D is onto.*

P r o o f. Define the bounded operator $D_0\colon Z/N(D) \to Z$ by $D_0(x + N(D)) = Dx$. Define the operator $C\colon M \to Z/N(D)$ by $Cy = D^{-1}y + N(D)$. As $\|Cy\| \leqslant K\|y\|$, we can extend C to an operator from Z to $Z/N(D)$, which we denote by the same symbol C. For $y \in M$ we have $D_0Cy = D_0(D^{-1}y + N(D)) = y$. If $z \in \overline{R(D)}$ then there exist elements $y_n \in M$ converging to z. As the sequence $y_n = D_0Cy_n$ converges to D_0Cz, we obtain $z \in D_0Cz \in R(D)$. Thus $R(D)$ is closed, dense in Z and therefore D is onto. $\square$

Proposition 6. *Let X be a Banach space, let Ω be a bounded, non-empty open subset of $\mathbb{C}$. Then there exists a Banach space Z and operators $\widetilde{S_\Omega} \in B(Z)$, $A\colon Z \to X$ such that*
(i) $\sigma(\widetilde{S_\Omega}) = \overline{\Omega}$,
(ii) $\partial\overline{\Omega} \subset \sigma_\delta(\widetilde{S_\Omega}) \subset \partial\Omega$,
(iii) $A(N(\widetilde{S_\Omega} - \lambda)) = X$ *for any* $\lambda \in \Omega$.

P r o o f. Let H be a Hilbert space, S_Ω be an operator and let h_0 be an element of H satisfying conditions (i)–(iii) of Lemma 4. Denote by $Z_0 = X \otimes_a H$ the algebraic tensor product of X and H. Define the norm on Z_0 by

$$\|z\| = \inf\Bigl\{\sum_{i=1}^{n} \|x_i\| \cdot \|h_i\|\colon\ x_i \in X,\ h_i \in H,\ z = \sum_{i=1}^{n} x_i \otimes h_i\Bigr\}.$$

Let $Z = X\widehat{\otimes}H$ be the completion of Z_0. It is not difficult to show that Z is a Banach space. Define the operator $\widetilde{S_\Omega}$ on Z_0 by

$$\widetilde{S_\Omega}\Bigl(\sum_{i=1}^{n} x_i \otimes h_i\Bigr) = \sum_{i=1}^{n} x_i \otimes S_\Omega h_i.$$

This definition is correct and

$$\Bigl\|\sum_{i=1}^{n} x_i \otimes S_\Omega h_i\Bigr\| \leqslant \|S_\Omega\| \sum_{i=1}^{n} \|x_i\| \cdot \|h_i\|.$$

Thus we can extend $\widetilde{S_\Omega}$ to a bounded operator on Z which we shall denote by the same symbol $\widetilde{S_\Omega}$.

Let $\lambda \in \Omega$, i.e. $S_\Omega - \lambda$ is onto. There exists $K > 0$ such that for any $h \in H$ there is $g \in H$ satisfying $(S_\Omega - \lambda)g = h$ and $\|g\| \leqslant K \cdot \|h\|$. Let $z \in Z_0$, $\varepsilon > 0$. Find elements $x_i \in X$, $h_i \in H$ such that $z = \sum\limits_{i=1}^{n} x_i \otimes h_i$ and $\sum\limits_{i=1}^{n} \|x_i\| \cdot \|h_i\| \leqslant (1 + \varepsilon)\|z\|$. Choose $g_i \in H$ such that $(S_\Omega - \lambda)g_i = h_i$ and $\|g_i\| \leqslant K \cdot \|h_i\|$. Then

$$z = \sum_{i=1}^{n} x_i \otimes h_i = \sum_{i=1}^{n} x_i \otimes (S_\Omega - \lambda)g_i = (\widetilde{S_\Omega} - \lambda)\Bigl(\sum_{i=1}^{n} x_i \otimes g_i\Bigr)$$

and

$$\left\|\sum_{i=1}^{n} x_i \otimes g_i\right\| \leqslant \sum_{i=1}^{n} \|x_i\| \cdot \|g_i\| \leqslant K \sum_{i=1}^{n} \|x_i\| \cdot \|h_i\| \leqslant K(1+\varepsilon)\|z\|.$$

Thus $\widetilde{S_\Omega} - \lambda$ is onto by Lemma 5.

If $\lambda \notin \sigma(S_\Omega)$ then $(\widetilde{S_\Omega} - \lambda)^{-1} = (\widetilde{(S_\Omega - \lambda)^{-1}})$, i.e. $\lambda \notin \sigma(\widetilde{S_\Omega})$. Define the operator $A\colon X_0 \to X$ by

$$A\left(\sum_{i=1}^{n} x_i \otimes h_i\right) = \sum_{i=1}^{n} (h_i, h_0) x_i.$$

This definition is correct and

$$\left\|\sum_{i=1}^{n} (h_i, h_0) x_i\right\| \leqslant \|h_0\| \sum_{i=1}^{n} \|x_i\| \cdot \|h_i\|.$$

Thus we can extend A to a bounded operator from Z to X which we shall denote by the same symbol A. Let $x \in X$, $\lambda \in \Omega$. By Lemma 4 there is a nonzero $g \in N(S_\Omega - \lambda)$ such that $(g, h_0) = 1$. Then $x \otimes g \in N(\widetilde{S_\Omega} - \lambda)$ and $A(x \otimes g) = (g, h_0)x = x$. Thus $A(N(\widetilde{S_\Omega} - \lambda)) = X$. □

Consider an operator $T \in B(X)$. By Lemma 3 the defect spectrum of any extension S of T to a larger Banach space Y such that $\sigma_\delta(S) \subset \sigma_\delta(T)(\sigma(S) \subset \sigma(T))$ contains $\partial\widehat{\sigma}(T)$ $(\partial(\sigma(T))$. Let F be a compact subset of $\mathbb{C}$ such that $\partial\widehat{\sigma}(T) \subset F \subset \sigma_\delta(T)$ $(\partial\sigma(T) \subset F \subset \sigma(T))$. We show that it is possible to find a Banach space $Y \supset X$ and an extension S of T to Y such that $\sigma_\delta(S) = F(\sigma_\delta(S) = F$ and $\sigma(S) \subset \sigma(T))$. In particular, Y and S can be constructed such that the defect spectrum of S is the smallest possible, i.e. $\sigma_\delta(S) = \partial\widehat{\sigma}(T)(\sigma_\delta(S) = \partial\sigma(T))$.

Theorem 7. *Let X be a Banach space, $T \in B(X)$. Then*

(i) *for any compact subset F of $\mathbb{C}$, $\partial\widehat{\sigma}(T) \subset F \subset \sigma_\delta(T)$, there are a Banach space Y containing X and an extension S of T to Y such that $\sigma_\delta(S) = F$,*

(ii) *for any compact subset F of $\mathbb{C}$, $\partial\sigma(T) \subset F \subset \sigma(T)$, there are a Banach space Y containing X and an extension S of T to Y such that $\sigma_\delta(S) = F$ and $\sigma(S) \subset \sigma(T)$.*

Proof. (i) Denote by Ω the interior of $\widehat{\sigma}(T)$. If Ω is an empty set then set $Z = \{0\}$. Otherwise by Proposition 6 there are a Banach space Z and the operators $\widetilde{S_\Omega} \in B(Z)$, $A\colon Z \to X$ such that $\sigma(\widetilde{S_\Omega}) = \overline{\Omega}$, $\sigma_\delta(\widetilde{S_\Omega}) = \partial\Omega$ and $A(N(\widetilde{S_\Omega} - \lambda)) = X$ for $\lambda \in \Omega$. Set $Y_0 = X \oplus Z$. If Ω is non-empty then define the operator S_0 on Y_0 by

$$S_0(x \oplus z) = (Tx + Az) \oplus \widetilde{S_\Omega} z,$$

otherwise we set $S_0 = T$. It is easy to see that for $\lambda \in \Omega$ we have $S_0 - \lambda$ is onto and that $S_0 - \lambda$ is invertible for $\lambda \notin \overline{\Omega}$. Find a normal operator N on the Hilbert space ℓ_2 with $\sigma(N) = \sigma_\delta(N) = F$. Set $Y = Y_0 \oplus \ell_2$ and define $S \in B(Y)$ by $S = S_0 \oplus N$. Then $\sigma_\delta(S) = F$ and (i) is proved.

For the proof of (ii) denote by Ω the interior of $\sigma(T)$. The rest of the proof is the same as in (i). □

Remark 8. In the case when X is even a Hilbert space it is easy to see that the space Y in the previous theorem can be also constructed as a Hilbert space.

Lemma 9. *Let A be an operator on a Banach space X. Then $\partial\sigma_\pi(A) \subset \sigma_\delta(A)$ and $\partial\sigma_\delta(A) \subset \sigma_\pi(A)$.*

Proof. Let $\mu \in \partial\sigma_\pi(A)$. Then there exist elements μ_n converging to μ such that $A - \mu_n$ is bounded below. If $\mu \notin \sigma_\delta(A)$ then $A - \mu_n$ is onto and consequently invertible for n large enough. But then $\mu \in \partial\sigma(A) \subset \sigma_\delta(A)$ and this is a contradiction. Thus $\mu \in \sigma_\delta(A)$. The second inclusion can be proved similarly. □

Theorem 10. *Let X be a Banach space, $T \in B(X)$. Let F_1, F_2 are subsets of the complex plane. The following conditions are equivalent:*

(i) *there exist a Banach space Y containing X and an extension S of T to Y such that $\sigma_\pi(S) = F_1$ and $\sigma_\delta(S) = F_2$,*

(ii) *the sets F_1, F_2 are compact, $\partial F_1 \subset F_2$, $\partial F_2 \subset F_1$, $\sigma_\pi(T) \subset F_1$, $\sigma(T) \subset \widehat{F}_2$.*

Proof. (i) ⇒ (ii). Clearly F_1, F_2 are compact. By Lemma 9 we have $\partial F_1 \subset F_2$ and $\partial F_2 \subset F_1$. Further we have $\sigma_\pi(T) \subset \sigma_\pi(S) = F_1$, so that by Lemma 2

$$\sigma(T) \subset \widehat{\sigma}(T) = \widehat{\sigma}_\pi(T) \subset \widehat{\sigma}_\pi(S) = \widehat{\sigma}_\delta(S) = \widehat{F}_2.$$

(ii) ⇒ (i). By [3] or [5] there exist a Banach space $X_0 \supset X$ and an operator T_0 such that $T = T_0|X$ and $\sigma(T_0) = \sigma_\pi(T_0) = \sigma_\pi(T)$. Denote by Ω the interior of F_1. If Ω is an empty set then set $Z = \{0\}$. Otherwise by Proposition 6 there exist a Banach space Z and operators $\widetilde{S_\Omega} \in B(Z)$, $A\colon Z \to X$ such that $\sigma(S_\Omega) = \overline{\Omega}$, $\sigma_\delta(\widetilde{S_\Omega}) = \partial\Omega$, $A(N(\widetilde{S}_\Omega - \lambda)) = X$ for $\lambda \in \Omega$. Set $Y_0 = X \oplus Z$. If Ω is non-empty then define $S_0 \in B(Y_0)$ by

$$S_0(x \oplus z) = (T_0 x + Az) \oplus \widetilde{S_\Omega} z,$$

otherwise set $S_0 = T_0$. Then

$$\sigma_\pi(S_0) \subset \sigma_\pi(T_0) \cup \sigma_\pi(\widetilde{S_\Omega}) \subset F_1 \cup \overline{\Omega} \subset F_1.$$

Further,

$$\sigma_\delta(S_0) \subset \partial\Omega \cup [(\mathbb{C} \setminus \overline{\Omega}) \cap \sigma_\delta(T_0)] \subset \partial\Omega \cup [(\mathbb{C} \setminus \Omega) \cap F_1] \subset \partial\Omega \cup \partial F_1 \subset \partial F_1 \subset F_2.$$

By Proposition 6 there are a Hilbert space W_1 and an operator $S_1 \in B(W_1)$ such that $\sigma_\pi(S_1) = \sigma(S_1) = \overline{\mathrm{int} F_1}$ and $\sigma_\delta(S_1) = \partial(\mathrm{int} F_1) \subset \partial F_1 \subset F_2$. Similarly there are a Hilbert space W_2 and an operator $S_2 \in B(W_2)$ such that $\sigma_\delta(S_2) = \sigma(S_2) = \overline{\mathrm{int} F_2}$ and $\sigma_\pi(S_2) = \partial(\mathrm{int} F_2) \subset \partial F_2 \subset F_1$. Find a normal operator $N \in B(\ell_2)$ with

$$\sigma(N) = \sigma_\pi(N) = \sigma_\delta(N) = \partial F_1 \cup \partial F_2.$$

Set $Y = Y_0 \oplus W_1 \oplus W_2 \oplus \ell_2$ and $S = S_0 \oplus S_1 \oplus S_2 \oplus N$. Then $\sigma_\pi(S) = F_1$ and $\sigma_\delta(S) = F_2$. □

Remark 11. The same as Remark 8.

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