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Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 4, 677–685

Persistent URL: <http://dml.cz/dmlcz/127446>

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NONOSCILLATION AND ASYMPTOTIC BEHAVIOUR FOR THIRD
ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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(Received November 8, 1995)

Abstract. In this paper we consider the equation

$$y''' + q(t)y'^{\alpha} + p(t)h(y) = 0,$$

where p, q are real valued continuous functions on $[0, \infty)$ such that $q(t) \geq 0$, $p(t) \geq 0$ and $h(y)$ is continuous in $(-\infty, \infty)$ such that $h(y)y > 0$ for $y \neq 0$. We obtain sufficient conditions for solutions of the considered equation to be nonoscillatory. Furthermore, the asymptotic behaviour of these nonoscillatory solutions is studied.

MSC 2000: 34C15, 34D05

Keywords: Third order nonlinear differential equations, nonoscillatory solutions, asymptotic properties of solutions

1. INTRODUCTION

Finding sufficient conditions for nonoscillation of solutions is a problem of general interest in the theory of ordinary and delay differential equations ([6], [13], [16]). In this paper, we consider

$$(1) \quad y''' + q(t)y'^{\alpha} + p(t)h(y) = 0,$$

where p, q are real valued continuous functions on $[0, \infty)$ such that $p(t) \geq 0$, $q(t) \geq 0$ and $h(y)$ is continuous on $(-\infty, \infty)$ subject to $h(y)y > 0$ for $y \neq 0$ while $\alpha > 0$ is the ratio of odd integers.

In recent years, third order homogeneous differential equations (linear and nonlinear) have been the main stream of investigations for many authors. Barrett [1], Bobrowski [2], Cecchi and Marini [3], Erbe ([4], [5]), Greguš ([7], [8]) are just some

of them. They have given sufficient conditions for oscillation and nonoscillation and studied asymptotic behaviour of the solutions. The study of properties of nonlinear equations is not as extensive as for the linear case. Moreover, some results for linear cases fail in some nonlinear cases [12].

For the third order equation (1) much less work has been done. Motivation for the study of this equation comes from the works of Greguš, M. and Greguš Jr.M ([9], [10]) and Heidel [11] who studied qualitative behaviour of solutions of the special cases of (1), and the work of Parhi, N. and Parhi S. [14] who studied the nonhomogeneous equations with the remark that some of their results are not valid for the homogeneous case.

In the present paper, first we give sufficient conditions under which solutions of (1) are nonoscillatory. Secondly, we are concerned with asymptotic behaviour of nonoscillatory solutions.

We restrict our considerations to those real solutions y of (1) which exist on the half line $[T, \infty)$, where $T \geq 0$ depends on the particular solution, and are nontrivial in any neighbourhood of infinity. We may recall that a solution y of (1) on $[T, \infty)$ is said to be nonoscillatory if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for $t \geq t_1$; $y(t)$ is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$; it is said to be of Z -type if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

2.

In this section we give sufficient conditions under which solutions of (1) are nonoscillatory.

Theorem 2.1. *Suppose that $q(t) \leq p(t)$ for $t \in [0, \infty)$ and $q(t) \not\equiv 0$ on any subinterval of $[0, \infty)$. If y is a solution of (1) defined on $[T, \infty)$, $T \geq 0$, such that it satisfies $z'^\alpha + h(z) < 0$ on the subinterval of $[T, \infty)$ on which $y'(t) > 0$, while $z'^\alpha + h(z) > 0$ on the subinterval of $[T, \infty)$ on which $y'(t) < 0$, then y is nonoscillatory on $[T, \infty)$.*

Proof. Let y be a solution of (1) on $[T, \infty)$, $T \geq 0$. Multiplying (1) by $y'(t)$ we get

$$(2) \quad y'''(t)y'(t) = -q(t)y'^\alpha(t)y'(t) - p(t)h(y(t))y'(t)$$

Let us assume that $y'(t) > 0$ on some subinterval of $[T, \infty)$. Then the condition $y'^\alpha + h(y) < 0$ yields $h(y) < 0$, and so $y(t) < 0$. Thus we cannot have simultaneously

$y(t) > 0$ and $y'(t) > 0$. Therefore oscillatory or nonnegative Z -type solutions cannot exist, and in the proof it is sufficient to exclude nonpositive Z -type solutions.

Let y be a nonpositive Z -type solution with consecutive double zeros at a and b ($a < b$). So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and hence $y'(t) > 0$ for $t \in (c, b)$. Now integrating (2) from c to b yields

$$\begin{aligned} 0 &= y''(t)y'(t) \Big|_c^b \\ &\geq \int_c^b y''^2(t) dt - \int_c^b q(t)[y'^\alpha(t) + h(y(t))]y'(t) dt > 0, \end{aligned}$$

which is a contradiction.

When $y'(t) < 0$, the above argument can be repeated to complete the proof of the theorem. \square

Now we give an example to illustrate the above theorem.

Example 1.

$$y''' + \frac{1}{2}y'^3 + \frac{2e^{3t} + e^t}{2(e^{3t} + e^t)}(y + y^3) = 0, \quad t \geq 0.$$

$y(t) = e^{-t}$ and $y(t) = -e^{-t}$ are nonoscillatory solutions of this equation.

Theorem 2.2. *Let $1 \leq q(t)$. If y is a solution of (1) on an interval on which $y''^2(t) - y'^{\alpha+1}(t) < 0$ holds, then y is nonoscillatory.*

Proof. Let $y(t)$ be of nonnegative Z -type with consecutive double zeros at a and b ($a < b$). So there exists a $c \in (a, b)$ such that $y'(c) = 0$. Integrating (2) from a to c , we get

$$\begin{aligned} 0 &= y''(t)y'(t) \Big|_a^c \\ &\leq \int_a^c q(t)[y''^2(t) - y'^{\alpha+1}(t)] dt - \int_a^c p(t)h(y(t))y'(t) dt. \end{aligned}$$

Since $h(y(t))$ is positive for $y(t) > 0$ we arrive at

$$0 \leq \int_a^c q(t)[y''^2(t) - y'^{\alpha+1}(t)] dt - \int_a^c p(t)h(y(t))y'(t) dt < 0,$$

which is a contradiction.

Next suppose that $y(t)$ is nonpositive Z -type with consecutive double zeros at a and b ($a < b$). So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) < 0$ for

$t \in (a, c)$. Now integrating (2) from a to c , we obtain

$$\begin{aligned} 0 &= y''(t)y'(t) \Big|_a^c \\ &\leq \int_a^c q(t)[y''^2(t) - y'^{\alpha+1}(t)] dt - \int_a^c p(t)h(y(t))y'(t) dt. \end{aligned}$$

As in the above case we get a contradiction.

Finally, to complete the proof, suppose that $y(t)$ is oscillatory. Let a, b and a' ($a < b < a'$) be any three consecutive zeros of $y(t)$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$; so $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0 = y'(c')$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. We consider two cases, namely, $y''(b) \leq 0$ and $y''(b) > 0$. First, let $y''(b) \leq 0$. Integrating (2) between b and c' , we obtain

$$\begin{aligned} 0 &\leq y''(t)y'(t) \Big|_b^{c'} \\ &\leq \int_b^{c'} q(t)[y''^2(t) - y'^{\alpha+1}(t)] dt - \int_b^{c'} p(t)h(y(t))y'(t) dt < 0, \end{aligned}$$

a contradiction. Hence $y''(b) > 0$. Since $y''(t)$ is continuous, $y''(t) > 0$ for $t \in [b, b + \delta_1)$, $0 < \delta_1 < c' - b$. So $y'(t)$ is increasing on $[b, b + \delta_1)$. Again $y'(c') = 0$ and $y'(t) > 0$ for $t \in (b, c')$ imply that $y'(t)$ is decreasing on $[c' - \delta_2, c']$, where $0 < \delta_2 < c' - b$. This in turn implies that $y''(t) < 0$ for $t \in [c' - \delta_2, c']$. Hence $y''(d) = 0$ for some $d \in (b, c')$. Integrating (2) from d to c' we get a contradiction again. \square

Example 2.

$$y''' + t^{10/3}y'^{5/3} + \frac{t^4 + 6}{2t^4 + 4t^3 + 3t^2 + t}(y + y^3) = 0, \quad t > 8.$$

Here $y(t) = 1 + 1/t$ is a nonoscillatory solution of the above equation satisfying the conditions of Theorem 2.2.

Theorem 2.3. Any solution y of (1) satisfying the inequality

$$z''^2 - q(t)z'^{\alpha+1} - p(t)h(z)z' > 0$$

on an interval on which $y'(t) > 0$, is nonoscillatory.

Proof. Let y be a solution of (1) satisfying conditions of the theorem. If possible, assume that y is of nonnegative Z -type. Let a and b ($a < b$) be any two consecutive double zeros of y . So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and

$y'(t) > 0$ for $t \in (a, c)$. Now multiplying (1) by $y'^2(t)$ and integrating the resulting identity from a to c , we get

$$\begin{aligned} 0 &= y''(t)y'^2(t) \Big|_a^c \\ &= - \int_a^c q(t)y'^{\alpha}(t)y'^2(t) dt - \int_a^c p(t)h(y(t))y'^2(t) dt + 2 \int_a^c y'(t)y''^2(t) dt \\ &> 0, \end{aligned}$$

a contradiction.

Similarly we can show that $y(t)$ cannot be of nonpositive Z -type.

Suppose that $y(t)$ is oscillatory. Let a, b and a' ($a < b < a'$) be any three consecutive zeros of $y(t)$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$; so $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. Thus there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0 = y'(c')$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. We consider two cases, namely, $y''(b) \leq 0$ and $y''(b) > 0$. Integrating (2) from b to c' in case $y''(b) > 0$ and from c to b in case $y''(b) \leq 0$, we get the required contradictions. \square

Example 3.

$$y''' + \frac{1}{27}t^{-3}y'^3 + \frac{1}{4}t^{-6}y^3 = 0, \quad t > 0.$$

This example illustrates the above theorem. Clearly $y(t) = t^{3/2}$ is a nonoscillatory solution of the equation.

3.

In this section we are concerned with the asymptotic behaviour of nonoscillatory solutions of (1). In [11], Heidel gave sufficient conditions under which a nonoscillatory solution of the equation (1) with $h(y) = y^r$ tends to zero as $t \rightarrow \infty$. In the following we extend some of Heidel's results to (1).

First we recall.

Lemma 3.1. Consider

$$(3) \quad z'' + q(t)z = 0$$

where $q(t) \geq 0$. If z is a nonoscillatory solution of (3) such that $z(t) \neq 0$ for $t \in [a, \infty)$, $a > 0$, and if u is a continuously differentiable function on $[a, \infty)$ such that $u(b) = 0 = u(c)$, $a < b < c$, and $u(t) \neq 0$ on $[b, c]$, then

$$\int_b^c [u'^2(t) - q(t)u^2(t)] dt > 0.$$

For the proof of this lemma the reader is referred to [15]. \square

Remark 3.1. $q(t)$ need not be nonnegative in Lemma 3.1. If $q(t) \leq 0$ then all solutions of (3) are nonoscillatory. For $q(t) \not\leq 0$, sufficient conditions were given by Moore [16, p. 73], Winter [17] and Potter [16, p. 81] for nonoscillation of all solutions of (3).

Now we prove

Lemma 3.2. *Let $\int_0^\infty p(t) dt = \infty$ and let $h(y)$ be nondecreasing. If y is a nonoscillatory solution of (1) and $y(t) \neq 0$ on $[t_0, \infty)$ then $|y|$ is not nondecreasing on $[t_0, \infty)$.*

Proof. Let $y(t) > 0$ and $y'(t) \geq 0$ for $t \geq t_0 > 0$. Integrating the inequality

$$y'''(t) \leq -p(t)h(y(t))$$

from t_0 to t , we get

$$y''(t) \leq y''(t_0) - h(y(t_0)) \int_{t_0}^t p(s) ds.$$

This in turn implies that $y'(t) < 0$ for large t , which yields a contradiction.

If $y(t) < 0$ and $y'(t) \leq 0$ for $t \geq t_0 > 0$, proceeding as above we obtain a similar contradiction. □

Theorem 3.1. *Let $\alpha = 1$ in equation (1). Let $q(t) \leq M$, $M > 0$ and let the hypothesis of Lemma 3.2 hold. If y is a nonoscillatory solution of (1) such that $y(t)y'(t) < 0$ for $t \in [t_0, \infty)$, $t_0 \geq 0$, then $\lim_{t \rightarrow \infty} y(t) = 0$.*

Proof. Let $y(t) > 0$ for $t \geq t_0$. So $y'(t) < 0$ for $t \geq t_0$ and hence $\lim_{t \rightarrow \infty} y(t)$ exists. Let us assume $\lim_{t \rightarrow \infty} y(t) > 0$. Integrating (1) from t_0 to t , we obtain

$$\begin{aligned} y''(t) &\leq y''(t_0) - M(y(t) - y(t_0)) - \int_{t_0}^t h(y(s))p(s) ds \\ &\leq y''(t_0) - M(y(t) - y(t_0)) - h(y(t)) \int_{t_0}^t p(s) ds. \end{aligned}$$

This yields that $y(t) < 0$ for large t , giving a contradiction.

Similarly, when $y(t) < 0$, $y'(t) > 0$ for $t \geq t_0$ and $\lim_{t \rightarrow \infty} y(t) < 0$ we get a contradiction.

This completes the proof of the theorem. □

In the above proof; the assumption employed is weaker than in Heidel [11, Theorem 3.7], if $t_0 > 1$.

Theorem 3.2. *Let $\alpha = 1$ in equation (1). Let the hypothesis of Lemma 3.2 hold and let equation (3) be nonoscillatory. If y is a nonoscillatory solution of (1) and $y(t) \neq 0$ on $[t_0, \infty)$ $t_0 \geq 0$, then $y(t)y'(t) < 0$ for all $t \in [t_0, \infty)$.*

Proof. Let $y(t) < 0$ for $t \geq t_0$. From Lemma 3.2 it follows that $y'(t) \not\leq 0$. Assume that $y'(t)$ is oscillatory or nonnegative Z -type with consecutive zeros at a and b ($t_0 < a < b$) such that $y'(t) > 0$ for $t \in (a, b)$. Integrating (2) from a to b , we get

$$\begin{aligned} 0 &> y''(t)y'(t) \Big|_a^b + \int_a^b p(t)h(y(t))y'(t) dt \\ &= \int_a^b [y''^2(t) - q(t)y'^2(t)] dt \end{aligned}$$

and by Lemma 3.1 we arrive at a contradiction. So $y'(t) > 0$.

Let $y(t) > 0$ for $t \geq t_0$. It is clear from Lemma 3.2 that $y'(t) \not\geq 0$. Let $y'(t)$ be oscillatory or nonpositive Z -type. Let a and b ($t_0 < a < b$) be consecutive zeros of $y'(t)$ such that $y'(t) < 0$ for $t \in (a, b)$. Integrating (2) from a to b , we obtain

$$\begin{aligned} 0 &> \int_a^b p(t)h(y(t))y'(t) dt \\ &= \int_a^b [y''^2(t) - q(t)y'^2(t)] dt \\ &> 0, \end{aligned}$$

which is a contradiction. So $y'(t) < 0$.

Hence the theorem. □

Example 4.

$$y''' + \frac{1}{4t^2}y' + \frac{4t^2\beta^3 + \beta}{4t^2(1 + e^{-2\beta t})}(y + y^3) = 0, \quad t > 1, \beta > 0 \text{ is a constant.}$$

Note that this example illustrates Lemma 3.2, Theorem 3.1 and Theorem 3.2. Clearly $y(t) = e^{-\beta t}$ is a positive solution of this equation. On the other hand $z'' + \frac{1}{4t^2}z = 0$ $t \geq 1$ is nonoscillatory because $z(t) = \sqrt{t} \ln t$ is a nonoscillatory solution of the equation.

In the following an attempt has been made to remove the restriction on α in Theorem 3.1.

Theorem 3.3. *Let the suppositions of Theorem 2.1 and Lemam 3.3 hold. If y is a solution of (1) having the properties as in Theorem 2.1, then $\lim_{t \rightarrow \infty} y(t) = 0$.*

Proof. Let $y(t) > 0$ for $t \geq t_0 > 0$. From Lemma 3.2 it follows that $y'(t) \not\geq 0$. Let $y'(t)$ be oscillatory or nonpositive Z -type with consecutive zeros at a and b ($t_0 < a < b$) such that $y'(t) < 0$ for $t \in (a, b)$. Now integration of (2) from a to b yields

$$\begin{aligned} 0 &= y''(t)y'(t) \Big|_a^b \\ &\geq - \int_a^b q(t)[y'^\alpha(t) + h(y(t))]y'(t) dt \\ &> 0, \end{aligned}$$

a contradiction. So $y'(t) < 0$. Consequently, $\lim_{t \rightarrow \infty} y(t)$ exists. Now let us assume $\lim_{t \rightarrow \infty} y(t) > 0$. From (1) it follows that $y''(t)$ is monotonic decreasing. We claim that there exists a $t_1 \geq t_0$ such that $y''(t) > 0$ for $t \geq t_1$. If not, for every $t_1 \geq t_0$ there exists a $t_2 \geq t_1$ such that $y''(t_2) \leq 0$. So $t \geq t_2$ implies that $y''(t) \leq 0$. Hence $y(t) < 0$ for large t . This is a contradiction. So our claim holds. Now integrating (1) from t_1 to t , we get

$$\begin{aligned} y''(t) &= y''(t_1) - \int_{t_1}^t q(s)y'^\alpha(s) ds - \int_{t_1}^t p(s)h(y(s)) ds \\ &\leq y''(t_1) - y'^\alpha(t_1) \int_{t_1}^t q(s) ds - h(y(t)) \int_{t_1}^t p(s) ds, \end{aligned}$$

which in turn implies $y''(t) < 0$ for large t , contradicting $y''(t) > 0$. Hence $\lim_{t \rightarrow \infty} y(t) = 0$.

When $y(t) < 0$ for $t \geq t_0 > 0$, the above argument can be repeated to complete the proof of the theorem. □

Remark 3.2. The results obtained here can be extended to the equation

$$(r(t)y'')' + q(t)y'^\alpha + p(t)h(y) = 0$$

where r is a continuous and positive function. The only assumption needed for minor manipulations is

$$\int_0^\infty \frac{dt}{r(t)} = \infty$$

to hold, whenever necessary.

Acknowledgment. The authors would like to thank the referee for his valuable suggestions and useful comments.

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