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ANTIATOMIC RETRACT VARIETIES OF MONOUNARY ALGEBRAS

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Retracts of monounary algebras were investigated in the papers [2]–[4].

The notion of the retract variety of monounary algebras was introduced in [5] by applying an analogy with the notion of the order variety of partially ordered sets studied in [1].

The collection $\mathfrak{R}$ of all retract varieties of monounary algebras was investigated in [5]. This collection is considered to be partially ordered by the class-theoretical inclusion. A retract variety $\mathscr{V}$ is called atomic if $\mathscr{V} \neq \emptyset$ and, whenever $\mathscr{V}'$ is a retract variety with $\emptyset \neq \mathscr{V}' \subseteq \mathscr{V}$, then $\mathscr{V}' = \mathscr{V}$. It was proved that there are exactly $2^{\aleph_0}$ atomic retract varieties in $\mathfrak{R}$.

A retract variety $\mathscr{V}$ of $\mathfrak{R}$ is said to be antiatomic if $\mathscr{V} \neq \emptyset$ and there is no atomic variety $\mathscr{V}_1$ of $\mathfrak{R}$ with $\mathscr{V}_1 \subseteq \mathscr{V}$.

In view of the relation $\subseteq$ for pairs of retract varieties $\mathscr{V}_1, \mathscr{V}_2$ of $\mathfrak{R}$, we apply also the symbols inf $\{\mathscr{V}_1, \mathscr{V}_2\}$ and sup $\{\mathscr{V}_1, \mathscr{V}_2\}$ in the usual way. Namely, if $\mathscr{V}_1$, $\mathscr{V}_2$, $\mathscr{V}_3$ belong to $\mathfrak{R}$ and

(i) $\mathscr{V}_1 \subseteq \mathscr{V}_3$, $\mathscr{V}_2 \subseteq \mathscr{V}_3$,

(ii) if $\mathscr{V}$ belongs to $\mathfrak{R}$, $\mathscr{V}_1 \subseteq \mathscr{V}$, $\mathscr{V}_2 \subseteq \mathscr{V}$, then $\mathscr{V}_3 \subseteq \mathscr{V}$,

then we write $\mathscr{V}_3 = \sup\{\mathscr{V}_1, \mathscr{V}_2\}$. The notion inf $\{\mathscr{V}_1, \mathscr{V}_2\}$ is defined dually.

The description of all antiatomic retract varieties of $\mathfrak{R}$ is given in Theorem 3.2. Further we investigate the collection *Ant* of all antiatomic retract varieties of $\mathfrak{R}$; the following results will be proved:

(a) *Ant* is closed with respect to the operations of inf and sup.

(b) There is a proper class $\mathscr{O}$ of ordinals with the following properties:

(b1) For each $\alpha \in \mathscr{O}$ there exists $\mathscr{W}_\alpha \in Ant$ such that, whenever $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$, then $\mathscr{W}_\alpha \nsubseteq \mathscr{W}_\beta$.

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(b2) For each $\alpha \in \mathscr{O}$ there exists $\mathscr{V}_\alpha \in Ant$ such that, whenever $\alpha, \beta \in \mathscr{O}$, $\beta < \alpha$, then $\mathscr{V}_\beta \subsetneqq \mathscr{V}_\alpha$.

1. Preliminaries

The symbol $\mathscr{U}$ will denote the class of all monounary algebras. Let $\mathscr{A} = (A, f) \in \mathscr{U}$. A nonempty subset M of A is said to be a retract of $\mathscr{A}$ if there is a mapping h of A onto M such that h is an endomorphism of $\mathscr{A}$ and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M.

For $\mathscr{K} \subseteq \mathscr{U}$ let $R(\mathscr{K})$ ($P(\mathscr{K})$) be the class of monounary algebras whose elements are all retracts (direct products) of members of $\mathscr{K}$ and their isomorphic images. A class $\mathscr{K}$ of monounary algebras is said to be retract (product) closed if it is closed with respect to isomorphisms and if it contains all retracts (direct products) of members of $\mathscr{K}$. A class $\mathscr{K}$ is said to be a retract variety if it is retract closed and product closed. By a retract variety $\mathscr{V}(\mathscr{K})$ generated by $\mathscr{K}$ (cf. [5]) we understand the class of all monounary algebras such that any of them is a member of every retract variety $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{K}$.

1.1. Lemma. ([5], 1.3) *If $\mathscr{K} \subseteq \mathscr{U}$, then $V(\mathscr{K}) = RP(\mathscr{K})$.*

In what follows we will use the notion of the degree of an element $x \in A$, where $(A, f) \in \mathscr{U}$; for this notion cf. e.g. [7], [6] and [2]. The degree of $x \in A$ is an ordinal or the symbol ∞ and is denoted by $s_f(x)$.

We will use without quotation the following properties of $s_f(x)$:

(A) Let $(A, f) = \prod\limits_{i \in I} (A_i, f), x \in A$. Then $s_f(x) \leqslant s_f(x(i))$ for each $i \in I$.

(B) If φ is a homomorphism of (A, f) into (B, g) and $x \in A$, then $s_g(\varphi(x)) \geqslant s_f(x)$.

1.2. Definition. Let (A, f) be a connected monounary algebra. We say that (A, f) is unbounded, if

(i) $s_f(x) \neq \infty$ for each $x \in A$,

(ii) if $x \in A$, $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^{-(m+n)}(f^m(x)) \neq \emptyset$.

1.3. Lemma. *Let (A, f) be a connected monounary algebra. Then (A, f) is unbounded if and only if*

(i) *$s_f(x) \neq \infty$ for each $x \in A$,*

(ii′) *there is $x \in A$ such that if $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ with $f^{-(m+n)}(f^m(x)) \neq \emptyset$.*

P r o o f. Obviously, if (i) and (ii) hold, then (i) and (ii′) hold. Suppose that (i) and (ii′) are valid. Let $x_1 \in A$, $n_1 \in \mathbb{N}$. The algebra (A, f) is connected, thus there are $k, l \in \mathbb{N} \cup \{0\}$ with

$$f^k(x) = f^l(x_1). \tag{1}$$

There is $j \in \mathbb{N}$ such that $l - k + j \geqslant 0$. Put $n = n_1 + l - k + j$. According to (ii′), there are $m \in \mathbb{N}$ and $y \in A$ such that

$$f^{m+n}(y) = f^m(x). \tag{2}$$

Denote $m_1 = m + l - k + j$. Then $m_1 \in \mathbb{N}$ and (1) and (2) imply

$$\begin{aligned} f^{m_1}(x_1) &= f^{m+l-k+j}(x_1) = f^{m+j}(x) \\ &= f^{m+n+j}(y) = f^{(m_1-l+k-j)+(n_1+l-k+j)+j}(y) \\ &= f^{m_1+n_1}(f^j(y)). \end{aligned}$$

Therefore

$$f^{-(m_1+n_1)}(f^{m_1}(x_1)) \neq \emptyset, \tag{3}$$

which implies that (ii) is valid. □

Remark. It can be shown that the conditions (ii) and (ii′) are equivalent for each connected monounary algebra.

We will apply the following notation introduced in [5]:

1.4. Notation. *Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Z}$ the set of all integers.*
$\underline{\mathbb{Z}} = (\mathbb{Z}, f)$, *where $f(k) = k + 1$ for each $k \in \mathbb{Z}$,*
$\underline{\mathbb{N}} = (\mathbb{N}, f)$, *where $f(k) = k + 1$ for each $k \in \mathbb{N}$.*

1.5. Lemma. *Let (A, f) be a monounary algebra such that $s_f(x) \neq \infty$ for each $x \in A$. If (B, f) is a connected component of (A, f), then (B, f) fails to be unbounded if and only if there exist distinct elements $e_k \in B$ for $k \in \mathbb{N}$ such that $f(e_k) = e_{k+1}$ and $f^{-k}(e_k) = \emptyset$ for each $k \in \mathbb{N}$.*

P r o o f. Suppose that (B, f) is a connected component of (A, f) and that (B, f) fails to be unbounded. Then in view of 1.2, either

$$s_f(x) = \infty \quad \text{for some } x \in B \tag{1}$$

or

(2) there are $x \in B$, $n \in \mathbb{N}$ such that if $m \in \mathbb{N}$, then
$$f^{-(m+n)}(f^m(x)) = \emptyset.$$

The assumption yields that (2) is valid. We can suppose that n is the least positive integer with the above property, i.e., that

(2′) there are $x \in B, n \in \mathbb{N}$ such that if $m \in \mathbb{N}$, then
$$f^{-(m+n)}(f^m(x)) = \emptyset \text{ and } f^{-(m+n-1)}(f^m(x)) \neq \emptyset.$$

Let x and n satisfy (2′). Take a fixed $m \in \mathbb{N}$ and an arbitrary element

$$e_1 \in f^{-(m+n-1)}(f^m(x)).$$

Further, put

$$e_k = f^{k-1}(e_1) \text{ for each } k \in \mathbb{N}, k > 1. \tag{3}$$

The elements e_k (for $k \in \mathbb{N}$) are then distinct by (1) and

$$f(e_k) = e_{k+1} \text{ for each } k \in \mathbb{N}. \tag{4}$$

Suppose that there is $y \in f^{-k}(e_k)$ for some $k \in \mathbb{N}$. We get

$$f^k(y) = e_k = f^{k-1}(e_1) \in f^{-n}(f^k(x)),$$
$$y \in f^{-(k+n)}(f^k(x)),$$

a contradiction to (2). Hence $f^{-k}(e_k) = \emptyset$ for each $k \in \mathbb{N}$.

Conversely, let there exist distinct elements $\{e_k\colon k \in \mathbb{N}\}$ with $f(e_k) = e_{k+1}$ and $f^{-k}(e_k) = \emptyset$ for each $k \in \mathbb{N}$. If we take $x = e_1, n = 1$, then, for $m \in \mathbb{N}$,

$$\begin{aligned} f^{-(m+1)}(f^m(x)) &= f^{-(m+1)}(f^m(e_1)) \\ &= f^{-(m+1)}(e_{m+1}) = \emptyset. \end{aligned}$$

Therefore (B, f) satisfies (2) and we obtain that (B, f) fails to be unbounded. □

1.6. Corollary. *Let (A, f) be a monounary algebra such that $s_f(x) \neq \infty$ for each $x \in A$. If (B, f) is a connected component of (A, f), then (B, f) is unbounded if and only if $\underline{\mathbb{N}} \notin R(B, f)$.*

P r o o f. The assertion is a consequence of 1.5 and of [3], 3.2. □

1.7. Lemma. *Let (A, f) be a monounary algebra such that $s_f(x) \neq \infty$ for each $x \in A$. The following conditions are equivalent:*

(i) *if (B, f) is a connected component of (A, f), then (B, f) fails to be unbounded;*

(ii) $\underline{\mathbb{N}} \in R(A, f)$.

P r o o f. Suppose that $\underline{\mathbb{N}} \in R(A, f)$. Then there is a subalgebra (E, f) of (A, f) with $(E, f) \cong \underline{\mathbb{N}}$ such that E is a retract of (A, f).

Hence $E = \{e_n \colon n \in \mathbb{N}\}$, where $e_i \neq e_j$ for $i, j \in \mathbb{N}, i \neq j$ and $f(e_n) = e_{n+1}$ for each $n \in \mathbb{N}$. Let (B, f) be a connected component of (A, f). Then there is an endomorphism φ of (B, f) into (E, f) and there is $i \in \mathbb{N}$ such that

$$\varphi^{-1}(e_j) \cap B = \emptyset \text{ for each } j \in \mathbb{N}, j < i, \tag{1}$$

$$\varphi^{-1}(e_j) \cap B \neq \emptyset \text{ for each } j \in \mathbb{N}, j \geqslant i. \tag{2}$$

Denote by b_1 an arbitrary element of $\varphi^{-1}(e_i) \cap B$ and, for each $n \in \mathbb{N}, n > 1$, put $b_k = f^{k-1}(b_1)$. Obviously,

$$f(b_k) = b_{k+1} \text{ for each } k \in \mathbb{N}. \tag{3}$$

Let $k \in \mathbb{N}$ and suppose that there is $y \in f^{-k}(b_k)$. Then $f^k(y) = b_k$, which implies

$$\begin{aligned} f^k(\varphi(y)) = \varphi(f^k(y)) &= \varphi(b_k) = \varphi(f^{k-1}(b_1)) \\ &= f^{k-1}(\varphi(b_1)) = f^{k-1}(e_i) = f^{k-1}(f^{i-1}(e_1)) \\ &= f^{k+i-2}(e_1). \end{aligned}$$

Thus

$$\varphi(y) \in f^{i-2}(e_1),$$

which yields that $i > 1$ and

$$\varphi(y) = e_{i-1}.$$

Therefore

$$y \in \varphi^{-1}(e_{i-1}) \cap B,$$

which contradicts (1). According to (3) and 1.5 we obtain that (i) is valid.

Now let (i) hold. By 1.6, for each connected component (B, f) of (A, f) we have $\underline{\mathbb{N}} \in R(B, f)$, hence $\underline{\mathbb{N}} \in R(A, f)$. $\square$

2. $s_f(x) = \infty$ FOR SOME x

In 2.1 – 2.6 we suppose that $\mathscr{V} \in \mathfrak{R}$ and that there exist $\mathscr{A} = (A, f) \in \mathscr{V}$ and $x_0 \in A$ such that $s_f(x_0) = \infty$.

The definition of $s_f(x)$ implies

2.1. Lemma. *There is a subalgebra of $\mathscr{A}$ such that either it is isomorphic to $\underline{\mathbb{Z}}$ or it is a cycle.*

2.2. Lemma. *Assume that each connected component of $\mathscr{A}$ is a cycle. Further suppose that*

(i) *if $(B, f), (C, f)$ are cycles of $\mathscr{A}$, card $B \neq$ card C, then card B does not divide card C.*

Then $\mathscr{V} \notin Ant$.

Proof. Let the assumption hold. There is a subalgebra (E, f) of (A, f) such that

$$\text{if } (B, f), (C, f) \text{ are cycles of } (E, f) \text{ then card } B \neq \text{ card } C; \tag{1}$$

$$\text{if } (B, f) \text{ is a cycle of } \mathscr{A}, \text{ then there is a cycle } (C, f) \text{ of } (E, f) \text{ with card } B = \text{ card } C. \tag{2}$$

By [2], 1.3 we obtain

$$(E, f) \in R(A, f), \tag{3}$$

which implies

$$V(E, f) \subseteq V(A, f) \subseteq \mathscr{V}. \tag{4}$$

Further, (E, f) satisfies (i), hence [5], 3.9 yields that $V(E, f)$ is atomic. According to (4) we get that $\mathscr{V} \notin Ant$. $\square$

2.3. Lemma. *If each connected component of $\mathscr{A}$ is a cycle, then $\mathscr{V} \notin Ant$.*

Proof. Suppose that each connected component of $\mathscr{A}$ is a cycle. For $x \in A$ let $n(x)$ be the cardinality of the cycle $\mathscr{C}(x)$ with $x \in \mathscr{C}(x)$. Further let E be the set of all $x \in A$ such that if $y \in A - \mathscr{C}(x)$, then either $n(y) = n(x)$ or $n(y)$ does not divide $n(x)$. The algebra (E, f) satisfies the condition (i) of 2.2, hence

$$V(E, f) \notin Ant. \tag{1}$$

Further, the definition of (E, f) implies (according to [2],1.3) that we have

$$(E, f) \in R(A, f). \tag{2}$$

By (2), $V(E, f) \subseteq V(A, f) \subseteq \mathscr{V}$, hence (1) yields that $\mathscr{V} \notin Ant$. □

2.4. Lemma. *If each connected component of $\mathscr{A}$ contains a cycle and there is $x \in A$ which does not belong to any cycle, then $\mathscr{V} \notin Ant$.*

Proof. Let the assumption hold and let D be the set–theoretical union of all cycles of $\mathscr{A}$. Then [2], 1.3 implies $(D, f) \in R(A, f)$, thus

$$V(D, f) \subseteq V(A, f) \subseteq \mathscr{V}.$$

Further, $V(D, f) \notin Ant$ with respect to 2.3, hence we obtain that $\mathscr{V} \notin Ant$, either. □

2.5. Lemma. *Suppose that (A, f) contains a subalgebra which is a cycle. Then $\mathscr{V} \notin Ant$.*

Proof. Let B be the set of all elements of connected components of (A, f) which contain a cycle. By the assumption, $B \neq \emptyset$. From 2.3 and 2.4 we get

$$V(B, f) \notin Ant. \tag{1}$$

Further, [2], 1.3 implies $(B, f) \in R(A, f)$, hence

$$V(B, f) \subseteq V(A, f) \subseteq \mathscr{V}. \tag{2}$$

Then (1) and (2) yield that $\mathscr{V} \notin Ant$. □

2.6. Lemma. *Suppose that no subalgebra of (A, f) is a cycle. Then $\mathscr{V} \notin Ant$.*

Proof. By 2.1, there is a subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{B} \cong \underline{\mathbb{Z}}$. Then [2], 1.3 implies $\underline{\mathbb{Z}} \in R(\mathscr{A})$, thus

$$V(\underline{\mathbb{Z}}) \subseteq V(\mathscr{A}) \subseteq \mathscr{V}.$$

Since $V(\underline{\mathbb{Z}})$ is atomic, we obtain that $\mathscr{V} \notin Ant$. □

2.7. Proposition. *If $\mathscr{V} \in \mathfrak{R}$ and there are $(A, f) \in \mathscr{V}$ and $x \in A$ with $s_f(x) = \infty$, then $\mathscr{V} \notin Ant$.*

Proof. Let the assumption hold. By 2.1, there is a subalgebra $\mathscr{B}$ of (A, f) such that either

(1) $$\mathscr{B} \text{ is a cycle of } (A, f)$$

or

(2) $$B \cong \underline{\mathbb{Z}}.$$

If (1) is valid, then 2.5 yields that $\mathscr{V} \notin Ant$. If no subalgebra of (A, f) is a cycle, then the required assertion is obtained by virtue 2.6. $\square$

3. The collection *Ant*

In this section we will describe all antiatomic retract varieties of $\mathfrak{R}$. Further, it will be proved that *Ant* is closed with respect to the operations of inf and sup. It is obvious that if $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, then

$$\inf\{\mathscr{V}_1, \mathscr{V}_2\} = \mathscr{V}_1 \cap \mathscr{V}_2$$

and that inf $\{\mathscr{V}_1, \mathscr{V}_2\}$ belongs to *Ant*. We have to show that if $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, then sup $\{\mathscr{V}_1, \mathscr{V}_2\}$ belongs to *Ant*, as well.

3.1. Lemma. *Let $\mathscr{V}_1, \mathscr{V}_2 \in Ant$. Then sup $\{\mathscr{V}_1, \mathscr{V}_2\} = RP(\mathscr{V}_1 \cup \mathscr{V}_2)$.*

Proof. The assertion is a consequence of 1.1. $\square$

3.2. Theorem. *Let $\mathscr{V} \in \mathfrak{R}$. The following conditions are equivalent:*

(i) *$\mathscr{V} \in Ant$;*
(ii) *if $(A, f) \in \mathscr{V}$, then $s_f(x) \neq \infty$ for each $x \in A$ and there is a connected component $\mathscr{B}$ of (A, f) such that $\mathscr{B}$ is unbounded.*

Proof. Let (i) hold and let (ii) be not valid. There exists $(A, f) \in \mathscr{V}$ such that either

(1) $$\text{there is } x \in A \text{ with } s_f(x) = \infty$$

or

(2) if $\mathscr{B}$ is a connected component of (A, f), then $\mathscr{B}$ fails to be unbounded.

From 2.7 it follows that (1) is not valid, thus (2) holds. Then 1.7 implies that $\underline{\mathbb{N}} \in R(A, f)$, therefore

$$V(\underline{\mathbb{N}}) \subseteq V(A, f) \subseteq \mathscr{V}.$$

Since $V(\underline{\mathbb{N}})$ is atomic, we obtain that $\mathscr{V} \notin Ant$, a contradiction.

Conversely, suppose that (ii) is valid and that (i) fails to hold. Then there is an atomic retract variety $V(\mathscr{A})$ in $\mathfrak{R}$ such that $V(\mathscr{A}) \subseteq \mathscr{V}$. By (ii),

(1) $$s_f(x) \neq \infty \text{ for each } x \in A.$$

Therefore [5], 3.9 implies

(2) $$V(\mathscr{A}) = V(\underline{\mathbb{N}}).$$

According to (2) we have $\underline{\mathbb{N}} \in \mathscr{V}$. Moreover, $\underline{\mathbb{N}}$ is the unique connected component of $\underline{\mathbb{N}}$ and $\underline{\mathbb{N}}$ fails to be unbounded, which is a contradiction to (ii). □

3.3. Lemma. *Let $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, $(A, f) \in P(\mathscr{V}_1 \cup \mathscr{V}_2)$. Then $s_f(x) \neq \infty$ for each $x \in A$.*

Proof. By assumption, there are $(A_1, f) \in \mathscr{V}_1$ and $(A_2, f) \in \mathscr{V}_2$ with $(A, f) = (A_1, f) \times (A_2, f)$. Let $x = (x_1, x_2) \in A_1 \times A_2$. Then

(1) $$s_f(x) \leqslant s_f(x_1).$$

Since $x_1 \in A_1, (A_1, f_1) \in \mathscr{V}_1 \in Ant$, we obtain according to 3.2

(2) $$s_f(x_1) \neq \infty.$$

Hence (1) and (2) yield $s_f(x) \neq \infty$. □

3.4. Lemma. *Let the assumption of 3.3 hold. There is $x \in A$ such that if $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ with $f^{-(m+n)}(f^m(x)) \neq \emptyset$.*

Proof. Analogously as in 3.3, $(A, f) = (A_1, f) \times (A_2, f)$, where $(A_1, f) \in \mathscr{V}_1$, $(A_2, f) \in \mathscr{V}_2$. Further, 3.2 implies that there are connected components (B_1, f) of (A_1, f) and (B_2, f) of (A_2, f) which are unbounded. Hence

(1) there is $x_1 \in B_1$ such that if $n \in \mathbb{N}$, then there is $m_1 \in \mathbb{N}$ with $f^{-(m_1+n)}(f^{m_1}(x_1)) \neq \emptyset$.

Let the analogous assertion for B_2 be denoted by (2). Put

$$x = (x_1, x_2) \in B_1 \times B_2.$$

Let $n \in \mathbb{N}$. According to (1) and (2) there are $m_1, m_2 \in \mathbb{N}$ with

$$f^{-(m_1+n)}(f^m(x_1)) \neq \emptyset, f^{-(m_2+n)}(f^{m_2}(x_2)) \neq \emptyset;$$

take $y_1 \in f^{-(m_1+n)}(f^{m_1}(x_1))$, $y_2 \in f^{-(m_2+n)}(f^{m_2}(x_2))$. Denote $y = (y_1, y_2), m = \max\{m_1, m_2\}$. Then we have

$$\begin{aligned} &f^{m+n}(y_1) = f^{(m-m_1)+(m_1+n)}(y_1) = f^{m-m_1}(f^{m_1}(x_1)) = f^m(x_1),\\ &f^{m+n}(y_2) = f^m(x_2), \end{aligned}$$

i.e., $y \in f^{-(m+n)}(f^m(x)) \neq \emptyset$. □

3.5. Corollary. *Let the assumption of* 3.3 *hold. Then there is a connected component* $\mathscr{B}$ *of* (A, f)*, which is unbounded.*

3.6. Lemma. *Let* $\mathscr{V}_1, \mathscr{V}_2 \in Ant$*,* $(A, f) \in RP(\mathscr{V}_1 \cup \mathscr{V}_2)$*. There is a connected component* $\mathscr{B}$ *of* (A, f)*, which is unbounded.*

Proof. By assumption, $(A, f) \in R(A', f)$, where $(A', f) \in P(\mathscr{V}_1 \cup \mathscr{V}_2)$. Then 3.3 implies that $s_f(x') \neq \infty$ for each $x' \in A'$, thus $s_f(x) \neq \infty$ for each $x \in A$, because $(A, f) \in R(A', f)$. Further, 3.5 yields that there is a connected component $\mathscr{B}'$ of (A', f) such that $\mathscr{B}'$ is unbounded. Consider a retraction endomorphism φ of (A', f) onto (A, f) and let $\mathscr{B}$ be the connected component of (A, f) with $\varphi(\mathscr{B}') \subseteq \mathscr{B}$. Then it is obvious that $\mathscr{B}$ is unbounded. □

3.7. Theorem. *The collection Ant is closed with respect to the operations of* inf *and* sup.

Proof. It suffices to prove that if $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, then $\sup\{\mathscr{V}_1, \mathscr{V}_2\} \in Ant$. Let $\mathscr{V}_1, \mathscr{V}_2 \in \mathscr{V}$. By 1.1, $\sup\{\mathscr{V}_1, \mathscr{V}_2\} = RP(\mathscr{V}_1 \cup \mathscr{V}_2)$. Let $(A, f) \in RP(\mathscr{V}_1 \cup \mathscr{V}_2)$. We have $(A, f) \in R(A', f)$ for some $(A', f) \in P(\mathscr{V}_1 \cup \mathscr{V}_2)$. Then 3.3 implies that $s_f(x') \neq \infty$ for each $x' \in A'$ and since (A, f) is isomorphic to a subalgebra of (A', f), $s_f(x) \neq \infty$ for each $x \in A$ as well. Further, there is a connected component $\mathscr{B}$ of (A, f) which is unbounded in view of 3.6. Hence 3.2 yields that $\sup\{\mathscr{V}_1, \mathscr{V}_2\} \in Ant$. □

In [5], for each ordinal α, a connected monounary algebra $\mathscr{P}_\alpha = (P_\alpha, g)$ was described such that there are distinct elements $p_\alpha, c_\alpha \in P_\alpha$ with the following properties:

(a) $g(p_\alpha) = g(c_\alpha) = c_\alpha$;
(b) if $x \in P_\alpha - \{c_\alpha\}$, then there is $n \in \mathbb{N} \cup \{0\}$ with $g^n(x) = p_\alpha$;
(c) $s_g(p_\alpha) = \alpha$.

4.1. Notation. *For $\alpha \in$ Ord and $n \in \mathbb{N}$ we denote $n_\alpha = (n, \alpha)$. Let $\alpha \in$ Ord. Put $\mathbb{N}_\alpha = \{n_\alpha \colon n \in \mathbb{N}\}$. Further, we denote by $\mathscr{A}_\alpha = (A_\alpha, f)$ the monounary algebra such that*

$$\begin{aligned} A_\alpha &= \mathbb{N}_\alpha \cup P_\alpha, \\ f(n_\alpha) &= (n+1)_\alpha \text{ for each } n \in \mathbb{N}, \\ f(x) &= \begin{cases} g(x) & \text{if } x \in P_\alpha - \{c_\alpha\}, \\ 1_\alpha & \text{if } x = c_\alpha. \end{cases} \end{aligned}$$

4.2. Lemma. *If α is an infinite ordinal, then $\mathscr{A}_\alpha$ is unbounded.*

P r o o f. The assertion follows from the definition of $\mathscr{A}_\alpha$ and from the definition of the degree. $\square$

4.3. Lemma. *If α is an infinite ordinal, then $V(\mathscr{A}_\alpha) \in$ Ant.*

P r o o f. Let α be an infinite ordinal. We will apply 3.2. Suppose that $(A, f) \in V(\mathscr{A}_\alpha)$. By 1.1, $(A, f) \in RP(\mathscr{A}_\alpha)$, i.e., there are a nonempty set I and a monounary algebra (D, f) such that

$$(A, f) \in R(D, f), \tag{1}$$

$$(D, f) = \mathscr{A}_\alpha^{\operatorname{card} I}. \tag{2}$$

Let $x \in A$. By (1), (A, f) is isomorphic to a subalgebra of (D, f), thus the property (B) of $s_f(x)$ implies that there is $y \in D$ with

$$s_f(x) \leqslant s_f(y). \tag{3}$$

Let $y(i)$ be the i–th projection of y into A_α. We have

$$s_f(y) \leqslant s_f(y(i)). \tag{4}$$

Further, 4.1 implies $s_f(y(i)) \neq \infty$, thus (3) and (4) yield

(5) $$s_f(x) \neq \infty \text{ for each } x \in A.$$

Define an element $d \in D$ by putting $d(i) = 1_\alpha$ for each $i \in I$. The algebra $\mathscr{A}_\alpha$ is unbounded, thus for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with

(6) $$f^{-(m+n)}(f^m(1_\alpha)) \neq \emptyset.$$

Since $\operatorname{card} f^m(1_\alpha) = 1$ for each $m \in \mathbb{N}$, the relation (6) implies

(6') $$f^{-n}(1_\alpha) \neq \emptyset,$$

i.e.,

(6″) $$f^{-n}(d(i)) \neq \emptyset.$$

Thus we obtain

(7) $$f^{-n}(d) \neq \emptyset \text{ for each } n \in \mathbb{N}.$$

Then obviously $f^{-(n+1)}(f(d)) \neq \emptyset$ for each $n \in \mathbb{N}$, i.e.,

(8) $$\text{there is } d \in D \text{ such that if } n \in \mathbb{N}, \text{ then there is } m(=1) \text{ with } f^{-(n+m)}(f^m(d)) \neq \emptyset.$$

According to (1), (8) holds in (A, f) as well, thus in view of 3.2, (5) and (8) yield that $\mathscr{V}$ is antiatomic. □

4.4. Lemma. *Let α be an ordinal, $x \in A_\alpha$. Then there is $n \in \mathbb{N} \cup \{0\}$ such that $s_f(x) \leqslant \alpha + n$.*

Proof. We have $x \in \mathbb{N}_\alpha \cup P_\alpha$. If $x \in P_\alpha - \{c_\alpha\}$, then the definition of $s_f(x)$ implies $s_f(x) = s_g(x) \leqslant s_g(p_\alpha) = \alpha$. If $x = c_\alpha$, then $s_f(x) = s_f(c_\alpha) = s_f(p_\alpha) + 1 = \alpha + 1$. If $x = k_\alpha, k \in \mathbb{N}$, then $s_f(x) = \alpha + k + 1$. □

4.5. Lemma. *Let α be an ordinal. If $(C, f) \in V(\mathscr{A}_\alpha), x \in C$, then there is $n \in \mathbb{N} \cup \{0\}$ such that $s_f(x) \leqslant \alpha + n$.*

Proof. Since $V(\mathscr{A}_\alpha) = RP(\mathscr{A}_\alpha)$, it suffices to show that if $x \in A_\alpha^{\operatorname{card} I}$, $I \neq \emptyset$, then $s_f(x) \leqslant \alpha + n$ for some $n \in \mathbb{N} \cup \{0\}$. Let $x \in A_\alpha^{\operatorname{card} I}$, $I \neq \emptyset$. Then

$$s_f(x) \leqslant s_f(x(i)) \text{ for each } i \in I$$

and 4.4 yields that there is $n \in \mathbb{N} \cup \{0\}$ with $s_f(x) \leqslant \alpha + n$. □

4.6. Lemma. *There is a collection $\mathscr{O}$ of ordinals such that*

(i) *$\mathscr{O}$ is a proper class,*
(ii) *if $\alpha \in \mathscr{O}$, then α is infinite,*
(iii) *if $\alpha, \beta \in \mathscr{O}, \alpha \neq \beta, k \in \mathbb{N}$, then $\alpha \neq \beta + k$.*

Proof. The assertion is obvious. □

In what follows let $\mathscr{O}$ be as in 4.6.

4.7. Lemma. *Let $\alpha, \beta \in \mathscr{O}$. If $\alpha < \beta$, then $\mathscr{A}_\beta \notin V(\mathscr{A}_\alpha)$.*

Proof. If $\mathscr{A}_\beta \in V(\mathscr{A}_\alpha)$, then according to 4.5 the relation $p_\beta \in \mathscr{A}_\beta$ implies

(1) $$s_f(p_\beta) \leqslant \alpha + n \text{ for some } n \in \mathbb{N} \cup \{0\}.$$

Further, by the definition of $\mathscr{A}_\beta$, we have

(2) $$s_f(p_\beta) = \beta.$$

Then(1), (2) and the assumption yield

(3) $$\alpha < \beta \leqslant \alpha + n.$$

Thus $\beta = \alpha + k$ for some $k \in \mathbb{N}$, which contradicts 4.6. □

4.8. Lemma. *Let $\alpha, \beta \in \mathscr{O}$. If $\alpha > \beta$, then $\mathscr{A}_\beta \notin V(\mathscr{A}_\alpha)$.*

Proof. Suppose that $\alpha > \beta$ and $\mathscr{A}_\beta \in V(\mathscr{A}_\alpha)$. Then $\mathscr{A}_\beta \in RP(\mathscr{A}_\alpha)$ and there are $I \neq \emptyset$ and $\mathscr{A}'_\beta = (A'_\beta, f) \cong (A_\beta, f)$ such that $\mathscr{A}'_\beta$ is a retract of $\mathscr{A}_\alpha^{\operatorname{card} I}$. Denote by φ a retraction endomorphism of $\mathscr{A}_\alpha^{\operatorname{card} I}$ onto $\mathscr{A}'_\beta$. Further let $p \in A_\alpha^{\operatorname{card} I}$, where $p(i) = p_\alpha$ for each $i \in I$. Then

(1) $$s_f(p) = \alpha.$$

Since φ is an endomorphism, (1) implies

(2) $$s_f(\varphi(p)) \geqslant s_f(p) = \alpha.$$

Further, $\mathscr{A}'_\beta \cong \mathscr{A}_\beta$, thus 4.4 yields

(3) $$\text{if } x \in \mathscr{A}'_\beta, \text{ then } s_f(x) \leqslant \beta + n \text{ for some } n \in \mathbb{N} \cup \{0\}.$$

By (2) and (3) we obtain

$$\alpha \leqslant s_f(\varphi(p)) \leqslant \beta + n \text{ for some } n \in \mathbb{N} \cup \{0\}. \tag{4}$$

According to (4) the assumption $\alpha > \beta$ implies

$$\beta < \alpha \leqslant \beta + n \text{ for some } n \in \mathbb{N} \cup \{0\},$$

which contradicts the relation $\{\alpha, \beta\} \subseteq \mathscr{O}$. □

4.9. Corollary. *If $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$, then $\mathscr{A}_\beta \notin V(\mathscr{A}_\alpha)$.*

4.10. Corollary. *If $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$, then $V(\mathscr{A}_\beta) \nsubseteq V(\mathscr{A}_\alpha)$.*

4.11. Theorem. *There is a proper class $\mathscr{O}$ of ordinals such that for each $\alpha \in \mathscr{O}$ there exists $\mathscr{W}_\alpha \in Ant$ with the property that if $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$, then $\mathscr{W}_\alpha \nsubseteq \mathscr{W}_\beta$.*

P r o o f. Let $\mathscr{O}$ be as in 4.6. Consider all $\mathscr{A}_\alpha$, $\alpha \in \mathscr{O}$ and put, for $\alpha \in \mathscr{O}$,

$$\mathscr{W}_\alpha = V(\mathscr{A}_\alpha).$$

According to 4.3 and 4.6, if $\alpha \in \mathscr{O}$, then $\mathscr{W}_\alpha \in Ant$. Let $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$. By 4.10 we have $\mathscr{W}_\alpha \nsubseteq \mathscr{W}_\beta$. □

5. Large chain of antiatomic retract varieties

In this section we will apply the algebras $\mathscr{A}_\alpha$ for $\alpha \in \mathscr{O}$ defined in the previous section.

5.1. Notation. *If $\alpha \in \mathscr{O}$, then we denote*

$$\mathscr{V}_\alpha = V(\{\mathscr{A}_\beta \colon \beta \leqslant \alpha\}).$$

5.2. Lemma. *If $\alpha, \beta \in \mathscr{O}, \beta < \alpha$, then $\mathscr{V}_\beta \subseteq \mathscr{V}_\alpha$.*

P r o o f. The assertion follows from 5.1. □

5.3. Lemma. *If $\alpha \in \mathscr{O}$, then $\mathscr{V}_\alpha \in Ant$.*

Proof. We shall prove the assertion by means of 3.2. Let $\alpha \in \mathscr{O}, (A,f) \in \mathscr{V}_\alpha$. By 1.1,

$$(A,f) \in RP(\{\mathscr{A}_\beta\colon\ \beta \leqslant \alpha\}),$$

i.e., there are a nonempty set I and monounary algebras (D,f) and (B_i,f) for each $i \in I$ such that

$$(A,f) \in R(D,f), \tag{1}$$

$$(D,f) = \prod_{i\in I}(B_i,f), \tag{2}$$

$$\{(B_i,f)\colon\ i \in I\} \subseteq \{\mathscr{A}_\beta\colon\ \beta \leqslant \alpha\}. \tag{3}$$

Let $x \in A$. Since (A,f) is isomorphic to a subalgebra of (D,f), by (B) there is $y \in D$ with $s_f(x) \leqslant s_f(y)$. Further, (A) yields that $s_f(y) \leqslant s_f(y(i))$ for each $i \in I$. According to 3.2, (B_i,f) is unbounded for $i \in I$, hence $s_f(y(i)) \neq \infty$, thus $s_f(x) \neq \infty$, too.

The relation (3) implies that if $i \in I$, then $(B_i,f) = \mathscr{A}_{\beta_i}$ for some $\beta_i \in \mathscr{O}$. Consider the element $d \in D$ such that $d(i) = 1_{\beta_i}$ for each $i \in I$. The remaining part of the proof is analogous as in 4.3. □

5.4. Lemma. *Let $\alpha \in \mathscr{O}$, $x \in D, (D,f) \in \mathscr{V}_\alpha$. Then there is $n \in \mathbb{N} \cup \{0\}$ such that $s_f(x) \leqslant \alpha + n$.*

Proof. It suffices to show that if $I \neq \emptyset$ and (D,f) satisfies (2) and (3) of 5.3, then the assertion is valid. In the case under consideration we have

$$s_f(x) \leqslant s_f(x(i)) \text{ for each } i \in I.$$

By 4.4, $s_f(x(i)) \leqslant \beta_i + n$ for some $n \in \mathbb{N} \cup \{0\}$, where $\beta_i \leqslant \alpha$, hence we obtain $s_f(x) \leqslant \alpha + n$ for some $n \in \mathbb{N} \cup \{0\}$. □

5.5. Lemma. *Let $\alpha, \beta \in \mathscr{O}$. If $\beta < \alpha$, then $\mathscr{A}_\alpha \notin \mathscr{V}_\beta$.*

Proof. Suppose that $\beta < \alpha$ and $\mathscr{A}_\alpha \in \mathscr{V}_\beta$. By the definition of $\mathscr{A}_\alpha$ we have

$$s_f(p_\alpha) = \alpha. \tag{1}$$

Further, 5.4 implies

$$s_f(p_\alpha) \leqslant \beta + n \text{ for some } n \in \mathbb{N} \cup \{0\}. \tag{2}$$

Thus $\beta < \alpha \leqslant \beta + n$, $\alpha = \beta + k$ for some $k \in \mathbb{N}$, which contradicts 4.6. □

5.6. Corollary. *If $\alpha, \beta \in \mathscr{O}, \beta < \alpha$, then $\mathscr{V}_\beta \subsetneqq \mathscr{V}_\alpha$.*

Proof. It is a consequence of 5.2 and 5.5. □

5.7. Theorem. *There is a proper class $\mathscr{O}$ of ordinals such that for each $\alpha \in \mathscr{O}$ there exists $\mathscr{V}_\alpha \in Ant$ with the property that if $\alpha, \beta \in \mathscr{O}$, $\beta < \alpha$, then $\mathscr{V}_\beta \subsetneqq \mathscr{V}_\alpha$.*

Proof. The assertion follows from 5.3 and 5.6. □

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