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SOME RESULTS ON PROJECTION OF PLANAR SETS

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Abstract. In this paper we define certain types of projections of planar sets and study some properties of such projections.

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Introduction

In 1920, Steinhaus [10] proved that if A and B are two linear sets of positive Lebesgue measure, then their difference set $A-B=\{a-b\colon\ a\in A, b\in B\}$ contains an interval. The category analogue of this result holds when the sets have the property of Baire. Ceder and Ganguly [3] strengthened the above results applying various kinds of projections of planar sets. Later some papers ([1], [6]) were devoted to study properties of various projections of planar sets. In 1962, Bose Majumder [2] proved that if A and B are two linear sets (with non-zero abscissae) having positive Lebesgue measure, then the ratio set $R(A,B)=\{\frac{a}{b}$ or $\frac{b}{a}\colon\ a\in A, b\in B\}$ of A and B contains at least one whole interval. In the present paper we introduce some new types of projections of big planar sets (in the sense of measure and category) and strengthen the above result of Bose Majumder. Some descriptive properties of such projections of planar sets are also obtained.

Terminology

Let $(a,b)\in\mathbb{R}^2$ and $E\subset\mathbb{R}^2$. By the (a,b)-projection of E we mean the set $P(a,b,E)=\{c\in\mathbb{R}\colon\ gr.\{y-b=c(x-a)\}\cap E\neq\emptyset\}$, i.e. the set of all real numbers c for which $y-b=c(x-a)$ holds for some $(x,y)\in E$. The (a,b)-measure projection of E is the set $M(a,b,E) = \{c \in \mathbb{R}\colon\ \mu(dom.([gr.\{y-b=c(x-a)\}] \cap E)) > 0\}$ where μ denotes the Lebesgue measure. The (a,b)-category projection of E is meant to be the set

$$C(a,b,E) = \{c \in \mathbb{R}\colon\ dom.(gr.\{y-b=c(x-a)\} \cap E) \text{ is of the second category}\}.$$

A set S is said to have the property of Baire if $S = G\Delta F$ where G is an open set, F is a set of the first category while Δ stands for the symmetric difference. If $t \in \mathbb{R}$ and $B \subset \mathbb{R}$ we define the set $tB = \{tb\colon\ b \in B\}$.

Note. In what follows, we assume that $a \notin A$.

We first prove some descriptive properties of $P(a,b,A\times B)$.

Theorem 1. *If A, B are compact subsets of $\mathbb{R}$ and $(a,b) \in \mathbb{R}^2$ with $a \notin A$, then $P(a,b,A\times B)$ is compact.*

P r o o f. Since A and B are compact sets, it follows in the same way as in paper [9] that $P(a,b,A\times B)$ is bounded. Let c be a limit point of $P(a,b,A\times B)$. Then there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of elements of $P(a,b,A\times B)$ such that $c_n \to c$ as $n \to \infty$. Now for each $c_n \in P(a,b,A\times B)$ there exist $x_n \in A$, $y_n \in B$ such that, $y_n - b = c_n(x_n - a)$. Since A, B are compact, we get subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that $x_{n_k} \to p \in A$ and $y_{n_k} \to q \in B$ as $k \to \infty$.

Also $y_{n_k} - b = c_{n_k}(x_{n_k} - a)$. Taking limit as $k \to \infty$ we have $q - b = c(p-a)$. Thus $c \in P(a,b,A\times B)$. Hence the result is proved. □

Remark. If A, B are closed sets, then $P(a,b,A\times B)$ need not be closed. This fact follows from the following example. Let $A = \{3^n\colon\ n \in \mathbb{N}\}$, $B = \{1\}$ ($\mathbb{N}$ is the set of natural numbers). Then A, B are closed sets, but $P(0,0,A\times B) = \{\frac{1}{3^n}\colon\ n \in \mathbb{N}\}$ is not a closed set.

Theorem 2. *If A and B are measurable subsets of $\mathbb{R}$ such that $\mu(A) > 0$ and B is of full measure, then $P(a,b,A\times B) = \mathbb{R}\setminus\{0\}$ for any $(a,b) \in \mathbb{R}^2$ where $a \notin A$ and $b \notin B$.*

P r o o f. For any $c \neq 0$, the set $b + c(A-a)$ is of positive measure. Since B is of full measure, we have $\mu(B') = 0$, where $''$ denotes the complement with respect to $\mathbb{R}$. It follows directly from definition that $P(a,b,A\times B)' = \{c\colon\ b + c(A-a) \subseteq B'\} = \{0\}$. Therefore $P(a,b,A\times B) = \mathbb{R}\setminus\{0\}$. □

The category analogue of the above theorem is also true.

Theorem 3. *If $(a,b) \in \mathbb{R}^2$ with $a \notin A$, $b \notin B$, A is a set of the second category and B is residual in $\mathbb{R}$ then $P(a,b,A\times B) = \mathbb{R}\setminus\{0\}$.*

Proof. As A is of the second category, so is $b + c(A-a)$ for any real number $c \neq 0$. Since B is residual in $\mathbb{R}$, its complement B' is of the first category. Hence, $P(a,b,A\times B)' = \{c\colon\ b+c(A-a) \subseteq B'\} = \{0\}$. Therefore, $P(a,b,A\times B) = \mathbb{R}\setminus\{0\}$. □

Note. The conclusion of Theorem 3 is not valid if A and B are sets of just the second category. This is evident from the following example.

Example. Let $\{G_\alpha\colon\ \alpha < \omega_c\}$ be a well ordering of all residual G_δ subsets of $\mathbb{R}, \omega_c$ being the first uncountable ordinal. Let us choose for all $\beta < \alpha$, two subsets A_β and B_β of G_β such that no three points of $A_\beta \times B_\beta$ are collinear. Since the set of all lines joining pairs of points of $A_\beta \times B_\beta$ with $\beta < \alpha$ has power less than c (the power of the continuum) we can find a direction not parallel to any of these lines. Hence by [8] some line in this direction meets $G_\alpha \times G_\alpha$ in a set of the second category and therefore in a set of power c (see [8]). We can choose therefore two subsets A_α and B_α of G_α such that no point of $A_\alpha \times B_\alpha$ is collinear with any two points of $A_\beta \times B_\beta$ with $\beta < \alpha$. The sets A_α and B_α are of the second category since their complements contain no residual G_δ subset of $\mathbb{R}$. Thus we get two linear sets A_α, B_α of the second category such that no three points of $A_\alpha \times B_\alpha$ are collinear and hence the sets A_α, B_α serve our purpose.

Theorem 4. *If A, B are non-empty open subsets of $\mathbb{R}$, then $P(a,b,A\times B)$ is a non-empty open set for any $(a,b) \in \mathbb{R}^2$ with $a \notin A$.*

Proof. Let $d \in P(a,b,A\times B)$. Then $\exists (x,y) \in A\times B$ such that $y-b = d(x-a)$. Since A and B are open sets we can find open intervals I_x and I_y of length 2ε with centres at x and y respectively such that $I_x \subseteq A$ and $I_y \subseteq B$.

Let $I = (d - \frac{\varepsilon}{|x-a|},\ d + \frac{\varepsilon}{|x-a|})$. By routine calculation we see that $b + c(x-a) \in I_y \subseteq B$ for all $c \in I$. This implies that $I \subseteq P(a,b,A\times B)$. Hence the result. □

Theorem 5. *If A, B are compact subsets of $\mathbb{R}$ and $(a,b) \in \mathbb{R}^2$ then the (a,b)-measure projection of $A\times B$ is a Borel set of additive class one, provided $a \notin A$.*

Proof. $P(a,b,A\times B)$ is compact by Theorem 1. Let γ and δ be the g.l.b. and l.u.b. of $P(a,b,A\times B)$, respectively. Let us define a function $f\colon\ [\gamma,\delta] \to \mathbb{R}$ by $f(c) = \mu(\{x \in A\colon\ \exists y \in B \text{ with } y-b = c(x-a)\})$. Then $M(a,b,A\times B) = \{c \in [\gamma,\delta]\colon\ f(c) > 0\}$. Let $\{c_k\}_{k=1}^{\infty}$ be a sequence in $[\gamma,\delta]$ such that $c_k \to c \in [\gamma,\delta]$ as

$k \to \infty$. Let us choose sets $Z_k = \{x \in A\colon\ \exists y \in B$ with $y - b = c_k(x-a)\}$ for every $k = 1, 2, \ldots$. We first show that

$$(1) \qquad LsZ_k \subset \{x \in A\colon\ \exists y \in B \ \text{ with } y - b = c(x-a)\}$$

where LsZ_k is the Upper Limit of the sequence $\{Z_k\}_{k=1}^{\infty}$ of sets as defined in [7]. Also, $LsZ_k \supset \overline{\lim} Z_k$ [see 7, p. 337].

Let $p \in LsZ_k$. Then there is a subsequence $\{c_{k_i}\}_{i=1}^{\infty}$ of $\{c_k\}_{k=1}^{\infty}$ such that $x_i \in Z_{k_i}$ for each i and $x_i \to p$ as $i \to \infty$.

Now $x_i \in Z_{k_i}$ implies that there exists $y_i \in B$ such that $y_i - b = c_{k_i}(x_i - a)$ for each i. Since B is compact, $\{y_i\}_{i=1}^{\infty}$ converges to $b + c(p-a) = q$ (say) $\in B$. Hence, $p \in \{x \in A\colon\ \exists\ y \in B$ with $y - b = c(x-a)\}$.

Also,

$$\begin{aligned} f(c) &= \mu(\{x \in A\colon\ \exists y \in B \ \text{ with } y - b = c(x-a)\}) \\ &\geqslant \mu(LsZ_k) \ \text{ by (1)} \\ &\geqslant \mu(\overline{\lim} Z_k) \\ &\geqslant \overline{\lim} \mu(Z_k) \ \text{ (By Fatou's lemma)} \\ &= \overline{\lim} f(c_k). \end{aligned}$$

Thus f is an upper semicontinuous function. Hence, there is a decreasing sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions defined over $[\gamma, \delta]$ such that $f(x) = \lim f_n(x)$, $\forall x \in [\gamma, \delta]$. Then, $M(a, b, A \times B) = \{c \in [\gamma, \delta]\colon\ f(c) > 0\} = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \{c \in [\gamma, \delta]\colon f_n(c) \geqslant \frac{1}{m}\}$ [5].

Since each f_n is continuous, the set $\{c \in [\gamma, \delta]\colon\ f_n(c) \geqslant \frac{1}{m}\}$ is a closed set. It follows that $M(a, b, A \times B)$ is a Borel set of additive class one [7]. □

Note. If A, B are compact subsets of $\mathbb{R}$ and $(a, b) \in \mathbb{R}^2$, $M(a, b, A \times B)$ need not be compact. For example if we consider $A = B = [0, 1]$, then $M(2, 0, A \times B) = (-1, 0]$, which is not compact.

Theorem 6. *If A, B are measurable subsets of $\mathbb{R}$ with finite positive Lebesgue measure and $(a, b) \in \mathbb{R}^2$ such that $a \notin A$ and $b \notin B$, then $M(a, b, A \times B)$ is an open subset of $P(a, b, A \times B)$.*

Proof. Since $b \notin B$, we have $0 \notin P(a, b, A \times B)$. Let us define a function $\varphi\colon\ \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $\varphi(c) = \mu(A \cap [a + c^{-1}(B - b)])$. Then φ is a continuous function (See [4]).

Now $M(a, b, A \times B) = \{c\colon\ \varphi(c) > 0\}$. Also $c \in M(a, b, A \times B)$ implies that $\varphi(c) > 0$. Since φ is continuous, there is a neighbourhood I_c of 'c' such that $\varphi(x) > 0$ for all $x \in I_c$. Then $I_c \subseteq M(a, b, A \times B)$. Hence the result. □

The result of Bose Majumder [2] mentioned earlier follows from the following corollary.

Corollary. *If A, B are linear sets (of points with non-zero abscissae) having positive Lebesgue measure, then $R(A,B)$ contains a non-empty open set.*

P r o o f. The proof follows from the above theorem and the fact that

$$R(A,B) = P(0,0,A\times B)\cup P(0,0,B\times A) \supseteq M(0,0,A\times B)\cup M(0,0,B\times A).$$

□

Theorem 7. *Let A and B be two linear sets of the second category such that each of A and B has the property of Baire. Then for any $(a,b)\in\mathbb{R}^2$ with $a\notin A$, the (a,b)-category projection of $A\times B$ is a non-empty open set.*

P r o o f. Without any loss of generality, let us assume that A has the Baire property. Then $A = G\Delta F$, where G is a non-empty open set and F is a set of the first category. Then

$$C(a,b,A\times B) = C(a,b,G\times B). \tag{1}$$

Let $Q = \{x\in\mathbb{R}\colon\ B \text{ is of the first category at } x\}$ and $J = \operatorname{Int}(\mathbb{R}\setminus Q)$. Then B is of the second category at each point of the open set J. Let $B_1 = B\cap J$. Then B_1 is of the second category at each point of J and $B\setminus B_1$ is a set of the first category. Hence,

$$C(a,b,G\times B) = C(a,b,G\times B_1). \tag{2}$$

In view of (1), (2) and Theorem 4, it is sufficient to prove that $C(a,b,G\times B_1) = P(a,b,G\times J)$. Indeed, since $B_1\subset J$, it is evident that $C(a,b,G\times B_1)\subseteq P(a,b,G\times B_1)\subseteq P(a,b,G\times J)$. Also, $c\in P(a,b,G\times J)\Rightarrow (c(G-a)+b)\cap J\neq\emptyset\Rightarrow (c(G-a)+b)\cap B_1$ is a set of the second category because B_1 is of the second category at each point of J. Therefore, $c\in C(a,b,G\times B_1)$. Thus $P(a,b,G\times J)\subseteq C(a,b,G\times B_1)$.

Hence the theorem is proved. □

Remark. The conclusion of the above theorem becomes false if we delete the property of Baire. This fact is evident from the example after Theorem 3.

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