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Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 1, 1–12

Persistent URL: `http://dml.cz/dmlcz/127461`

Czechoslovak Mathematical Journal, 49 (124) (1999), 1–12

CERTAIN TRANSFORMATIONS T_ω AND LEBESGUE MEASURABLE SETS OF POSITIVE MEASURE

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(Received October 30, 1995)

Suppose that for every ω belonging to a metric space Ω, there is a certain transformation T_ω transforming a Lebesgue measurable set in $\mathbb{R}_N$ (N-dimensional Euclidean space) into a Lebesgue measurable set in $\mathbb{R}_N$.

In [1] T. Neubrunn and T. Šalát introduced this type of transformations T_ω for measurable sets of the real line satisfying certain conditions.

In [2] M. Pal considered such transformations T_ω for measurable sets in $\mathbb{R}_N$ satisfying the following conditions which are equivalent to the conditions as introduced by T. Neubrunn and T. Šalát for transformations T_ω transforming a measurable set of the real line into a measurable set of the real line provided $N = 1$.

(I) There exists $\omega_0 \in \Omega$ such that for every closed ball $K = B[a, r] \subset \mathbb{R}_N$ with centre a and radius r and for every sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0,

$$\lim_{n\to\infty} \big[\sup\big\{|a - T_{\omega_n}(K)|\big\}\big] = r$$

holds where the symbol $\big\{|a - A|\big\}$, $a \in A$, $A \in \mathbb{R}_N$ denotes the set of all numbers $|a - x|$, $x \in A$.

(II) If E and F are measurable sets in $\mathbb{R}_N$ with $F \subset E$ then $T_\omega(F) \subset T_\omega(E)$ for every $\omega \in \Omega$.

(III) For $\omega_0 \in \Omega$ as in (I), for every sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0 and for every measurable set E,

$$\lim_{n\to\infty} \big|T_{\omega_n}(E)\big| = \big|T_{\omega_0}(E)\big| = |E|$$

Keywords: N-dimensioanal Euclidean space, metric space, Lebesque measure, T_ω transformation

MSC 1991: 28A05

where $|E|$ denotes the Lebesgue measure of the set E. Then among other results, M. Pal in [2] proved the following theorem which extends a theorem [Theorem 1.1] proved by T. Neubrunn and T. Šalát in [1].

Theorem 1 [2]. *Let $T_{\omega_n}(\omega_n \in \Omega)$ be transformations satisfying the conditions* (I), (II), (III) *and let the sequence $\{\omega_n\}$ converge to ω_0 (in Ω). Let A be a set of positive measure in $\mathbb{R}_N$. Then there exists a natural number N_0 such that for $n \geqslant N_0$, $A \cap T_{\omega_n}(A)$ is a set of positive measure.*

In [4] N.G. Saha and K.C. Ray also extended the above theorem considering a family of transformations like T_ω transforming a set in α^N (the collection of all measurable subsets of $\mathbb{R}_N$) into a set α^N and satisfying (II) and (III) mentioned above and the following condition stated in the form we consider here:

(I′) Let a, $b \in \mathbb{R}_N$ and let there exist a point $\omega_0 \in \Omega$ such that for every sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_0,

$$\lim_{n\to\infty} \left[\sup\left\{|b - T_{\omega_n}(K)|\right\}\right] = r$$

holds for every ball $K = B[a, r]$, $r > 0$.

The purpose of the paper is to study some properties of sets in $\mathbb{R}_N$ under transformations like T_ω which transform a measurable set in $\mathbb{R}_N$ into a measurable set in $\mathbb{R}_N$.

In this paper we relax the conditions (I) and (II) as considered by M. Pal in [2] and extend the result of Theorem 1 of [2] as stated earlier. Theorem 2 of this paper gives an extension of the result of a theorem in [4] by relaxing the condition (I′) as considered by N.G. Saha and K.C. Ray. We prove Theorem 3 in which a further extension of Theorem 2 is achieved. Before going into details we explain some notation used in the sequel.

Notation.

1) $B[c, \varrho]$ stands for the closed ball with centre c and radius ϱ while $B(c, \varrho)$ denotes the open ball with the same centre and radius.
2) $|x|$ denotes the norm of the vector $x \in \mathbb{R}_N$, while $|x|$ stands for the absolute value of the real number x.
3) $A \setminus B$ denotes the set of all points of the set A which do not belong to the set B.
4) For a set $A \subset \mathbb{R}_N$ and a non-zero real number α, αA is the set $\{\alpha x\colon\ x \in A\}$.
5) For sets A and B, $A - B$ denotes the difference set

$$\{a - b\colon\ a \in A, b \in B\}.$$

6) For a ball $K = B[a, r] \subset \mathbb{R}_N$, d_n stands for $\sup\left\{\left|a - T_{\omega_n}(K)\right|\right\}$.

Throughout the paper Ω is a metric space and the set under consideration is a set in $\mathbb{R}_N$. Now we introduce the conditions (i) and (iii) which are less restrictive than the conditions (I) and (III) as considered by M. Pal in [2].

(i) Let there exist a point $\omega_0 \in \Omega$ such that for every sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_0 and for every sequence $\{\alpha_n\}$ of non-zero real numbers converging to a non-zero real number α_0 and for every ball $K = B[a, r]$,

$$\lim_{n\to\infty} \left[\sup\left\{|\alpha_n a - T_{\omega_n}(\alpha_n K)|\right\}\right] = |\alpha_0| r.$$

(ii) For every $\omega \in \Omega$ and for sets E and F with $F \subset E$, let

$$T_\omega(F) \subset T_\omega(E).$$

(iii) For $\omega_0 \in \Omega$ as in (i), for every sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_0, for every sequence $\{\alpha_n\}$ of non-zero real numbers converging to a non-zero real number α_0 and for any measurable set E,

$$\lim_{n\to\infty} \left|T_{\omega_n}(\alpha_n E)\right| = \left|T_{\omega_0}(\alpha_0 E)\right| - |\alpha_0 E|.$$

If we take $\alpha_n = i$, $n = 1, 2, \ldots$, then (i) and (iii) reduce to the conditions (I) and (II).

In proving the results we follow the method as adopted by K. C. Ray in [3] with the necessary modifications. In this connection we note a well known result [5], viz. if T is a linear transformation in $\mathbb{R}_N$ given by $x'_i = \sum\limits_{j=1}^{N} a_{ij} x_j$, $i = 1, 2, \ldots, N$ and a_{ij}'s are real numbers and if E is a measurable set in $\mathbb{R}_N$, then $\left|T(E)\right| = \delta |E|$ where δ is the absolute value of the determinant of T.

As a Corollary of this result, it can be easily deduced that if α is a real number and E is a measurable set in $\mathbb{R}_N$, then $|\alpha E| = |\alpha|^N E$.

To substantiate our conditions (i) and (ii) as introduced above we present the following examples:

Examples. Let $\mathbb{R}$ be the real line. Then $\mathbb{R}$ is a metric space with the usual metric.

(i) Let $T_{\omega_n}(E) = E + \frac{1}{2n}$, E being a measurable subset of $\mathbb{R}$ and let $\alpha_n = \left(2 - \frac{1}{2n}\right)$. Then $\lim\limits_{n\to\infty} \alpha_n = 2$. So, for a closed interval $[1, 5]$

$$K = \sup_{x\in K} \left|3(\alpha_n) - T_{\omega_n}\left[\left(2 - \frac{1}{2n}\right)x\right]\right|,$$

where 3 is the middle point of the interval $[1, 5]$.

$$= \sup_{x\in K} \left|3\left(2 - \frac{1}{2n}\right) - \left[\left(2 - \frac{1}{2n}\right)x + \frac{1}{2n}\right]\right| = \left|6 - \left[\left(2 - \frac{1}{2n}\right)5 + \frac{1}{2n} + \frac{3}{2n}\right]\right|$$

$$= \left|6 - 10 + \frac{1}{n}\right| = \left|-4 + \frac{1}{n}\right| = \left|4 - \frac{1}{n}\right|.$$

So, $\lim_{n\to\infty} d_n = 4 = 2\cdot 2 = |\alpha_0|\cdot r$, where $r = \sup_{x\in[1,5]} \{|3-x|\}$.

(ii) Let $E=[0,1]$ and let $\{\alpha_n\}$ be a sequence of non-zero real numbers converging to a non-zero real number α_0. Then $\lim_{n\to\infty} \big|T_{\omega_n}\big(\alpha_n[0,1]\big)\big|$, where

$$\begin{aligned} T_{\omega_n}(E) = E + \frac{1}{n} &= \lim_{n\to\infty}\Big|[0,\alpha_n]+\frac{1}{n}\Big| \text{ or } \lim_{n\to\infty}\Big|[\alpha_n,0]+\frac{1}{n}\Big| \\ &\quad \text{provided } \alpha_n > 0 \text{ or } \alpha_n < 0 \\ &= \lim_{n\to\infty}|\alpha_n| = |\alpha_0|. \end{aligned}$$

Theorem 1. *Let there exist an element $\omega_0 \in \Omega$ such that for a sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0 and for a sequence $\{\alpha_n\}$ of non-zero real numbers converging to a non-zero real number α_0 such that the sequence $\{T_{\omega_n}\}$ of transformations satisfies the conditions* (I), (II), (III).

Let A be a set of positive Lebesgue measure in $\mathbb{R}_N$. Then there exists a positive integer N_0 such that for a system of p positive integers $N_1, N_2, \dots, N_p$ with $N_i > N_0$, the set

$$\frac{1}{\alpha_0}A \cap T_{\omega_{N_1}}\Big(\frac{1}{\alpha_{N_1}}A\Big) \cap T_{\omega_{N_2}}\Big(\frac{1}{\alpha_{N_2}}A\Big) \cap \ldots \cap T_{\omega_{N_p}}\Big(\frac{1}{\alpha_{N_p}}A\Big)$$

is a set of positive measure.

Proof. Since A is a set of positive Lebesgue measure, there exists a ball $K_1 = B[a,r]$, $a \neq 0$ such that

$$|K_1 \setminus A| < \varepsilon |K_1| \text{ where } 0 < \varepsilon < \frac{1}{(1+p)(1+2p)}.$$

Let

$$d_n = \sup\Big\{\Big|\frac{1}{\alpha_n}a - T_{\omega_n}\Big(\frac{1}{\alpha_n}K_2\Big)\Big|\Big\}$$

where $K_2 = B[a,s]$, $s = \big(\frac{p}{1+p}\big)^{1/N} r$.

Since $\lim_{n\to\infty} d_n = \frac{1}{|\alpha_0|}s$ and $\lim_{n\to\infty}\alpha_n = \alpha_0$, there exists a positive integer N_1 such that for every $n > N_1$

$$\Big|d_n - \frac{1}{|\alpha_0|}s\Big| < \frac{r-s}{2|\alpha_0|} \quad \text{and} \quad \Big|\frac{1}{\alpha_n} - \frac{1}{\alpha_0}\Big| < \frac{r-s}{2|a||\alpha_0|}.$$

According to (iii), $\lim_{n\to\infty}\Big|T_{\omega_n}\Big[\frac{1}{\alpha_n}(A\cap K_2)\Big]\Big| = \Big|\frac{1}{\alpha_0}(A\cap K_2)\Big|$. So there exists a positive integer N_2 such that for $n > N_2$,

$$\Big|\Big|T_{\omega_n}\Big[\frac{1}{\alpha_n}(A\cap K_2)\Big]\Big| - \Big|\frac{1}{\alpha_0}(A\cap K_2)\Big|\Big| < \varepsilon\frac{|K_1|}{|\alpha_0|^N}.$$

Let $N_0 = \max(N_1, N_2)$. So for $x \in K_2$ and for $n > N_0$, we have

$$\begin{aligned}\Big|\frac{1}{\alpha_0}a - T_{\omega_n}\Big(\frac{1}{\alpha_0}x\Big)\Big| &= \Big|\frac{1}{\alpha_0}a - \frac{1}{\alpha_n}a + \frac{1}{\alpha_n}a - T_{\omega_n}\Big(\frac{1}{\alpha_0}x\Big)\Big| \\ &\leqslant |a|\Big|\frac{1}{\alpha_0} - \frac{1}{\alpha_n}\Big| + \Big|\frac{1}{\alpha_n}a - T_{\omega_n}\Big(\frac{1}{\alpha_0}x\Big)\Big| \\ &\leqslant |a|\Big|\frac{1}{\alpha_0} - \frac{1}{\alpha_n}\Big| + d_n \leqslant |a|\frac{r-s}{2|a||\alpha_0|} + \frac{r+s}{2|\alpha_0|} = \frac{1}{|\alpha_0|}r.\end{aligned}$$

So, $T_{\omega_n}\big(\frac{1}{\alpha_n}K_2\big) \subset \frac{1}{\alpha_0}K_1$ for $n > N_0$ and hence

$$T_{\omega_n}\Big(\frac{1}{\alpha_n}(K_2 \cap A)\Big) \subset \frac{1}{\alpha_0}K_1 \qquad \text{for } n > N_0.$$

Let $N_1, N_2, \ldots, N_p$ be p positive integers with $N_i > N_0$. Also let

$$\begin{aligned}X = \Big[\frac{1}{\alpha_0}(A \cap K_1)\Big] \cap T_{\omega_{N_1}}\Big[\frac{1}{\alpha_{N_1}}(A \cap K_2)\Big] \cap T_{\omega_{N_2}}\Big[\frac{1}{\alpha_{N_2}}(A \cap K_2)\Big] \\ \cap \ldots \cap T_{\omega_{N_p}}\Big[\frac{1}{\alpha_{N_p}}(A \cap K_2)\Big].\end{aligned}$$

So,

$$X = \Big(\frac{1}{\alpha_0}K_1\Big) \setminus \Big[\Big(\Big(\frac{1}{\alpha_0}K_1\Big) \setminus \frac{1}{\alpha_0}A\Big) \cup \bigcup_{i=1}^{p}\Big\{\frac{1}{\alpha_0}K_1 \setminus T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(A \cap K_2)\Big)\Big\}\Big].$$

Hence

$$\begin{aligned}|X| &\geqslant \Big|\frac{1}{\alpha_0}K_1\Big| - \Big[\Big|\frac{1}{\alpha_0}K_1 \setminus \frac{1}{\alpha_0}A\Big| + \sum_{i=1}^{p}\Big\{\frac{1}{\alpha_0}K_1 \setminus T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(A \cap K_2)\Big)\Big\}\Big] \\ &= \Big|\frac{1}{\alpha_0}K_1\Big| - \Big[\Big|\frac{1}{\alpha_0}(K_1 \setminus A)\Big| + p\Big|\frac{1}{\alpha_0}K_1\Big| - \sum_{i=1}^{p}\Big|T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(A \cap K_2)\Big)\Big|\Big] \\ &= \Big|\frac{1}{\alpha_0}K_1\Big| - \Big|\frac{1}{\alpha_0}(K_1 \setminus A)\Big| - p\Big|\frac{1}{\alpha_0}K_1\Big| + \sum_{i=1}^{p}\Big|T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(A \cap K_2)\Big)\Big| \\ &> \Big|\frac{1}{\alpha_0}K_1\Big| - \Big|\frac{1}{\alpha_0}(K_1 \setminus A)\Big| - p\Big|\frac{1}{\alpha_0}K_1\Big| + p\Big|\frac{1}{\alpha_0}(A \cap K_2)\Big| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\ &= \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}|K_1 \setminus A| - p\frac{1}{|\alpha_0|^N}|K_1| + p\frac{1}{|\alpha_0|^N}|A \cap K_2| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\ &> \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\frac{1}{|\alpha_0|^N}|K_1| + p\frac{1}{|\alpha_0|^N}\big|K_2 \setminus (K_2 \setminus A)\big| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\ &= \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\frac{1}{|\alpha_0|^N}|K_1| + p\frac{1}{|\alpha_0|^N}\big[|K_2| - |K_2 \setminus A|\big] - p\varepsilon\frac{|K_1|}{|\alpha_0|^N}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\frac{1}{|\alpha_0|^N}\big[|K_1| - |K_2|\big] - p\frac{1}{|\alpha_0|^N}|K_2 \setminus A| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\
&> \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - \frac{1}{|\alpha_0|^N}|K_2| - p\frac{1}{|\alpha_0|^N}|K_1 \setminus A| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\
&> \frac{1}{|\alpha_0|^N}\big[|K_1| - |K_2|\big] - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\
&= \frac{1}{|\alpha_0|^N}\frac{1}{p}|K_2| - \frac{1}{|\alpha_0|^N}[1+2p]\varepsilon|K_1| \\
&= \frac{1}{|\alpha_0|^N}\frac{1}{p}\frac{p}{1+p}|K_1| - \frac{1}{|\alpha_0|^N}(1+2p)\varepsilon|K_1| \\
&= \frac{1}{|\alpha_0|^N}\Big[\frac{1}{1+p} - (1+2p)\varepsilon\Big]|K_1| \\
&= \frac{1+2p}{|\alpha_0|^N}\Big[\frac{1}{(1+p)(1+2p)} - \varepsilon\Big]|K_1| \\
&> 0, \quad \text{since } 0 < \varepsilon < \frac{1}{(1+p)(1+2p)}.
\end{aligned}
$$

Hence X is a set of positive measure and so by (II′), for $N_1, N_2, \dots, N_p \geqslant N_0$ the set

$$
\frac{1}{\alpha_0}A \cap T_{\omega_{N_1}}\Big(\frac{1}{\alpha_{N_1}}A\Big) \cap T_{\omega_{N_2}}\Big(\frac{1}{\alpha_{N_2}}A\Big) \cap \dots \cap T_{\omega_{N_p}}\Big(\frac{1}{\alpha_{N_p}}A\Big)
$$

is a set of positive measure.

This completes the proof. □

Corollary. *Let $\alpha_n = 1$, $n = 1, 2, \dots$. Then Theorem 1 of [2] follows immediately.*

Now we introduce the following condition which is equivalent to (I′).

Condition (i′)**.** Let $a, b \in \mathbb{R}_N$, let there exist $\omega_0 \in \Omega$ (a metric space) and a sequence $\{\omega_n\}$ converging to ω_0 such that for every ball $K = B[b, r]$ ($r > 0$) and for every sequence $\{\alpha_n\}$ of non-zero numbers converging to a non-zero real number α_0,

$$
\limsup_{n\to\infty}\big\{\big|\alpha_n a - T_{\omega_n}(\alpha_n K)\big|\big\} = |\alpha_0| r
$$

holds. For the following theorems we denote the condition (ii) as the condition (ii′), and the condition (iii) is replaced by the condition (iii′) which is the condition (iii) with $\omega_0 \in \Omega$ as in (i′).

Theorem 2. *Let A and B be sets of positive Lebesgue measure in $\mathbb{R}_N$ and let a and b be points of density of A and B, respectively. Let there exist an element $\omega_0 \in \Omega$ and a sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0 such that for a sequence*

$\{\alpha_n\}$ of non-zero real numbers converging to $\alpha_0 (\neq 0)$ the transformations T_{ω_n} satisfy the conditions (i′), (ii′), (iii′) *with respect to (a, b, ω_0). Then there exists a natural number N_0 such that for a system of p elements $\omega_{N_1}, \omega_{N_2}, \ldots, \omega_{N_p}$ of the sequence $\{\omega_n\}$ with $N_i > N_0$,*

$$\frac{1}{\alpha_0} A \cap T_{\omega_{N_1}}\Big(\frac{1}{\alpha_{N_1}} B\Big) \cap T_{\omega_{N_2}}\Big(\frac{1}{\alpha_{N_2}} B\Big) \cap \ldots \cap T_{\omega_{N_p}}\Big(\frac{1}{\alpha_{N_p}} B\Big)$$

is a set of positive measure.

Proof. Since A and B are sets of positive measure, there exist balls $K_A = B[a, r]$ $(a \neq 0)$ and $K_B = [b, r]$ such that $|K_A \setminus A| < \varepsilon |K_A|$, $|K_B \setminus B| < \varepsilon |K_B|$ where $0 < \varepsilon < \frac{1}{2p^2+2p+1}$. Let $\sup \big\{ \big| \frac{1}{\alpha_n} a - T_{\omega_n}\big(\frac{1}{\alpha_n} K_2\big) \big| \big\} = d_n$ where $K_2 = B[b, s]$,

$$s = \Big(\frac{p}{1+p}\Big)^{1/N} r.$$

Since $\lim\limits_{n\to\infty} d_n = \frac{1}{|\alpha_0|} s$ and $\lim\limits_{n\to\infty} \alpha_n = \alpha_0$, there exists a positive integer N_1 such that for every $n > N_1$ we have

$$\Big| d_n - \frac{1}{|\alpha_0|} s \Big| < \frac{r-s}{2|\alpha_0|} \quad \text{and} \quad \Big| \frac{1}{\alpha_n} - \frac{1}{\alpha_0} \Big| < \frac{r-s}{2|a||\alpha_0|}.$$

In virtue of (iii) $K_2 = B[a, s]$,

$$\lim_{n\to\infty} \Big| T_{\omega_n}\Big[\frac{1}{\alpha_n}(A \cap K_2')\Big] \Big| = \Big| \frac{1}{\alpha_0}(A \cap K_2') \Big|.$$

So, there exists a positive integer N_2 such that for $n > N_2$ we have

$$\Big| \Big| T_{\omega_n}\Big[\frac{1}{\alpha_n}(A \cap K_2')\Big] \Big| - \Big| \frac{1}{\alpha_0}(A \cap K_2') \Big| \Big| < \varepsilon_1$$

where $0 < \varepsilon_1 < \varepsilon \big| \frac{1}{\alpha_0} K_2 \big|$. Let $N_0 = \max(N_1, N_2)$. Then, for $x \in K_2$ and for $n > N_0$,

$$\begin{aligned} \Big| \frac{1}{\alpha_0} a - T_{\omega_n}\Big(\frac{1}{\alpha_0} x\Big) \Big| &= \Big| \frac{1}{\alpha_0} a - \frac{1}{\alpha_n} a + \frac{1}{\alpha_n} a - T_{\omega_n}\Big(\frac{1}{\alpha_n} x\Big) \Big| \\ &\leqslant |a| \Big| \frac{1}{\alpha_0} - \frac{1}{\alpha_n} \Big| + \Big| \frac{1}{\alpha_n} a - T_{\omega_n}\Big(\frac{1}{\alpha_n} x\Big) \Big| \\ &\leqslant |a| \frac{r-s}{2|a||\alpha_0|} + d_n \leqslant |a| \frac{r-s}{2|a||\alpha_0|} + \frac{r+s}{2|\alpha_0|} = \frac{1}{|\alpha_0|} r. \end{aligned}$$

So $T_{\omega_n}\big(\frac{1}{\alpha_n} K_2\big) \subset \frac{1}{\alpha_0} K_A$ and hence

$$T_{\omega_n}\Big(\frac{1}{\alpha_n}(K_2 \cap B)\Big) \subset \frac{1}{\alpha_0} K_A \qquad \text{for } n > N_0.$$

Let $N_1, N_2, \ldots, N_p$ be positive integers with $N_i > N_0$. Also let

$$X = \Big[\frac{1}{\alpha_0}(A \cap K_A)\Big] \cap T_{\omega_{N_1}}\Big[\frac{1}{\alpha_{N_1}}(B \cap K_2)\Big] \cap T_{\omega_{N_2}}\Big[\frac{1}{\alpha_{N_2}}(B \cap K_2)\Big] \cap \ldots \cap T_{\omega_{N_p}}\Big[\frac{1}{\alpha_{N_p}}(B \cap K_2)\Big].$$

Then

$$X = \Big(\frac{1}{\alpha_0}K_A\Big) \setminus \Big[\Big(\Big(\frac{1}{\alpha_0}K_A\Big) \setminus \frac{1}{\alpha_0}(K_A \cap A)\Big) \cup \bigcup_{i=1}^{p}\Big\{\frac{1}{\alpha_0}K_A \setminus T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(B \cap K_2)\Big)\Big\}\Big].$$

Hence

$$\begin{aligned}
|X| &\geqslant \Big|\frac{1}{\alpha_0}K_A\Big| - \Big[\Big|\frac{1}{\alpha_0}K_A \setminus \frac{1}{\alpha_0}A\Big| + \sum_{i=1}^{p}\Big\{\Big|\frac{1}{\alpha_0}K_A \setminus T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(B \cap K_2)\Big)\Big|\Big\}\Big] \\
&= \Big|\frac{1}{\alpha_0}K_A\Big| - \Big[\Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| + p\Big|\frac{1}{\alpha_0}K_A\Big| - \sum_{i=1}^{p}\Big|T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(B \cap K_2)\Big)\Big|\Big] \\
&= \Big|\frac{1}{\alpha_0}K_A\Big| - \Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| - p\Big|\frac{1}{\alpha_0}K_A\Big| + \sum_{i=1}^{p}\Big|T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(B \cap K_2)\Big)\Big| \\
&> \Big|\frac{1}{\alpha_0}K_A\Big| - \Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| - p\Big|\frac{1}{\alpha_0}K_A\Big| + p\Big|\frac{1}{\alpha_0}(B \cap K_2)\Big| - p\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}|K_A \setminus A| - p\frac{1}{|\alpha_0|^N}|K_A| + p\frac{1}{|\alpha_0|^N}|B \cap K_2| - p\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p\frac{1}{|\alpha_0|^N}|K_A| + p\frac{1}{|\alpha_0|^N}\big|K_2 \setminus (K_2 \setminus B)\big| - p\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p\frac{1}{|\alpha_0|^N}|K_A| + p\frac{1}{|\alpha_0|^N}\big[|K_2| - |K_2 \setminus B|\big] - p\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p\frac{1}{|\alpha_0|^N}\big[|K_A| - |K_2|\big] - p\frac{1}{|\alpha_0|^N}|K_2 \setminus B| - p\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - \frac{1}{|\alpha_0|^N}|K_2| - p\frac{1}{|\alpha_0|^N}|K_B \setminus B| - p\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}\big[|K_A| - |K_2|\big]\frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p\frac{1}{|\alpha_0|^N}\varepsilon|K_B| - p\varepsilon\frac{|K_2|}{|\alpha_0|^N} \\
&= \frac{1}{|\alpha_0|^N}\frac{1}{p}|K_2| - \frac{1}{|\alpha_0|^N}\varepsilon\frac{1+p}{p}|K_2| - p\frac{1}{|\alpha_0|^N}\varepsilon\frac{1+p}{p}|K_2| - \frac{p}{|\alpha_0|^N}\varepsilon|K_2| \\
&= \frac{1}{|ao|^N}\Big[\frac{1}{p} - \Big(\frac{1+p}{p} + 1 + p + p\Big)\varepsilon\Big]|K_2|
\end{aligned}$$

$$= \frac{1}{|\alpha_0|^N}\Big[\frac{1}{p} - \Big(\frac{1+p+p+2p^2}{p}\Big)\varepsilon\Big]|K_2|$$
$$= \frac{1+2p+2p^2}{|\alpha_0|^N p}\Big[\frac{1}{1+2p+2p^2} - \varepsilon\Big]|K_2|$$
$$> 0, \qquad \text{since } 0 < \varepsilon < \frac{1}{2p^2+2p+1}.$$

Hence X is a set of positive measure and so by (ii), for $N_1, N_2, \ldots, N_p \geqslant N_0$, the set

$$\frac{1}{\alpha_0}A \cap T_{\omega_{N_1}}\Big(\frac{1}{\alpha_{N_1}}B\Big) \cap T_{\omega_{N_2}}\Big(\frac{1}{\alpha_{N_2}}B\Big) \cap \ldots \cap T_{\omega_{N_p}}\Big(\frac{1}{\alpha_{N_p}}B\Big)$$

is a set of positive measure.

This completes the proof. □

Theorem 3. *Let A and $B_1, B_2, \ldots, B_p$ be sets of positive measure in $\mathbb{R}_N$ and let a and b_i $(i = 1, 2, \ldots, p)$ be points of density of A and B_i $(i = 1, 2, \ldots, p)$, respectively. Let there exist an element $\omega_0 \in \Omega$ and a sequence $\{\omega_n^i\}(\omega_n^i \in \Omega)$ $(i = 1, 2, \ldots, p)$ converging to ω_0 such that for a sequence $\{\alpha_n\}$ of non-zero real number converging to a non-zero real number α_0, the sequence of transformations $\{T_{\omega_n}i\}$ $(i = 1, 2, \ldots, p)$ satisfies the conditions* (i′), (ii′), (iii′) *with respect to (a, b_i, ω_0). Then there exists a natural number N_0 such that for a system of p^2 elements $\omega_{N_1}i$, $\omega_{N_2}i$, …, $\omega_{N_p}i$ of the sequence $\{T_{\omega_n}i\}$ with $N_k^i > N_0$ and for a system of p numbers α_{N_1}, α_{N_2}, …, α_{N_p} of the sequence $\{\alpha_n\}$ with $N_k > N_0$ the set*

$$\begin{aligned}\frac{1}{\alpha_0}A &\cap T_{\omega^1_{N_1^1}}\Big(\frac{1}{\alpha_{N_1}}B_1\Big) \cap T_{\omega^1_{N_2^1}}\Big(\frac{1}{\alpha_{N_2}}B_1\Big) \cap \ldots \cap T_{\omega^1_{N_p^1}}\Big(\frac{1}{\alpha_{N_p}}B_1\Big)\\ &\cap T_{\omega^2_{N_1^2}}\Big(\frac{1}{\alpha_{N_1}}B_2\Big) \cap T_{\omega^2_{N_2^2}}\Big(\frac{1}{\alpha_{N_2}}B_2\Big) \cap \ldots \cap T_{\omega^2_{N_p^2}}\Big(\frac{1}{\alpha_{N_p}}B_2\Big) \cap \ldots\\ &\cap T_{\omega^p_{N_1^p}}\Big(\frac{1}{\alpha_{N_1}}B_p\Big) \cap T_{\omega^p_{N_2^p}}\Big(\frac{1}{\alpha_{N_2}}B_p\Big) \cap \ldots \cap T_{\omega^p_{N_p^p}}\Big(\frac{1}{\alpha_{N_p}}B_p\Big)\end{aligned}$$

is a set of positive measure.

P r o o f. Since A and B_i $(i = 1, 2, \ldots, p)$ are sets of positive measure, there exist balls $K_A = B[a, r]$, $a \neq 0$ and $K_{B_i} = B[b_i, r]$ $(i = 1, 2, \ldots, p)$ such that

$$|K_A \setminus A| < \varepsilon|K_A| \quad \text{ and } \quad |K_{B_i} \setminus B_i| < \varepsilon|K_{B_i}|, \quad i = 1, 2, \ldots, p,$$

where $0 < \varepsilon < \frac{1+p}{1+p+2p^2}$. Let $\sup\big\{\big|\frac{1}{\alpha_n}a - T_{\omega_n}\big(\frac{1}{\alpha_n}K_2^i\big)\big|\big\} = d_n^i$ where $K_2^i = B[b_i, s]$,

$$s = \Big(\frac{p}{1+p}\Big)^{1/N} r.$$

Since $\lim_{n\to\infty} d_n^i = \frac{1}{|\alpha_0|}s$ and $\lim_{n\to\infty} \alpha_n = \alpha_0$, there exists a positive integer N_1 such that for every $n > N_1$,

$$\Big|d_n^i - \frac{1}{|\alpha_0|}s\Big| < \frac{r-s}{2|\alpha_0|} \quad \text{and} \quad \Big|\frac{1}{\alpha_n} - \frac{1}{\alpha_0}\Big| < \frac{r-s}{2|a||\alpha_0|}.$$

In virtue of (iii) we have

$$\lim_{n\to\infty} \Big|T_{\omega_n^i}\Big[\frac{1}{\alpha_n}(A\cap K_2^i)\Big]\Big| = \Big|\frac{1}{\alpha_0}(A\cap K_2^i)\Big|.$$

So there exists a positive integer N_2 such that for $n > N_2$,

$$\bigg|\Big|T_{\omega_n^i}\Big[\frac{1}{\alpha_n}(A\cap K_2^i)\Big]\Big| - \Big|\frac{1}{\alpha_0}(A\cap K_2^i)\Big|\bigg| < \varepsilon_1$$

where $0 < \varepsilon_1 < \varepsilon\big|\frac{1}{\alpha_0}K_2^i\big|$. Let $N_0 = \max(N_1, N_2)$. Then, for $x \in K_2^i$ and for $n > N_0$ we obtain

$$\begin{aligned}\Big|\frac{1}{\alpha_0}a - T_{\omega_n^i}\Big(\frac{1}{\alpha_0}x\Big)\Big| &= \Big|\frac{1}{\alpha_0}a - \frac{1}{\alpha_n}a + \frac{1}{\alpha_n}a - T_{\omega_n^i}\Big(\frac{1}{\alpha_0}x\Big)\Big| \\ &\leqslant |a|\Big|\frac{1}{\alpha_0} - \frac{1}{\alpha_n}\Big| + \Big|\frac{1}{\alpha_n}a - T_{\omega_n^i}\Big(\frac{1}{\alpha_n}x\Big)\Big| \\ &\leqslant |a|\frac{r-s}{2|a||\alpha_0|} + d_n^i \leqslant |a|\frac{r-s}{2|a||\alpha_0|} + \frac{r+s}{2|\alpha_0|} = \frac{1}{|\alpha_0|}r.\end{aligned}$$

Hence $T_{\omega_n^i}\big(\frac{1}{\alpha_n}K_2^i\big) \subset \frac{1}{\alpha_0}K_A$ and for $n > N_0$ and for $i = 1, 2, \ldots, p$. Let $N_1^i, N_2^i, \ldots, N_p^i$ and N_i $(i = 1, 2, \ldots, p)$ be positive integers with $N_k^i > N_0$ and $N_i > N_0$. Also let

$$\begin{aligned}X = \Big[\frac{1}{\alpha_0}(A\cap K_A)\Big] &\cap T_{\omega^1_{N_1^1}}\Big[\frac{1}{\alpha_{N_1}}(B\cap K_2^1)\Big] \cap T_{\omega^1_{N_2^1}}\Big[\frac{1}{\alpha_{N_2}}(B_1\cap K_2^1)\Big] \cap \ldots \\ &\cap T_{\omega^1_{N_p^1}}\Big[\frac{1}{\alpha_{N_p}}(B_1\cap K_2^1)\Big] \\ &\cap T_{\omega^2_{N_1^2}}\Big[\frac{1}{\alpha_{N_1}}(B_2\cap K_2^2)\Big] \cap T_{\omega^2_{N_2^2}}\Big[\frac{1}{\alpha_{N_2}}(B_2\cap K_2^2)\Big] \cap \ldots \\ &\cap T_{\omega^2_{N_p^2}}\Big[\frac{1}{\alpha_{N_p}}(B_2\cap K_2^2)\Big] \cap \ldots \\ &\cap T_{\omega^p_{N_1^p}}\Big[\frac{1}{\alpha_{N_2}}(B_p\cap K_2^p)\Big] \cap T_{\omega^p_{N_2^p}}\Big[\frac{1}{\alpha_{N_2}}(B_p\cap K_2^p)\Big] \cap \ldots \\ &\cap T_{\omega^p_{N_p^p}}\Big[\frac{1}{\alpha_{N_p}}(B_p\cap K_2^p)\Big].\end{aligned}$$

Then

$$X = \Big(\frac{1}{\alpha_0}K_A\Big)\backslash\Big\{\Big[\Big(\frac{1}{\alpha_0}K_A\Big)\backslash\frac{1}{\alpha_0}(K_A\cap A)\Big] \cup \bigcup_{i=1}^{p}\bigcup_{j=2}^{p}\Big[\frac{1}{\alpha_0}K_A\backslash T_{\omega^i_{N_j^i}}\Big(\frac{1}{\alpha_{N_j}}(B_i\cap K_2^i)\Big)\Big]\Big\}.$$

Hence

$$
\begin{aligned}
|X| &\geqslant \Big|\frac{1}{\alpha_0}K_A\Big| - \Big[\Big|\frac{1}{\alpha_0}K_A \setminus \frac{1}{\alpha_0}A\Big| + \sum_{i=1}^{p}\sum_{j=1}^{p}\Big\{\Big|\frac{1}{\alpha_0}K_A \setminus T_{\omega^i_{N^i_j}}\Big(\frac{1}{\alpha_{N_j}}(B_i \cap K_2^i)\Big)\Big|\Big\}\Big] \\
&= \Big|\frac{1}{\alpha_0}K_A\Big| - \Big[\Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| + p^2\Big|\frac{1}{\alpha_0}K_A\Big| - \sum_{i=1}^{p}\sum_{j=1}^{p}\Big|T_{\omega^i_{N^i_j}}\Big(\frac{1}{\alpha_{N_j}}(B_i \cap K_2^i)\Big)\Big|\Big] \\
&= \Big|\frac{1}{\alpha_0}K_A\Big| - \Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| - p^2\Big|\frac{1}{\alpha_0}K_A\Big| + \sum_{i=1}^{p}\sum_{j=1}^{p}\Big|T_{\omega^i_{N^i_j}}\Big(\frac{1}{\alpha_{N_j}}(B_i \cap K_2^i)\Big)\Big| \\
&> \Big|\frac{1}{\alpha_0}K_A\Big| - \Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| - p^2\Big|\frac{1}{\alpha_0}K_A\Big| + p\frac{1}{|\alpha_0|^N}\sum_{i=1}^{p}\big|[K_2^i \setminus \{K_2^i \setminus B_i\}]\big| - p^2\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}|K_A \setminus A| - p^2\frac{1}{|\alpha_0|^N}|K_A| \\
&\quad + p\frac{1}{|\alpha_0|^N}\sum_{i=1}^{p}\big[|K_2^i| - |K_2^i \setminus B_i|\big] - p^2\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p^2\frac{1}{|\alpha_0|^N}|K_A| + p\frac{1}{|\alpha_0|^N}\big[p|K_2| - \sum_{i=1}^{p}\varepsilon|K_{B_i}|\big] - p^2\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p^2\frac{1}{|\alpha_0|^N}|K_A| + p^2\frac{1}{|\alpha_0|^N}|K_2| - \frac{p^2\varepsilon}{|\alpha_0|^N}|K_A| - p^2\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - \frac{p^2}{|\alpha_0|^N}\big[|K_A| - |K_2|\big] - \frac{p^2}{|\alpha_0|^N}|K_A| - p^2\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}\frac{p}{1+p}|K_2| - \frac{1}{|\alpha_0|^N}\varepsilon\frac{p}{1+p}|K_2| - \frac{p^2}{|\alpha_0|^N}\Big[\frac{p}{1+p}|K_2| - |K_2|\Big] \\
&\quad - \frac{p^2\varepsilon}{|\alpha_0|^N}\frac{p}{1+p}|K_2| - p^2\frac{1\varepsilon}{\alpha_0^N}|K_2| \\
&= \frac{1}{|\alpha_0|^N}\frac{p}{1+p}|K_2| - \frac{1}{|\alpha_0|^N}\,\varepsilon\,\frac{p}{1+p}|K_2| - p\frac{p^2}{|\alpha_0|^N}\Big[\frac{p}{1+p} - 1\Big]|K_2| \\
&\quad - \frac{p^3}{|\alpha_0|^N}\,\varepsilon\,\frac{|K_2|}{1+p} - \frac{p^2}{|\alpha_0|^N}\varepsilon|K_2| \\
&= \frac{1}{|ao|^N}\Big[\frac{p}{1+p} + \frac{p^2}{1+p}\Big]|K_2| - \frac{1}{|\alpha_0|^N}\varepsilon\Big[\frac{p}{1+p} + \frac{p^3}{1+p} + p^2\Big]|K_2| \\
&= \frac{p}{|\alpha_0|^N}|K_2| - \frac{p}{|\alpha_0|^N}\varepsilon\frac{1+p+2p^2}{1+p}|K_2| \\
&= \frac{p}{|\alpha_0|^N}\frac{1+p+2p^2}{1+p}\Big[\frac{1+p}{1+p+2p^2} - \varepsilon\Big]|K_2| \\
&> 0, \qquad \text{since } 0 < \varepsilon < \frac{1+p}{1+p+2p^2}.
\end{aligned}
$$

Hence X is a set of positive measure and so by (II), for $N_1^i, N_2^i, \ldots, N_p^i$, $N_i \geqslant N_0$ $(i = 1, 2, \ldots, p)$ the set

$$\begin{aligned}\frac{1}{\alpha_0} A \cap T_{\omega^1_{N^1_1}}\Big(\frac{1}{\alpha_{N_1}} B_1\Big) \cap T_{\omega^1_{N^1_2}}\Big(\frac{1}{\alpha_{N_2}} B_1\Big) \cap \ldots \cap T_{\omega^1_{N^1_p}}\Big(\frac{1}{\alpha_{N_p}} B_1\Big) \\ \cap T_{\omega^2_{N^2_1}}\Big(\frac{1}{\alpha_{N_1}} B_2\Big) \cap T_{\omega^2_{N^2_2}}\Big(\frac{1}{\alpha_{N_2}} B_2\Big) \cap \ldots \cap T_{\omega^2_{N^2_p}}\Big(\frac{1}{\alpha_{N_p}} B_2\Big) \cap \ldots \\ \cap T_{\omega^p_{N^p_1}}\Big(\frac{1}{\alpha_{N_1}} B_p\Big) \cap T_{\omega^p_{N^p_2}}\Big(\frac{1}{\alpha_{N_2}} B_p\Big) \cap \ldots \cap T_{\omega^p_{N^p_p}}\Big(\frac{1}{\alpha_{N_p}} B_p\Big)\end{aligned}$$

is a set of positive measure.

This completes the proof. □

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