Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 1, 63–66

Persistent URL: `http://dml.cz/dmlcz/127467`

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ON AN EXTENSION OF FEKETE'S LEMMA

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(Received February 19, 1996)

Abstract. We show that if a real $n \times n$ non-singular matrix ($n \geqslant m$) has all its minors of order $m-1$ non-negative and has all its minors of order m which come from consecutive rows non-negative, then all mth order minors are non-negative, which may be considered an extension of Fekete's lemma.

MSC 2000: 15A15

Fekete's lemma (see [2] or [4, p. 59]) states that if an $n \times m$ matrix ($n \geqslant m$) has all its minors of order $m-1$ which come from the last $m-1$ columns and all mth order minors which come from consecutive rows positive, then all mth order minors are positive. In this note we find sufficient conditions for all mth order minors of an $n \times n$ non-singular square matrix ($n \geqslant m$) to be non-negative, which may be considered an extension of Fekete's lemma.

Definition. A rectangular matrix $A = \|a_{ik}\|$ ($i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$) over $\mathbb{R}$ is called *totally positive* (or *strictly totally positive*)—hereafter denoted by TP (or STP)—if all its minors of any order are non-negative (or positive). An $n \times n$ matrix over $\mathbb{R}$ is called *totally positive of order* m (or *strictly totally positive of order* m) and is denoted by TP_m (or STP_m) if all its minors of order $j \leqslant m$ are non-negative (or positive). Here $\mathbb{R}$ denotes the set of all real numbers and hereafter we shall use this notation.

This paper was supported in part by the research fund of the Natural Science Institute of Seoul Women's University, 1994.

We will denote the determinant formed from elements of the given matrix $A = \|a_{ik}\|$ $(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$ as follows:

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ k_1 & k_2 & \ldots & k_p \end{pmatrix} = \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_p} \\ a_{i_2k_1} & a_{i_2k_2} & \ldots & a_{i_2k_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_pk_1} & a_{i_pk_2} & \ldots & a_{i_pk_p} \end{vmatrix}.$$

We need the following well known Cauchy-Binet formula (see [3, p. 9]) for the proof of our main Theorem 2.

Cauchy-Binet formula. *Let A, B and C denote matrices of real numbers of orders $n \times m$, $n \times k$ and $k \times m$, respectively. If $A = BC$, then*

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ j_1 & j_2 & \ldots & j_p \end{pmatrix} = \sum_{1 \leqslant k_1 < \ldots < k_p \leqslant n} B\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ k_1 & k_2 & \ldots & k_p \end{pmatrix} C\begin{pmatrix} k_1 & k_2 & \ldots & k_p \\ j_1 & j_2 & \ldots & j_p \end{pmatrix}.$$

Lemma 1. *Suppose $n \geqslant m$. If a real $n \times n$ matrix $A = \|a_{ij}\|$ has all its minors of order $m - 1$ positive and all its minors of order m which come from consecutive rows positive, then all mth order minors are positive.*

P r o o f. Follows immediately from Fekete's lemma. $\square$

Theorem 2. *Suppose $n \geqslant m$. If a real $n \times n$ non-singular matrix $A = \|a_{ij}\|$ has all its minors of order $m - 1$ non-negative and all its minors of order m which come from consecutive rows non-negative, then all mth order minors are non-negative.*

P r o o f. Let H be an auxiliary $n \times n$ matrix such that

$$H = H(q) = \|q^{(i-j)^2}\| \quad (i, j = 1, 2, \ldots, n) \ \text{ for } \ 0 < q < 1.$$

$H \in STP$ follows from a theorem of Pólya (see [6, p. 49]). Let $U = AH$. Then

$$(1) \qquad U\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ j_1 & j_2 & \ldots & j_p \end{pmatrix} = \sum_{1 \leqslant r_1 < \ldots < r_p \leqslant n} A\begin{pmatrix} i_1 & \ldots & i_p \\ r_1 & \ldots & r_p \end{pmatrix} H\begin{pmatrix} r_1 & \ldots & r_p \\ j_1 & \ldots & j_p \end{pmatrix}$$

for $p = 1, 2, \ldots, n$ by the Cauchy-Binet formula. Since $A \in TP_{m-1}$ and $H \in STP$, $U \in TP_{m-1}$.

From the hypothesis,

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_{m-1} \\ r_1 & r_2 & \ldots & r_{m-1} \end{pmatrix} \geqslant 0.$$

Suppose that

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_{m-1} \\ r_1 & r_2 & \dots & r_{m-1} \end{pmatrix} = 0$$

for every $r_1, r_2, \dots, r_{m-1}$ such that $1 \leqslant r_1 < r_2 < \dots < r_{m-1} \leqslant n$.

Let

$$A_1 = \begin{pmatrix} a_{i_1 1} & a_{i_1 2} & \dots & a_{i_1 n} \\ a_{i_2 1} & a_{i_2 2} & \dots & a_{i_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_{m-1} 1} & a_{i_{m-1} 2} & \dots & a_{i_{m-1} n} \end{pmatrix}.$$

If the row rank of A_1 is $m-1$, there are $m-1$ linearly independent columns in A_1. By contradiction, the row rank of A_1 is strictly less than $m-1$, and consequently the row rank of A is strictly less than n. This contradicts our hypothesis. Thus

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_{m-1} \\ r_1 & r_2 & \dots & r_{m-1} \end{pmatrix} > 0$$

for some $r_1, \dots, r_{m-1}$ such that $1 \leqslant r_1 < \dots < r_{m-1} \leqslant n$. Hence $U \in STP_{m-1}$.

Similarly we may show that

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_m \\ r_1 & r_2 & \dots & r_m \end{pmatrix} > 0$$

for some $r_1, \dots, r_m$ such that $1 \leqslant r_1 < \dots < r_m \leqslant n$.

Since the order of the rows of U is the same as that of the rows of A in the equation (1), $U \in STP_m$ based on consecutive rows follows from the assumption that $A \in TP_m$ based on consecutive rows. Since $U \in STP_{m-1}$ and $U \in STP_m$ based on consecutive rows, $U \in STP_m$ by Lemma 1.

From the Cauchy-Binet formula,

$$\begin{aligned} u_{ij} = U\begin{pmatrix} i \\ j \end{pmatrix} &= \sum_{1 \leqslant r \leqslant n} A\begin{pmatrix} i \\ r \end{pmatrix} H\begin{pmatrix} r \\ j \end{pmatrix} = a_{i1} q^{(j-1)^2} + \dots + a_{ij} 1 + \dots + a_{in} q^{(n-j)^2} \\ &= a_{ij} + q \cdot (\text{a sum of nonnegative terms}). \end{aligned}$$

As $q \to 0$, $u_{ij} \to a_{ij}$. That is, $U \to A$ as $q \to 0$. Since the set of all strictly totally positive matrices is dense in the set of all totally positive matrices (see [7, p. 88]), $A \in TP_m$. $\square$

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