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ON AN EXTENSION OF FEKETE’S LEMMA

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Abstract. We show that if a real $n \times n$ non-singular matrix ($n \geq m$) has all its minors of order $m - 1$ non-negative and has all its minors of order $m$ which come from consecutive rows non-negative, then all $m$th order minors are non-negative, which may be considered an extension of Fekete’s lemma.

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Fekete’s lemma (see [2] or [4, p. 59]) states that if an $n \times m$ matrix ($n \geq m$) has all its minors of order $m - 1$ which come from the last $m - 1$ columns and all $m$th order minors which come from consecutive rows positive, then all $m$th order minors are positive. In this note we find sufficient conditions for all $m$th order minors of an $n \times n$ non-singular square matrix ($n \geq m$) to be non-negative, which may be considered an extension of Fekete’s lemma.

Definition. A rectangular matrix $A = \|a_{ik}\| \ (i = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ over $\mathbb{R}$ is called totally positive (or strictly totally positive)—hereafter denoted by TP (or STP)—if all its minors of any order are non-negative (or positive). An $n \times n$ matrix over $\mathbb{R}$ is called totally positive of order $m$ (or strictly totally positive of order $m$) and is denoted by $TP_m$ (or $STP_m$) if all its minors of order $j \leq m$ are non-negative (or positive). Here $\mathbb{R}$ denotes the set of all real numbers and hereafter we shall use this notation.

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We will denote the determinant formed from elements of the given matrix \( A = \|a_{ik}\| \) \((i = 1, 2, \ldots, m; k = 1, 2, \ldots, n)\) as follows:

\[
A(i_1 \ i_2 \ \ldots \ i_p) = \begin{vmatrix}
   a_{i_1 k_1} & a_{i_1 k_2} & \ldots & a_{i_1 k_p} \\
   a_{i_2 k_1} & a_{i_2 k_2} & \ldots & a_{i_2 k_p} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{i_p k_1} & a_{i_p k_2} & \ldots & a_{i_p k_p}
\end{vmatrix}.
\]

We need the following well known Cauchy-Binet formula (see [3, p. 9]) for the proof of our main Theorem 2.

**Cauchy-Binet formula.** Let \( A, B \) and \( C \) denote matrices of real numbers of orders \( n \times m, n \times k \) and \( k \times m \), respectively. If \( A = BC \), then

\[
A(i_1 \ i_2 \ \ldots \ i_p) = \sum_{1 \leq k_1 < \ldots < k_p \leq n} B(i_1 \ i_2 \ \ldots \ i_p) C(k_1 \ k_2 \ \ldots \ k_p).
\]

**Lemma 1.** Suppose \( n \geq m \). If a real \( n \times n \) matrix \( A = \|a_{ij}\| \) has all its minors of order \( m - 1 \) positive and all its minors of order \( m \) which come from consecutive rows positive, then all \( m \)th order minors are positive.

**Proof.** Follows immediately from Fekete’s lemma. \( \square \)

**Theorem 2.** Suppose \( n \geq m \). If a real \( n \times n \) non-singular matrix \( A = \|a_{ij}\| \) has all its minors of order \( m - 1 \) non-negative and all its minors of order \( m \) which come from consecutive rows non-negative, then all \( m \)th order minors are non-negative.

**Proof.** Let \( H \) be an auxiliary \( n \times n \) matrix such that

\[
H = H(q) = \|q^{(i-j)^2}\| \ (i, j = 1, 2, \ldots, n) \text{ for } 0 < q < 1.
\]

\( H \in STP \) follows from a theorem of Pólya (see [6, p. 49]). Let \( U = AH \). Then

\[
(1) \quad U(i_1 \ i_2 \ \ldots \ i_p) = \sum_{1 \leq r_1 < \ldots < r_p \leq n} A(i_1 \ \ldots \ i_p) H(r_1 \ \ldots \ r_p)
\]

for \( p = 1, 2, \ldots, n \) by the Cauchy-Binet formula. Since \( A \in TP_{m-1} \) and \( H \in STP \), \( U \in TP_{m-1} \).

From the hypothesis,

\[
A(i_1 \ i_2 \ \ldots \ i_{m-1}) \geq 0.
\]
Suppose that
\[ A \left( \begin{array}{cccc} i_1 & i_2 & \ldots & i_{m-1} \\ r_1 & r_2 & \ldots & r_{m-1} \end{array} \right) = 0 \]
for every \( r_1, r_2, \ldots, r_{m-1} \) such that \( 1 \leq r_1 < r_2 < \ldots < r_{m-1} \leq n \).

Let
\[
A_1 = \begin{pmatrix}
a_{i_11} & a_{i_12} & \ldots & a_{i_1n} \\
a_{i_21} & a_{i_22} & \ldots & a_{i_2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i_{m-1}1} & a_{i_{m-1}2} & \ldots & a_{i_{m-1}n}
\end{pmatrix}
\]

If the row rank of \( A_1 \) is \( m - 1 \), there are \( m - 1 \) linearly independent columns in \( A_1 \). By contradiction, the row rank of \( A_1 \) is strictly less than \( m - 1 \), and consequently the row rank of \( A \) is strictly less than \( n \). This contradicts our hypothesis. Thus
\[
A \left( \begin{array}{cccc} i_1 & i_2 & \ldots & i_{m-1} \\ r_1 & r_2 & \ldots & r_{m-1} \end{array} \right) > 0
\]
for some \( r_1, \ldots, r_{m-1} \) such that \( 1 \leq r_1 < \ldots < r_{m-1} \leq n \). Hence \( U \in STP_{m-1} \).

Similarly we may show that
\[
A \left( \begin{array}{cccc} i_1 & i_2 & \ldots & i_m \\ r_1 & r_2 & \ldots & r_m \end{array} \right) > 0
\]
for some \( r_1, \ldots, r_m \) such that \( 1 \leq r_1 < \ldots < r_m \leq n \).

Since the order of the rows of \( U \) is the same as that of the rows of \( A \) in the equation (1), \( U \in STP_m \) based on consecutive rows follows from the assumption that \( A \in TP_m \) based on consecutive rows. Since \( U \in STP_{m-1} \) and \( U \in STP_m \) based on consecutive rows, \( U \in STP_m \) by Lemma 1.

From the Cauchy-Binet formula,
\[
u_{ij} = U \binom{i}{j} = \sum_{1 \leq r \leq n} A \binom{i}{r} H \binom{r}{j} = a_{i1} q^{(j-1)^2} + \ldots + a_{ij} 1 + \ldots + a_{in} q^{(n-j)^2} = a_{ij} + q \cdot (a \text{ sum of nonnegative terms}).
\]

As \( q \to 0 \), \( u_{ij} \to a_{ij} \). That is, \( U \to A \) as \( q \to 0 \). Since the set of all strictly totally positive matrices is dense in the set of all totally positive matrices (see [7, p. 88]), \( A \in TP_m \). \(\square\)
References


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