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SEQUENTIAL COMPLETENESS OF SUBSPACES OF PRODUCTS OF TWO CARDINALS

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Abstract. Let κ be a cardinal number with the usual order topology. We prove that all subspaces of κ^2 are weakly sequentially complete and, as a corollary, all subspaces of ω_1^2 are sequentially complete. Moreover we show that a subspace of $(\omega_1+1)^2$ need not be sequentially complete, but note that $X = A \times B$ is sequentially complete whenever A and B are subspaces of κ.

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Sequentially complete spaces arise in connection with the extension of sequentially continuous maps as absolutely sequentially closed spaces [FK]. Since normal spaces are sequentially complete, it is interesting to compare the normality of subspaces of products of two cardinals, see [KOT], with the sequential completeness. The results are described in the abstract.

Throughout the paper, a *space* means a *Hausdorff completely regular topological space.* Denote by $C(X)$ the continuous real-valued functions on a space X. If X is a subspace of Y, then X is $C(X)$-*embedded* in Y if each $f \in C(X)$ can be continuously extended over Y.

Let X be a space. A *sequence* in X is a function from the set ω of all natural numbers to X; it will be denoted by $\langle x_n \colon n \in \omega\rangle$. Let $x \in X$. A sequence $\langle x_n \colon n \in \omega\rangle$ *converges* to a point x in X if the set $\{n \in \omega \colon x_n \in V\}$ is cofinite in ω, i.e. its complement in ω is finite for each neighborhood V of x. A real-valued function f on X is *sequentially continuous* if the following implication is true: if a sequence

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$\langle x_n \colon n \in \omega \rangle$ converges in X to a point x, then the sequence $\langle f(x_n) \colon n \in \omega \rangle$ converges in R to $f(x)$. Denote by $C_s(X)$ the set of all sequentially continuous real-valued functions on X. Obviously $C(X) \subset C_s(X)$. In accordance with [Ko], denote by $\mathbf{P}$ the class of all spaces X for which $C(X) = C_s(X)$. If X is sequential (in particular metrizable), then $X \in \mathbf{P}$. But not all spaces in $\mathbf{P}$ are sequential [Ko]. Denote by X_s the underlying set of X carrying the weak topology with respect to $C_s(X)$. Then X_s is a space and $C_s(X) = C(X_s)$. Observe that a sequence $\langle x_n \colon n \in \omega \rangle$ converges in X to a point x if and only if it converges in X_s to x. If X is sequentially closed in every space Y in which it is $C(X)$-embedded, then X is said to be *sequentially complete* (cf. [FK]). For easier reference, we call a space X *weakly sequentially complete* if X_s is sequentially complete. We shall abbreviate sequential completeness and weak sequential completeness to SC and WSC, respectively.

Let X be a space. A sequence $\langle x_n \colon n \in \omega \rangle$ is said to be *fundamental* if the sequence $\langle f(x_n) \colon n \in \omega \rangle$ converges in R for each $f \in C(X)$. For the reader's convenience, we recall here the following characterizations of SC spaces (cf. [FK]).

Theorem 0. *Let X be a space. Then the following are equivalent.*

(1) *X is SC.*
(2) *Each fundamental sequence in X is convergent.*
(3) *X is sequentially closed in its Čech-Stone compactification βX.*
(4) *X is sequentially closed in its Hewitt realcompactification υX.*

Observe that X is WSC if and only if, for every sequence $\langle x_n \colon n \in \omega \rangle$ in X, if $\langle f(x_n) \colon n \in \omega \rangle$ converges in R for every $f \in C_s(X)$, then $\langle x_n \colon n \in \omega \rangle$ converges in X.

In [F2], the following assertion was proved.

Proposition 1. *All normal spaces are SC.*

Moreover, it is also well-known that all subspaces of a cardinal κ with the usual order topology are normal. Therefore we have

Corollary 2. *All subspaces of a cardinal κ are SC.*

Note that ω_1^2 is normal. But according to [KOT], if A and B are disjoint stationary sets of ω_1, then $X = A \times B$ is not normal. So it is natural to ask whether such spaces are (W)SC or not. Our first result is

Theorem 3. *Let κ be a cardinal. Then all subspaces of the square κ^2 with the usual product topology are WSC.*

P r o o f. Assume $X \subset \kappa^2$ and $\langle x_n \colon n \in \omega \rangle$ is a sequence in X such that $\langle f(x_n) \colon n \in \omega \rangle$ converges for each $f \in C_s(X)$. We shall show that $\langle x_n \colon n \in \omega \rangle$ converges. By retaking a suitably large κ, we may assume κ is a successor cardinal. Let $\alpha = \min\{\gamma < \kappa \colon \{n \in \omega \colon x_n \in [0, \gamma] \times \kappa\}$ is infinite$\}$ and $\beta = \min\{\delta < \kappa \colon \{n \in \omega \colon x_n \in [0, \alpha] \times [0, \delta]\}$ is infinite$\}$. Since κ is a successor cardinal, such α and β always exist. Then $T = \{n \in \omega \colon x_n \in [0, \alpha] \times [0, \beta]\}$ is infinite, $T_{\alpha'} = \{n \in \omega \colon x_n \in [0, \alpha'] \times [0, \beta]\}$ is finite for each $\alpha' < \alpha$ and $T^{\beta'} = \{n \in \omega \colon x_n \in [0, \alpha] \times [0, \beta']\}$ is finite for each $\beta' < \beta$. Consider the function $f \colon X \to I$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in X \cap [0, \alpha] \times [0, \beta], \\ 1, & \text{otherwise.} \end{cases}$$

Since $X \cap [0, \alpha] \times [0, \beta]$ is clopen in X, f is continuous. Note that $f(x_n) = 0$ for each $n \in T$ and $f(x_n) = 1$ for each $n \in \omega \backslash T$. So, by our assumption, T must be cofinite. Moreover, since $T_{\alpha'}$ and $T^{\beta'}$ are finite for each $\alpha' < \alpha$ and $\beta' < \beta$ and T is cofinite, $\langle x_n \colon n \in \omega \rangle$ converges to $\langle \alpha, \beta \rangle$ in κ^2. We shall show that $\langle x_n \colon n \in \omega \rangle$ converges to $\langle \alpha, \beta \rangle$ in X. It suffices to show the next claim.

Claim. $\langle \alpha, \beta \rangle \in X$.

P r o o f o f C l a i m. Assume $\langle \alpha, \beta \rangle \notin X$. Put $Z = \{x_n \colon n \in \omega\} \cap [0, \alpha] \times [0, \beta]$, $Z(0) = \{x_n \colon n \in \omega\} \cap \alpha \times \beta$, $Z(1) = \{x_n \colon n \in \omega\} \cap \{\alpha\} \times [0, \beta]$ and $Z(2) = \{x_n \colon n \in \omega\} \cap [0, \alpha] \times \{\beta\}$. Note that $Z = \{x_n \colon n \in T\}$ and Z is the disjoint union of $Z(0)$, $Z(1)$ and $Z(2)$. Moreover, put $T(i) = \{n \in T \colon x_n \in Z(i)\}$ for each $i \in 3 = \{0, 1, 2\}$. Then T is also the disjoint union of $T(0)$, $T(1)$ and $T(2)$.

Assume Z is finite. Then, since T is infinite, there is $z \in Z$ such that $\{n \in T \colon x_n = z\}$ is infinite, say $z = \langle \gamma, \delta \rangle$. By the minimality of α and β, we have $\gamma = \alpha$ and $\delta = \beta$. Thus $X \supset Z \ni z = \langle \gamma, \delta \rangle = \langle \alpha, \beta \rangle$, which contradicts the assumption $\langle \alpha, \beta \rangle \notin X$. This shows Z is an infinite subset of $X \cap [0, \alpha] \times [0, \beta]$.

Fact 1. *Z is closed discrete in X.*

P r o o f o f F a c t 1. Let $\langle \gamma, \delta \rangle \in X$. It suffices to find a neighborhood U of $\langle \gamma, \delta \rangle$ such that $U \cap Z$ is finite.

If $\langle \gamma, \delta \rangle \in U = X \backslash [0, \alpha] \times [0, \beta]$, then U is a neighborhood with $U \cap Z = \emptyset$. So assume $\langle \gamma, \delta \rangle \in X \cap [0, \alpha] \times [0, \beta]$. Then by our assumption $\langle \alpha, \beta \rangle \notin X$, we have $\gamma < \alpha$ or $\delta < \beta$. If $\gamma < \alpha$, then, by the minimality of α, $U = X \cap [0, \gamma] \times [0, \beta]$ is a neighborhood of $\langle \gamma, \delta \rangle$ such that $U \cap Z$ is finite. Similarly, if $\delta < \beta$, then $U = X \cap [0, \alpha] \times [0, \delta]$ is a desired one. This completes the proof of Fact 1. $\square$

To prove Claim, we consider three cases. In all cases, we shall derive contradictions.

Case 1. $\operatorname{cf}\alpha \geqslant \omega_1$ or α is a successor ordinal, where $\operatorname{cf}\alpha$ denotes the cofinality of α.

First assume $\operatorname{cf}\alpha \geqslant \omega_1$. Since $Z \cap \alpha \times \kappa$ is countable and $\operatorname{cf}\alpha \geqslant \omega_1$, there is $\alpha' < \alpha$ such that $Z \cap \alpha' \times \kappa = Z \cap \alpha \times \kappa$. Then by the minimality of α, $Z \cap \alpha \times \kappa$ must be finite. Next assume α is a successor ordinal. Then of course, by the minimality of α, $Z \cap \alpha \times \kappa$ is also finite. Thus in both cases, by the minimality of β and the infinity of Z, $Z(1)$ is infinite.

Put $Y = X \cap \{\alpha\} \times [0, \beta]$. Note that $Z(1)$ is an infinite closed discrete subset of Y and Y is homeomorphic to a subspace of $[0, \beta]$, thus Y is normal. Divide $Z(1)$ into two disjoint infinite sets $Z_0(1)$ and $Z_1(1)$. Then they are disjoint closed sets in the normal space Y. Put $T_i(1) = \{n \in \omega \colon x_n \in Z_i(1)\}$ for each $i \in 2 = \{0, 1\}$. Hence there is a continuous function $g \colon Y \to I$ such that $g(x) = i$ for each $x \in Z_i(1)$ and $i \in 2$. Moreover, define a function $f \colon X \to I$ by

$$f(x) = \begin{cases} g(x), & \text{if } x \in Y, \\ 1, & \text{otherwise.} \end{cases}$$

Fact 2. *f is sequentially continuous.*

Proof of Fact 2. Let $\langle y_n \colon n \in \omega \rangle$ be a sequence in X which converges to a point $y \in X$. We shall show $\langle f(y_n) \colon n \in \omega \rangle$ converges to $f(y)$.

First assume $y \notin Y$. Since $X \backslash Y = X \backslash \{\alpha\} \times [0, \beta]$ is an open neighborhood of y, $C = \{n \in \omega \colon y_n \in X \backslash Y\}$ is cofinite. By the definition of f, $f(y_n) = 1$ for each $n \in C$ and $f(y) = 1$. Therefore $\langle f(y_n) \colon n \in \omega \rangle$ converges to $f(y)$.

Next assume $y \in Y$. Since $X \cap [0, \alpha] \times [0, \beta]$ is an open neighborhood of y, $\{n \in \omega \colon y_n \in X \cap [0, \alpha] \times [0, \beta]\}$ is cofinite. Moreover, by $\operatorname{cf}\alpha \geqslant \omega_1$ or α successor, $C = \{n \in \omega \colon y_n \in Y\}$ is also cofinite. Note that $f(y_n) = g(y_n)$ for each $n \in C$. Let V be a neighborhood of $f(y) = g(y)$ in I. Since g is continuous and $\langle y_n \colon n \in \omega \rangle$ converges to y, $F = \{n \in C \colon g(y_n) \notin V\}$ is finite. Since $C \backslash F$ is also cofinite in ω and $f(y_n) = g(y_n) \in V$ for each $n \in C \backslash F$, $\langle f(y_n) \colon n \in \omega \rangle$ converges to $f(y)$. This completes the proof of Fact 2. □

By Fact 2 and our assumption, $\langle f(x_n) \colon n \in \omega \rangle$ must converge. But, since $f(x_n) = i$ for each $n \in T_i(1)$ and $i \in 2$ and $T_i(1)$'s are infinite, we have a contradiction. This completes Case 1.

The next case is similar to Case 1.

Case 2. $\operatorname{cf}\beta \geqslant \omega_1$ or β is a successor ordinal.

Finally we consider the following case.

Case 3. $\operatorname{cf}\alpha = \operatorname{cf}\beta = \omega$.

First fix two strictly increasing sequences $\langle \alpha(m)\colon m \in \omega\rangle$ and $\langle \beta(m)\colon m \in \omega\rangle$ cofinal in α and β, respectively.

Subcase 0. $Z(0)$ is infinite.

For each $\alpha' < \alpha$ and $\beta' < \beta$, since $T_{\alpha'}$ and $T^{\beta'}$ are finite, $\{z \in Z(0)\colon z \in [0,\alpha'] \times [0,\beta] \cup [0,\alpha] \times [0,\beta']\}$ is also finite. So, since $Z(0)$ is infinite, we can define, by induction, two strictly increasing sequences $\langle \gamma_m\colon m \in \omega\rangle$ in α and $\langle \delta_m\colon m \in \omega\rangle$ in β such that $\alpha(m) < \gamma_m$, $\beta(m) < \delta_m$ and $z_m = \langle \gamma_m, \delta_m\rangle \in Z(0)$ for each $m \in \omega$. Put $V_m = X \cap (\gamma_{m-1}, \gamma_m] \times (\delta_{m-1}, \delta_m]$ for each $m \in \omega$, where we consider $\gamma_{-1} = \delta_{-1} = -1$. Note that each V_m is a clopen neighborhood of z_m.

Fact 3. *$\mathcal{V} = \{V_m\colon m \in \omega\}$ is discrete in X.*

P r o o f o f F a c t 3. Note that, by the definition, $\mathcal{V}$ is disjoint. Let $\langle \gamma, \delta\rangle \in X$. If $\langle \gamma, \delta\rangle \in U = X \backslash [0,\alpha] \times [0,\beta]$, then U does not meet any member of $\mathcal{V}$. So we may assume $\langle \gamma, \delta\rangle \in X \cap [0,\alpha] \times [0,\beta]$. Since $\langle \alpha, \beta\rangle \notin X$, we have $\gamma < \alpha$ or $\delta < \beta$. If $\gamma < \alpha$ ($\delta < \beta$, resp.), then take the smallest $m_0 \in \omega$ with $\gamma \leqslant \gamma_{m_0}$ ($\delta \leqslant \delta_{m_0}$, resp.). Then $U = X \cap [0,\gamma] \times [0,\delta]$ is a neighborhood of $\langle \gamma, \delta\rangle$ which does not meet V_m's for $m > m_0$. This argument completes the proof of Fact 3. □

Consider the function $f\colon X \to I$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in V_{2m} \text{ for some } m \in \omega, \\ 1, & \text{otherwise.} \end{cases}$$

By Fact 3, f is continuous, so $f \in C_s(X)$. Therefore $\langle f(x_n)\colon n \in \omega\rangle$ must converge. But since $f(z_{2m}) = 0$ and $f(z_{2m+1}) = 1$ for each $m \in \omega$, $f(x_n) = 0$ for infinitely many $n \in \omega$ and $f(x_n) = 1$ for infinitely many $n \in \omega$, a contradiction. This completes the proof of Subcase 0.

Subcase 1. $Z(1)$ is infinite.

Similarly by induction, define a strictly increasing sequence $\langle \delta_m\colon m \in \omega\rangle$ in β such that $\beta(m) < \delta_m$ and $z_m = \langle \alpha, \delta_m\rangle \in Z(1)$ for each $m \in \omega$. Put $V_m = X \cap (\alpha(m), \alpha] \times (\delta_{m-1}, \delta_m]$ for each $m \in \omega$ and $\mathcal{V} = \{V_m\colon m \in \omega\}$. The rest is similar to Subcase 0.

Subcase 2. $Z(2)$ is infinite.

This subcase is also similar to Subcase 1.

Thus, in all subcases, we have contradictions. This completes the proof of Claim. □

This completes the proof of Theorem 3. □

Since the space ω_1^2 is first countable, we have $C(\omega_1^2) = C_s(\omega_1^2)$ and hence

Corollary 4. *All subspaces of ω_1^2 are SC.*

Now we will describe a subspace of $(\omega_1 + 1)^2$ which is not SC.

Example 5. Let $X = (\omega_1 + 1) \times (\omega + 1) \setminus \{\langle \omega_1, \omega \rangle\}$, and $x_n = \langle \omega_1, n \rangle$ for each $n \in \omega$. Evidently $\langle x_n \colon n \in \omega \rangle$ does not converge in X. Let $f \in C(X)$. Since f is continuous, for each $n \in \omega$, we can fix $\alpha_n < \omega_1$ such that f has the constant value $f(x_n)$ on $(\alpha_n, \omega_1] \times \{n\}$. Put $\alpha = \sup\{\alpha_n \colon n \in \omega\}$ and take $\gamma < \omega_1$ with $\alpha < \gamma$, and moreover put $y_n = \langle \gamma, n \rangle$ for each $n \in \omega$. Since $\langle y_n \colon n \in \omega \rangle$ converges to $\langle \gamma, \omega \rangle$, by the continuity of f, $\langle f(y_n) \colon n \in \omega \rangle$ must converge to $f(\langle \gamma, \omega \rangle)$. Since $f(x_n) = f(y_n)$ for each $n \in \omega$, $\langle f(x_n) \colon n \in \omega \rangle$ also converges to $f(\langle \gamma, \omega \rangle)$. This argument shows X is not SC.

The next theorem is in fact a corollary to Lemma 1.17 and Lemma 1.16 in [F2]. We give a simple direct proof.

Theorem 6. *The properties WSC and SC are hereditary with respect to sequentially closed subspaces and are productive.*

Proof. Let Y be a sequentially closed subspace of an SC space X. If a sequence is fundamental in Y, then it is fundamental in X and hence converges to a point in Y. This proves the first assertion.

Let $X = \prod_{\alpha \in \kappa} X_\alpha$ be the product space of WSC spaces X_α's and let $\langle x_n \colon n \in \omega \rangle$ be a fundamental sequence in X, say $x_n = \langle x_n(\alpha) \colon \alpha \in \kappa \rangle$. Then, for each $\alpha \in \kappa$, the sequence $\langle x_n(\alpha) \colon n \in \omega \rangle$ is fundamental in X_α (remember the composition of each projection p_α of X onto X_α and each $f \in C_s(X_\alpha)$ is sequentially continuous on X) and hence converges in X_α to a point $x(\alpha)$. Hence $\langle x_n \colon n \in \omega \rangle$ converges to $\langle x(\alpha) \colon \alpha \in \kappa \rangle$.

The same argument proves that also SC is productive. □

Corollary 7. *Let κ be a cardinal. If A and B are subspaces of κ, then $X = A \times B$ is SC.*

Historical Remarks. An extension theory for sequentially continuous functions analogous to the Čech-Stone compactification and the Hewitt real compactification was initiated by J. Novák in [No]. Absolutely sequentially closed spaces (in the class **P** of spaces for which sequentially continuous functions are continuous) were investigated in [F1] and in a very general setting in [FK]. Independently, sequential completeness has been defined and investigated in [Ki].

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