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A GENERALISATION OF A THEOREM OF KOLDUNOV WITH AN ELEMENTARY PROOF

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Abstract. We generalize a Theorem of Koldunov [2] and prove that a disjointness proserving quasi-linear operator between Resz spaces has the Hammerstein property.

Keywords: Riesz spaces (vector lattices), Hammerstein property and disjointness preserving operators

MSC 2000: 46A40, 47H30

Throughout this paper an operator $T\colon E\to F$ between Riesz spaces E, F (not necessarily the Archimedean) means a function from E into F with $T(0)=0$. Let $T\colon E\to F$ be an operator from a Riesz space E into another Riesz space F. We call T positive if $T(x)\leqslant T(y)$ for all $x,y\in E$ with $x\leqslant y$. We say that T is order bounded if $T([x,y])$ is an order bounded subset of F for all $x,y\in E.T$ is said to be disjointness preserving (or equivalently, T preserves disjointness) if $|T(x)|\wedge|T(y)|=0$ in F for all $x,y\in E$ with $|x|\wedge|y|=0$ in E. For unexplained definitions and notations we refer to ([2], [3] and [4]).

Koldunov ([2]) has modified the *Hammerstein* and *weak Hammerstein* property for operators between Riesz spaces as follows:

Definition 1 ([2]). Let $T\colon E\to F$ be an operator between Riesz spaces E and F. Consider the following statements:

(i) $T(u+v+x)-T(u+x)=T(v+x)-T(x)$ for all disjoint $u,v\in E$ and for all $x\in E$.

(ii) $|T(u+v+x)-T(u+x)|\leqslant|T(v+x)|+|T(x)|$ for all disjoint $u,v\in E$ and for all $x\in E$.

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We say that T satisfies the *weak Hammerstein property* (the *Hammerstein property*) whenever it satisfies statement (ii) (statement (i)), respectively.

It is obvious that operators with the Hammerstein property are disjointly additive and each linear operator satisfies the Hammerstein property. Clearly, operators with the Hammerstein property may be viewed as natural generalisation of linear operators on the Riesz spaces. The following characterisation of operators with the Hammerstein property is also given in [1].

Theorem 1 ([1]). *Let $T\colon E \to F$ be an operator between Riesz spaces E and F. The following are equivalent:*

(i) *T satisfies the Hammerstein property.*

(ii) *$T(x) + T(y) = T(x \vee y) + T(x \wedge y)$ for all $x, y \in E$.*

Proof. First assume that (i) holds. It follows from the equality

$$x \vee y = (x-y)^+ + (x-y)^- + (y - (x-y)^-)$$

and from the disjointness of $(x-y)^+$ and $(x-y)^-$ that $T(x)+T(y) = T(x\vee y)+T(x\wedge y)$ for all $x, y \in E$, holds. Now suppose that (ii) holds. Let $u, v, x \in E$ be given in which u, v are disjoint. First suppose that $u, v \geqslant 0$. Then the equalities

$$u + v + x = (u+x) \vee (v+x) \text{ and } (u+x) \wedge (v+x) = x$$

imply that

$$T(u+v+x) - T(u+x) = T(v+x) - T(x).$$

For the general case, we let $y = x - u^- - v^-$ and observe that

$$T(u+v+x) - T(u+x) = T(v^+ + y) - T(v^- + y) = T(v+x) - T(x).$$

This completes the proof. □

Koldunov ([2]) has defined quasi-linear operators between Riesz spaces as follows:

Definition 2 ([2]). An operator $T\colon E \to F$ between Riesz spaces E and F is said to be weakly quasi-linear if

(i) for each $w \in E$ there exists a real number $\lambda(w) > 0$ such that

$$|T(u) - T(v)| \leqslant \lambda(w)|T(u-v)|$$

whenever $|u|, |v| \leqslant |w|$ and quasi-linear if, moreover

(ii) for each $u \in E$ and $r \in \mathbb{R}$ there exists real numbers $k(r,u) > 0$ and $l(r,u) > 0$ such that for each $v \in [-|u|, |v|]$,

$$l(r,u)|T(v)| \leqslant |T(rv)| \leqslant k(r,u)|T(v)|$$

and also $\lim k(r,u) = 0$ as $r \to 0$.

Koldunov ([2], Theorem 2.6) has proved that an $(r_u - r_u)$-continuous operator $T\colon E \to F$ between Archimendean Riesz spaces E and F has the Hammerstein property whenever it satisfies the DQLH (= disjointness preserving, quasi-linear satisfying the weak Hammerstein) property. Koldunov's proof is based on representation theory and quite difficult. We can generalise his theorem with an easy and direct proof as follows:

Theorem 2. *Let $T\colon E \to F$ be a disjointness preserving, weak quasi-linear operator between Riesz spaces E and F (not necessarily Archimedean). Then T satisfies the Hammerstein property.*

Proof. Let $x, y \in E$ be given. Then we have that,

$$\begin{aligned} |T(x) - T(x \vee y) + T(y) - T(x \wedge y)| &\leqslant |T(x) - T(x \vee y| + |T(y) - T(x \wedge y)| \\ &\leqslant 2\lambda |T(y - x)^+|. \end{aligned}$$

Similarly

$$\begin{aligned} |T(x) - T(x \vee y) + T(y) - T(x \wedge y)| &\leqslant |T(x) - T(x \wedge y| + |T(y) - T(x \vee y)| \\ &\leqslant 2\lambda |T(x - y)^+| \end{aligned}$$

where $\lambda = \lambda(w) > 0$, $w = |x| \vee |y|$ (the same notation as in the Definition 2). Since T is disjointness preserving $|T(x-y)^+| \wedge |T(y-x)^+| = 0$. This yields $T(x)+T(y) = T(x \vee y) + T(x \wedge y)$ for all $x, y \in E$. Now by Theorem 1, T has the Hammerstein property. $\square$

When we restrict ourself to the order bounded operators we can generalise another theorem of Koldunov ([2], theorem 3.4) as follows:

Theorem 3. *Let $T\colon E \to F$ be an order bounded quasi-linear operator between Riesz spaces E and F. Then T is $(r_u - r_u)$-continuous. In particular, if F is Achimedean and T is disjointness preserving, then T is:*

$$\textit{additive} \Leftrightarrow \textit{homogeneous} \Leftrightarrow \textit{linear.}$$

Proof. Suppose that $x_n \to x(r.u)$ in E. Then there exists $0 \leqslant u \in E$ such that $x_n \to x(u)$ i.e. for each $\varepsilon > 0$ there exists a natural number n_0 such that $|x_n - x| \leqslant \varepsilon u$ for all $n \geqslant n_0$. Let $\varepsilon > 0$. Letting $\lambda = \lambda(u + |x|)$ (the same notation as in definition

2) we have that

$$\begin{aligned}|T(x_n) - T(x)| &\leqslant \lambda |T(x_n - x)| \\ &\leqslant \lambda |T(\varepsilon(\varepsilon^{-1}(x_n - x))| \\ &\leqslant \lambda k(\varepsilon, u + |x|)|T(\varepsilon^{-1}(x_n - x))| \\ &\leqslant \lambda k(\varepsilon, u + |x|)v\end{aligned}$$

for all $n \geqslant n_0$, where v is an upper bound of $T[-u, u]$ in F. Since $k(\varepsilon, u + |x|) \to 0$ as $\varepsilon \to 0 T(x_n) \to T(x)(r \cdot u)$. Hence T is $(r_u - r_u)$-continuous. For the rest of the proof, combine Theorem 2 and Theorem 2.6 in [2]. □

Let $T\colon E \to F$ be an operator between Riesz spaces E, F and suppose that $T(x) - T(y) \in A_{T(x-y)}$ (= the ideal generated by $T(x - y)$) for all $x, y \in E$. If T is also a disjointness preserving operator then $T(x) - T(x \wedge y)$ and $T(y) - T(x \wedge y)$ are disjoint for all $x, y \in E$. Then it is easy to see that $T(x \vee y) \leqslant T(x) \vee T(y)$ and $T(x) \wedge T(y) \leqslant T(x \wedge y)$ whenever T is disjointness preserving weak quasi-linear. It leads to the following Theorem.

Theorem 4. *Let $T\colon E \to F$ be a disjointness preserving weak quasi-linear operator between Riesz spaces E and F. Then the following are equivalent:*

(i) *T is positive.*

(ii) *T is a lattice homomorphism, i.e. $T(x \vee y) = T(x) \vee T(y)$ and $T(x \wedge y) = T(x) \wedge T(y)$ for all $x, y \in E$.*

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