Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 1, 213–224

Persistent URL: http://dml.cz/dmlcz/127480

Czechoslovak Mathematical Journal, 49 (124) (1999), 213–224

HULLS OF MIXED MODULES WITH FINITE QUOTIENT p-RANK

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(Received January 13, 1997)

1. Introduction

This paper deals with mixed modules over discrete valuation domains. In particular, the results hold for local mixed abelian groups. Let R denote a *discrete valuation domain*, i.e. a local principal ideal domain with prime p. All modules are always understood to be R-modules.

In Theorem 3.2 we use generators and relations to give a general representation of mixed modules over discrete valuation domains. We then introduce the concept of a hull and in Theorem 4.6 we give a classification of mixed modules with finite quotient p-rank in terms of an automorphism of their hull.

2. Representations of torsion modules

In this section we describe torsion modules by representations. If $\mathbf{t}$ is a torsion module then $\{x_i^v \mid i \in \mathbb{N}, v \in I_i\} \subset \mathbf{t}$ is called a *straight basis of* $\mathbf{t}$ as in [1, 1.4] if $\mathbf{t}[p^i]/\mathbf{t}[p^{i-1}] = \bigoplus_{v \in I_i} R(x_i^v + \mathbf{t}[p^{i-1}])$ for all $i \in \mathbb{N}$, where $\mathbf{t}[p^i] = \{x \in \mathbf{t} \mid \operatorname{ann} x \supset p^i R\}$ is the p^i-socle of $\mathbf{t}$. If $s_i = \dim_{R/pR} \mathbf{t}[p^i]/\mathbf{t}[p^{i-1}] = |I_i|$ then the sequence $s = (s_i \mid i \in \mathbb{N})$ is an invariant of $\mathbf{t}$. We call the quotient $\mathbf{t}[p^n]/\mathbf{t}[p^{n-1}]$ an *n-section of* $\mathbf{t}$ and s_n the *dimension of an n-section*.

Let T be a free module on $\{t_i^v \mid i \in \mathbb{N}, v \in I_i\}$. Then $\mathbf{t} \cong T/N$ where N is the kernel of the canonical epimorphism which maps $t_i^v \mapsto x_i^v$. Such a quotient is called a *representation of* $\mathbf{t}$ relative to a straight basis $\{x_i^v \mid i \in \mathbb{N}, v \in I_i\}$. Define a *decomposition* $(T_i \mid i \in \mathbb{N})$ *of* T by $T_0 = 0$ and $T_i = T_{i-1} \oplus \bigoplus_{v \in I_i} Rt_i^v$ for $i \in \mathbb{N}$. An endomorphism ν of T is said to stabilize the decomposition of T if $\nu T_i \subset T_i$ for all $i \in \mathbb{N}$. This decomposition of T depends on the cardinal numbers s_i and not on the particular choice of the straight basis.

Construction 2.1. Let $\mathbf{t}$ be a torsion module with two representations T/N_x and T/N_y. As above let $\{t_i^v \mid i \in \mathbb{N}, v \in I_i\}$ be a basis of T and let $\varrho_x \colon T \longrightarrow T/N_x$ and $\varrho_y \colon T \longrightarrow T/N_y$ be the canonical epimorphisms. Consider the isomorphism $\alpha \colon T/N_x \longrightarrow T/N_y$ defined by $\alpha(t_i^v + N_x) = \sum\limits_{j\in\mathbb{N}} \sum\limits_{u\in I_j} \lambda_{i,j}^{v,u}(t_j^u + N_y)$ for all $i \in \mathbb{N}, v \in I_i$, where the coefficients $\lambda_{i,j}^{v,u}$ are chosen to be 0 or $\lambda_{i,j}^{v,u} \in R \setminus pR$. One can use α to define an endomorphism φ of T given by $\varphi t_i^v = \sum\limits_{j\in\mathbb{N}} \sum\limits_{u\in I_j} \lambda_{i,j}^{v,u} t_j^u$ for all $i \in \mathbb{N}, v \in I_i$. We shall call the endomorphism φ a *semi-extension of* α. It is easy to see that $\alpha\varrho_x = \varrho_y\varphi$ and thus $\varphi N_x \subset N_y$. Since $\lambda_{i,j}^{v,u} \in R \setminus pR$ or 0 the definition of φ implies that $\varphi T_i \subset T_i$ for all $i \in \mathbb{N}$.

We prove a technical lemma on decompositions.

Lemma 2.2. *Let T/N be a representation of the torsion module $\mathbf{t}$ and let $(T_i \mid i \in \mathbb{N})$ be a decomposition of T. Then*

$$pT_i + T_{i-1} = (T_i \cap N) + T_{i-1} \quad \text{for all} \quad i \in \mathbb{N}.$$

In particular, $T_i/(pT_i + T_{i-1}) \cong (T/N)[p^i]/(T/N)[p^{i-1}]$.

P r o o f. We have by the definition of a decomposition of T that $(T_i + N)/N = (T/N)[p^i]$ for $i \in \mathbb{N}$. Further we have for all $i \in \mathbb{N}$

$$\begin{aligned} T_i/\big((T_i \cap N) + T_{i-1}\big) &\cong T_i/\big(T_i \cap (T_{i-1} + N)\big) \cong (T_i + N)/(T_{i-1} + N) \\ &\cong \big((T_i + N)/N\big)/\big((T_{i-1} + N)/N\big) \\ &\cong (T/N)[p^i]/(T/N)[p^{i-1}], \end{aligned}$$

hence $pT_i + T_{i-1} \subset (T_i \cap N) + T_{i-1}$. To show the reverse containment let $x = \sum\limits_{u\in I_i} \kappa_i^u t_i^u + t \in (T_i \cap N) + T_{i-1} = T_i \cap (N + T_{i-1}) \subset T_{i-1} + N$, where $t \in T_{i-1}$ and $\sum\limits_{u\in I_i} \kappa_i^u t_i^u \in T_i \cap N$. Then $\varrho x = x + N \in (T/N)[p^{i-1}]$, relative to the canonical epimorphism ϱ from T to T/N. But $\{t_i^v + N \mid i \in \mathbb{N}\}$ is a straight basis, hence $\varrho x \in (T/N)[p^{i-1}]$ if and only if $\kappa_i^u \in pR$ for all $u \in I_i$, i.e. $x \in pT_i + T_{i-1}$. This shows $pT_i + T_{i-1} = (T_i \cap N) + T_{i-1}$ for all $i \in \mathbb{N}$. The last statement follows from the identity just established and the isomorphism in the first half of the proof. □

Now we show that all isomorphisms between different representations of the torsion module $\mathbf{t}$ are induced by certain injective endomorphisms of T.

Theorem 2.3. *Let $\mathbf{t}$ be a torsion module with two representations T/N_x and T/N_y and let $(T_i \mid i)$ be a decomposition of T. Suppose $\alpha \colon T/N_x \longrightarrow T/N_y$ is* *an isomorphism and φ is any semi-extension of α to T. Then φ is an injective endomorphism of T which stabilizes the decomposition of T such that $\varphi N_x \subset N_y$ and $T = \varphi T + p^n T$ for all natural n. Moreover, φ induces an automorphism of $T_i/(pT_i + T_{i-1})$ for all $i \in \mathbb{N}$.*

Proof. Let $\alpha(t_i^v + N_x) = \sum_{j\in\mathbb{N}} \sum_{u\in I_j} \lambda_{i,j}^{v,u}(t_j^u + N_y)$ for all $i \in \mathbb{N}, v \in I_i$, where we assume $\lambda_{i,j}^{v,u} \in R \setminus pR$ or 0. Clearly the restriction of α to

$$\bar{\alpha}_i\colon\ (T/N_x)[p^i]/(T/N_x)[p^{i-1}] \longrightarrow (T/N_y)[p^i]/(T/N_y)[p^{i-1}]$$

is an isomorphism. If these quotients are considered as vector spaces over the field R/pR then $\bar{\alpha}_i$ is a vector space isomorphism given by the coefficients $\bar{\lambda}_{i,i}^{v,u} = \lambda_{i,i}^{v,u} + pR \in R/pR$ for this specific i, relative to the bases $\{t_i^v + N_x + (T/N_x)[p^{i-1}] \mid v \in I_i\}$ and $\{t_i^v + N_y + (T/N_y)[p^{i-1}] \mid v \in I_i\}$, respectively. Since $\varphi T_i \subset T_i$ the endomorphism φ induces an endomorphism $\bar{\varphi}_i$ of $T_i/(pT_i + T_{i-1})$ over the field R/pR. Note that all three vector spaces over R/pR under consideration are isomorphic by Lemma 2.2. Recall that the endomorphism φ is given by the coefficients $\lambda_{i,j}^{v,u}$ relative to the basis $\{t_i^v \mid i \in \mathbb{N}, v \in I_i\}$. Thus $\bar{\varphi}_i$ is given by the same coefficients $\bar{\lambda}_{i,i}^{v,u}$ relative to the basis $\{t_i^v + pT_i + T_{i-1} \mid v \in I_i\}$ and is therefore a vector space automorphism.

We now show the injectivity of φ. Assume $\ker\varphi \neq 0$ and let i be minimal with respect to $T_i \cap \ker\varphi \neq 0$. Then the kernel of the restriction φ_i of φ to T_i/T_{i-1} equals $K/T_{i-1} \supset \big((\ker\varphi \cap T_i) \oplus T_{i-1}\big)/T_{i-1}$ and is not 0. But K/T_{i-1} is pure in the free module T_i/T_{i-1} and hence not contained in $p(T_i/T_{i-1}) = (pT_i + T_{i-1})/T_{i-1}$, i.e. $K \not\subset pT_i + T_{i-1}$. Thus $0 \neq (K + pT_i + pT_{i-1})/(pT_i + T_{i-1})$ is contained in the kernel of $\bar{\varphi}_i$ contradicting the injectivity of $\bar{\varphi}_i$. Consequently φ is injective.

For the proof of $T = \varphi T + p^n T$ for all n it suffices to show $T = \varphi T + pT$. Note that the restrictions $\bar{\varphi}_i$ of φ to $T_i/(pT_i + T_{i-1})$ are epic hence $T_i = \varphi T_i + pT_i + T_{i-1}$ for all i. Recursively substituting into this equation for the terms $T_{i-1}, T_{i-2}, \ldots$, one obtains the identity $T_i = \varphi T_i + pT_i$. It is easy to see that $T = \varphi T + pT$ follows from $T_i = \varphi T_i + pT_i$ for all i and hence the proof is complete. □

Remark 2.4. If the isomorphism α maps $t_i^v + N_x$ to $t_i^v + N_y$ for all $i \in \mathbb{N}, v \in I_i$, then $\varrho_y = \alpha\varrho_x$ and $N_y = \ker\varrho_y = \ker\alpha\varrho_x = N_x$. Note that in this case φ becomes the identity map.

In this section we describe mixed modules G by generators and relations and use this later to find a representation for mixed modules as submodules of a vector space over K, the quotient field of R.

If L is any free pure submodule of the module G, then by [2, Section 32, Exercise 7] there is a free pure submodule F containing L such that $G/(\mathbf{t}G \oplus F)$ is torsion-free divisible. Such a pure free module F with torsion-free divisible quotient $G/(\mathbf{t}G \oplus F)$ is called *relatively maximal pure free in* G. A pure free submodule is relatively maximal if and only if $(F + \mathbf{t}G)/\mathbf{t}G$ is basic in $G/\mathbf{t}G$. For a torsion-free module X define the *p-rank of* X as the dimension of X/pX over the field R/pR. We use this to make two definitions. First we define the *quotient p-rank* of a mixed module G to be the p-rank of the torsion-free quotient $G/\mathbf{t}G$. Second we define the *relative divisibility dimension of* G *relative to* F to be the dimension of the vector space $G/(\mathbf{t}G \oplus F)$. By [2, Section 32, Exercise 5] the rank of a relatively maximal pure free submodule F in a mixed module equals the quotient p-rank of G and is an invariant of the mixed module. However the relative divisibility dimension of a module is invariant if and only if the quotient p-rank is finite.

Let G be a mixed module with torsion submodule $\mathbf{t}G$ with sequence $s = (s_i \mid i \in \mathbb{N})$ of section dimensions and quotient p-rank r. Then there is a relatively maximal pure free submodule F of G of rank r with quotient $G/(\mathbf{t}G \oplus F)$ a vector space over K of dimension d. The latter is the relative divisibility dimension of G relative to F.

Let I be a set of cardinality d, then the subset

$$B = \{x_i^v, a_{i-1}^k, b_l \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\} \subset G$$

is called a *basic generating system of* G *relative to a straight basis*, if

(1) $\{x_i^v \mid i \in \mathbb{N}, v \in I_i\}$ is a straight basis of $\mathbf{t}G$ with $\operatorname{ann} x_i^v = p^i R$ for all $i \in \mathbb{N}, v \in I_i$,
(2) $F = \bigoplus_{l \in P} R b_l$ is a relatively maximal pure free submodule,
(3) $G/(\mathbf{t}G \oplus F) = \bigoplus_{k \in I} K \bar{a}_0^k$,

where $\bar{a}_{i-1}^k = a_{i-1}^k + \mathbf{t}G \oplus F$ and $p\bar{a}_i^k = \bar{a}_{i-1}^k$ for all $i \in \mathbb{N}, k \in I$. The definition implies that the elements a_0^k, b_l, $k \in I, l \in P$, are independent and $r = |P|$ and $s_i = |I_i|$ for all $i \in \mathbb{N}$. It is easy to see that

$$G = \langle x_i^v, a_{i-1}^k, b_l \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\rangle.$$

A basic generating system intrinsically defines a series of equations with coefficients in R that describe the relations among the generators. Modeling upon these relations we introduce the concept of an abstract array $(\alpha,\eta)=(\alpha^{k,u}_{i-1,j},\eta^k_{i-1,l})$ with $i,j\in\mathbb{N},u\in I_j,k\in I,l\in P$ having entries in R with the condition that the entries of α are in $R\setminus pR$ or 0. Both arrays are assumed to be *row finite in j,u and l*, i.e. for a fixed pair (k,i) there is $\alpha^{k,u}_{i,j}=0$ for almost all pairs (j,u) and $\eta^k_{i,l}=0$ for almost all l. We shall refer to α,η and (α,η) as *relation arrays.*

Let $\mathbf{t}$ be a torsion module with sequence $s=(s_j\mid j\in\mathbb{N})$ of section dimensions and let r and d be cardinal numbers and I,I_i and P be sets, such that $|I_i|=s_i$, $|I|=d$ and $|P|=r$. Then the relation array (α,η) with $i,j\in\mathbb{N},u\in I_j,k\in I,l\in P$ is said to have *format* (s,r,d).

The following results give descriptions of mixed modules by generators and relations.

Proposition 3.1. *Every mixed module G has a basic generating system $B=\{x^v_i,a^k_{i-1},b_l\mid i\in\mathbb{N},v\in I_i,k\in I,l\in P\}$ and a relation array (α,η) relative to B given by*

$$pa^k_i=a^k_{i-1}+\sum_{l\in P}\eta^k_{i-1,l}b_l+\sum_{j\in\mathbb{N}}\sum_{u\in I_j}\alpha^{k,u}_{i-1,j}x^u_j\quad(i\in\mathbb{N},k\in I). \tag{1}$$

The array η and the torsion elements $\sum\limits_{j\in\mathbb{N}}\sum\limits_{u\in I_j}\alpha^{k,u}_{i-1,j}x^u_j$ are unique relative to B.

P r o o f. Using straight basis and [2, Section 32, Exercise 7] it is clear that every mixed module G has such a basic generating system B. Moreover, every basic generating system B of G determines a relation array. Consider equation (1) modulo $\mathbf{t}G$. The elements $b_l+\mathbf{t}G$ form a basis of $(F\oplus\mathbf{t}G)/\mathbf{t}G$, thus the coefficients $\eta^k_{i-1,l}$ are unique. Hence the torsion elements $\sum\limits_{j\in\mathbb{N}}\sum\limits_{u\in I_j}\alpha^{k,u}_{i-1,j}x^u_j$ are unique relative to B. □

Theorem 3.2. *Let $\mathbf{t}$ be a torsion module with sequence s of section dimensions. Every relation array (α,η) of format (s,r,d) can be realized by a mixed module with torsion submodule isomorphic to $\mathbf{t}$, quotient p-rank r and relative divisibility dimension d.*

P r o o f. Let $s=(s_i\mid i\in\mathbb{N})$ and sets I_i, P and I be given, where $|I_i|=s_i$, $|P|=r$ and $|I|=d$. Let $(\alpha,\eta)=(\alpha^{k,u}_{i-1,j},\eta^k_{i-1,l})$ with $i,j\in\mathbb{N},u\in I_j,k\in I,l\in P$ and let L be a free R-module on

$$\{t^v_i,w^k_{i-1},b_l\mid i\in\mathbb{N},v\in I_i,k\in I,l\in P\},$$

T the submodule of L generated by $\{t_i^v \mid i \in \mathbb{N}, v \in I_i\}$, T/N a representation of $\mathbf{t}$ and M the submodule of L generated by

$$\left\{N, pw_i^k - w_{i-1}^k - \sum_{l\in P}\eta_{i-1,l}^k b_l - \sum_{j\in\mathbb{N}}\sum_{u\in I_j}\alpha_{i-1,j}^{k,u} t_j^u \mid i \in \mathbb{N}, k \in I\right\}.$$

It is easy to see that $T\cap M = N$, hence $\mathbf{t} \cong T/N = T/(T\cap M) \cong (T+M)/M$ with straight basis $\{\bar{t}_i^v \mid i \in \mathbb{N}, v \in I_i\}$ where $\bar{x} = x+M$ for $x \in L$. Let $H = L/M$. We now show that $\mathbf{t}H = (T+M)/M$. One direction is trivial, namely, $(T+M)/M \subset \mathbf{t}H$. We write elements $\bar{h} \in H$ in the form

$$\bar{h} = \bar{t} + \sum_{l\in P} f_l \bar{b}_l + \sum_{i\in\mathbb{N}}\sum_{k\in I} m_{i-1}^k \overline{w}_{i-1}^k,$$

where $\bar{t} \in (T+M)/M$, $f_l, m_{i-1}^k \in R$, $i \in \mathbb{N}, k \in I, l \in P$, and $m_i^k = f_l = 0$ for almost all i, k and l. We may assume that for $i > 0$ the m_i^k are units or 0 in view of

$$p\overline{w}_i^k = \overline{w}_{i-1}^k + \sum_{l\in P}\eta_{i-1,l}^k \bar{b}_l + \sum_{j\in\mathbb{N}}\sum_{u\in I_j}\alpha_{i-1,j}^{k,u}\bar{t}_j^u \quad (i \in \mathbb{N}, k \in I). \tag{2}$$

For the reverse direction it suffices to show that if $\bar{h}$ is torsion and $\bar{h} = \sum_{i\in\mathbb{N}}\sum_{k\in I} m_{i-1}^k \overline{w}_{i-1}^k + \sum_{l\in P} f_l \bar{b}_l$ with $m_{i-1}^k, f_l \in R$, m_i^k units or 0 if $i > 0$, then $\bar{h} = 0$. Let $h = \sum_{i\in\mathbb{N}}\sum_{k\in I} m_{i-1}^k w_{i-1}^k + \sum_{l\in P} f_l b_l \in \bar{h}$. Without loss of generality we may assume $p\bar{h} = 0$, i.e. $ph \in M$. By renumbering, if necessary, there is a natural number q, $d_i^k \in R$ and $z \in N$ such that

$$\begin{aligned} &p\left(\sum_{i=1}^{q+1}\sum_{k=1}^{q} m_{i-1}^k w_{i-1}^k + \sum_{l=1}^{q} f_l b_l\right) \\ &\qquad = z + \sum_{i=1}^{q}\sum_{k=1}^{q} d_i^k\left(pw_i^k - w_{i-1}^k - \sum_{l=1}^{q}\eta_{i-1,l}^k b_l - \sum_{j=1}^{q}\sum_{u=1}^{q}\alpha_{i-1,j}^{k,u} t_j^u\right). \end{aligned} \tag{3}$$

Equation (3) holds in the free module L and so we may equate the coefficients of like terms. This gives the following:

$$\begin{gathered} pm_0^k = -d_1^k, \quad pm_i^k = pd_i^k - d_{i+1}^k, \quad (1 \leqslant i < q, 1 \leqslant k \leqslant q), \\ pm_q^k = pd_q^k, \quad (1 \leqslant k \leqslant q). \end{gathered}$$

Working down the equations in sequence we conclude that $d_i^k \in pR$ for all i and $1 \leqslant k \leqslant q$. Working the reverse direction and using the fact, that $pd_q^k \in p^2R$

and m_i^k are units or 0 for $1 \leqslant i$ we obtain that $m_i^k = d_i^k = 0$ for $1 \leqslant i$, hence $m_0^k = 0$ for $1 \leqslant k \leqslant q$. Thus $f_l = 0$ for $1 \leqslant l \leqslant q$ and hence $\bar{h} = 0$. This shows $\mathbf{t} \cong \mathbf{t}H = (T+M)/M$.

Let $U = \bigoplus_{l \in P} Rb_l$. By $M \cap U = 0$ we have $F = (U \oplus M)/M \cong U$ free. Moreover, $T + U + M = T \oplus U \oplus \langle pw_i^k - w_{i-1}^k \mid i \in \mathbb{N}, k \in I\rangle$ and

$$H/(\mathbf{t}H \oplus F) \cong (L/M)/\big((T+M+U)/M\big) \cong L/(T+U+M)$$
$$\cong \frac{\langle w_{i-1}^k \mid i \in \mathbb{N}, k \in I\rangle}{\langle pw_i^k - w_{i-1}^k \mid i \in \mathbb{N}, k \in I\rangle}.$$

Hence $H/(\mathbf{t}H \oplus F)$ is torsion-free and divisible of rank $|I| = d$. Thus the relative divisibility dimension of H is $|I| = d$ and the quotient p-rank of H is $|P| = r$.

From the above it follows that $\{t_i^v + M \mid i \in \mathbb{N}, v \in I_i\}$ is a straight basis of $\mathbf{t}H$ and that F is a relatively maximal pure free submodule in H and $\{w_0^k + \mathbf{t}H \oplus F \mid k \in I\}$ forms a basis of the vector space $H/(\mathbf{t}H \oplus F)$. Moreover, by (2) and the fact that $H/(\mathbf{t}H \oplus F)$ is torsion-free divisible, the set $B = \{\bar{t}_i^v, \overline{w}_{i-1}^k, \bar{b}_l \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$ is a basic generating system of H with relation array (α, η). □

4. Hulls of mixed modules with finite quotient p-rank

In this section we describe mixed modules as homomorphic images of suitable torsion-free modules which are embedded in a vector space over the field K. We give a condition on when two modules with isomorphic torsion submodules having the same finite quotient p-ranks and the same relative divisibility dimensions are isomorphic. If the modules are torsion-free this vector space is the usual divisible hull and all modules of the same (torsion-free) rank can be embedded in that vector space such that an isomorphism between any two can be lifted to an automorphism of the whole space. In an analogous way Theorem 4.6 shows that an isomorphism between two mixed modules having the same hull is induced by a special automorphism of this hull.

First we begin with a useful lemma.

Lemma 4.1. *Let V be a vector space over the field K and let H be an R-submodule of V with free submodule $U \subset H$. Then the following are equivalent:*

(1) *The divisible hull of H is V, i.e. $KH = V$, and U is a basic submodule of H;*
(2) *$H + KU = V$ and $H \cap KU = U$.*

P r o o f. Assume (1). Then $(H \cap KU)/U = H/U \cap KU/U = 0$ since KU/U is torsion and H/U is torsion-free by the purity of U in H. Hence $H \cap KU = U$. But

$$H/U \cong H/(H \cap KU) \cong (H + KU)/KU \subset V/KU$$

implies that $(H + KU)/KU$ is a subspace of V/KU. Hence the quotient

$$V/(H + KU) \cong (V/KU)/\big((H + KU)/KU\big)$$

is torsion-free. Since $V = KH$ it follows that $V/(H + KU)$ is torsion hence 0. Consequently $V = H + KU$. This shows (2).

Now assume (2). It is clear that $KH = V$. Moreover, $H/U \cong H/(H \cap KU) \cong (H + KU)/KU = V/KU$ implies that the submodule U is basic, since V/KU is a vector space. □

Recall that if F is a relatively maximal pure free submodule of a mixed module G with finite quotient p-rank then the quotient p-rank and the relative divisibility dimension equal rank F and $\dim_K\big(G/(\mathbf{t}G \oplus F)\big)$, respectively, and are invariants of G. Now we construct a hull of G.

Construction 4.2. Let $\mathbf{t}$ be a torsion module with representation T/N and let r and d be two cardinal numbers. Let E be a free module of rank r and A be a vector space over the field K of dimension d. Define $V = V(\mathbf{t}, r, d) = KT \oplus KE \oplus A$ to be a vector space over the field K, and a set $\mathcal{M}$ of submodules H in V by

$$\begin{aligned}\mathcal{M} &= \{H \subset V \mid H \cap K(T \oplus E) = T \oplus E \quad \text{and} \quad H + K(T \oplus E) = V\}\\ &= \mathcal{M}(\mathbf{t}, r, d).\end{aligned}$$

By Lemma 4.1 a submodule H of V is in $\mathcal{M}$ if and only if $V = KH$ and $T \oplus E$ is a basic submodule of H. In particular, for $H \in \mathcal{M}(\mathbf{t}, r, d)$ we have $H/(T \oplus E) \cong V/K(T \oplus E) \cong A$.

Note that since r and d are invariants of G, the vector space V is uniquely determined and is defined to be a hull of G.

Now we show that every mixed module with torsion submodule $\mathbf{t}$, quotient p-rank r and relative divisibility dimension d has a very specific description as a homomorphic image of a module in $\mathcal{M}(\mathbf{t}, r, d)$.

Theorem 4.3. *Let G be a mixed module with finite quotient p-rank r and relative divisibility dimension d. Let T/N be a representation of the torsion submodule $\mathbf{t}G$. Then there is a module H in $\mathcal{M} = \mathcal{M}(\mathbf{t}, r, d)$ with $G \cong H/N$.*

Conversely, let $\mathbf{t}$ *be a torsion module with representation* T/N*, let* r *and* d *be cardinal numbers with* r *finite, and let* H *be a torsion-free module in* $\mathcal{M}(\mathbf{t}, r, d)$*, then* H/N *is a mixed module with torsion submodule* $\mathbf{t}(H/N) = T/N$*, quotient* p*-rank* r *and relative divisibility dimension* d*.*

P r o o f. Let G be a mixed module satisfying the hypothesis. We can use the method of Theorem 3.2 to obtain a representation of G as a quotient L/M, where $\mathbf{t} = \mathbf{t}(L/M) = (T + M)/M$ and

$$B = \{t_i^v + M, w_{i-1}^k + M, b_l + M \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$$

a basic generating system with $r = |P|$ and $d = |I|$. We now define $\mathcal{M}(\mathbf{t}, r, d)$. Let $T = \bigoplus_{i\in\mathbb{N}} \bigoplus_{v\in I_i} Rt_i^v$ as given by Theorem 3.2 and define $E = \bigoplus_{l\in P} Rb_l$ and $A = \bigoplus_{k\in I} Kw_0^k$. Thus $V = V(\mathbf{t}, r, d) = KT \oplus KE \oplus A$ and let $\varphi\colon L \longrightarrow V$ be a homomorphism defined by $\varphi t_i^v = t_i^v$, $\varphi b_l = b_l$, $\varphi w_0^k = w_0^k$ and

$$\varphi w_h^k = p^{-h}\left(w_0^k + \sum_{l\in P}\left(\sum_{i=1}^{h} \eta_{i-1,l}^k p^{i-1}\right) b_l + \sum_{j\in\mathbb{N}} \sum_{u\in I_j} \left(\sum_{i=1}^{h} \alpha_{i-1,j}^{k,u} p^{i-1}\right) t_j^u\right)$$

for $i, h \in \mathbb{N}, v \in I_i, k \in I, l \in P$. The homomorphism φ is the identity on $T \oplus E \oplus (\bigoplus_{k\in I} Rw_0^k)$, hence $T \oplus E \oplus (\bigoplus_{k\in I} Rw_0^k) \subset \varphi L$, $N = \varphi N \subset \varphi M$ and $K(\varphi L) = V$. Moreover,

$$\varphi\big(pw_i^k - w_{i-1}^k - \sum_{l\in P} \eta_{i-1,l}^k b_l - \sum_{j\in\mathbb{N}} \sum_{u\in I_j} \alpha_{i-1,j}^{k,u} t_j^u\big) = 0 \quad \text{for} \quad i \in \mathbb{N}, k \in I.$$

Thus $\varphi M = N$ and φ induces an embedding $G \cong L/M \cong \varphi L/N \subset V/N$. This isomorphism implies that $\{t_i^v + N, w_{i-1}^k + N, b_l + N \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$ is a basic generating system of $\varphi L/N$ and

$$\varphi L/(T \oplus E) \cong (\varphi L/N)/\big(T/N \oplus (E \oplus N)/N\big)$$

is torsion-free divisible. Hence $T \oplus E$ is a basic submodule of φL. Therefore Lemma 4.1 holds with $H = \varphi L$ and $U = T \oplus E$. Thus by (2) we have $\varphi L \in \mathcal{M}$ and φL is the desired submodule H of $\mathcal{M}$.

Conversely, if $H \in \mathcal{M}(\mathbf{t}, r, d)$ and T/N is a representation of the torsion module $\mathbf{t}$, then H/N is a mixed module, where $H/(T \oplus E)$ is torsion-free divisible by Lemma 4.1 with $U = T \oplus E$. This shows that $\mathbf{t}(H/N) \subset \big(T/N\big) + \big((E \oplus N)/N\big)$. Since $T \cap (E \oplus N) = N$, the sum in the previous statement is direct and therefore $\mathbf{t}(H/N) =$

T/N. Moreover, $(E \oplus N)/N \cong E$ is a relatively maximal pure free submodule of H/N by

$$(H/N)/\big(\mathbf{t}(H/N) \oplus (E \oplus N)/N\big) = (H/N)/\big((T \oplus E)/N\big) \cong H/(T \oplus E) \cong A$$

with the last isomorphism shown after Construction 4.2, and the rank r of $(E \oplus N)/N$ equals the quotient p-rank of H/N. Moreover, the previous isomorphism shows that d is the relative divisibility dimension of H/N. □

Remark 4.4. Let $\{t_i^v + N, w_{i-1}^k + N, b_l + N \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$ be a basic generating system of H/N, as in Theorem 3.2. Then $\{t_i^v, w_0^k, b_l \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$ is a basis of the vector space $V(\mathbf{t}, r, d)$ because the set $C = \{t_i^v, w_0^k, b_l \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$ is independent in H, hence linearly independent in V. By $H \in \mathcal{M}$ we have $H + K(T \oplus E) = V$ and the set C generates V, i.e. C is a basis of V.

As a direct consequence of Theorem 4.3 we have a corollary which is analogous to that for divisible hulls of torsion-free modules.

Corollary 4.5. *Let G be a mixed module of torsion-free rank f with torsion submodule $\mathbf{t}$ given by a representation T/N. Let V be a vector space with free submodule $T \oplus U$ where U is of rank f and $V/(T \oplus U)$ torsion. Then there is a submodule H of V containing $T \oplus U$ such that $H/N \cong G$ and $T/N = \mathbf{t}(H/N)$.*

In particular, $f = r + d$ where r and d are the quotient p-rank and relative divisibility dimension of G, respectively. Moreover when r is finite, H may be chosen within the uniquely determined set of submodules $\mathcal{M}(\mathbf{t}, r, d)$.

If $\mathbf{t}$ is the torsion submodule of a mixed module with representation T/N, then by Theorem 4.3 the mixed module G is given as a quotient $G \cong H/N$ of a torsion-free module $H \in \mathcal{M}$. We will prove in the next theorem that two mixed modules $G = H/N$ and $G' = H'/N'$ are isomorphic if and only if there is an automorphism Γ of their hulls such that Γ maps H to H' modulo N'. This automorphism Γ induces an isomorphism of $T/N \longrightarrow T/N'$, as in Theorem 2.3.

Theorem 4.6. *Let $\mathbf{t}$ be a torsion module with representations T/N and T/N'. Let H and H' be modules in $\mathcal{M}(\mathbf{t}, r, d)$, where r and d are two cardinal numbers with r finite. The mixed modules H/N and H'/N' are isomorphic if and only if there is an automorphism Γ of $V(\mathbf{t}, r, d)$ which induces the given isomorphism and such that $\Gamma H + N' = H'$ and $\Gamma H \cap N' = \Gamma N$.*

Proof. If the automorphism Γ is given then $H'/N' = (\Gamma H + N')/N' \cong \Gamma H/(N' \cap \Gamma H) = \Gamma H/\Gamma N \cong H/N$.

Conversely, if $\psi\colon H/N \longrightarrow H'/N'$ is an isomorphism we have to show that ψ can be lifted to an automorphism Γ of $V = V(\mathbf{t}, r, d)$ with the prescribed properties. The restriction $\psi|_{T/N}\colon T/N \longrightarrow T/N'$ is an isomorphism between the torsion submodules of H/N and H'/N', respectively. By Theorem 2.3, $\psi|_{T/N}$ can be lifted to an injective endomorphism φ of T which stabilizes a decomposition of T. Let $\{t_i^v + N, w_{i-1}^k + N, b_l + N \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$ be a basic generating system of H/N. Then $\{\psi(t_i^v + N), \psi(w_{i-1}^v + N), \psi(b_l + N) \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$ is a basic generating system of H'/N'. These basic generating systems have equal relation arrays since the isomorphism ψ does not change the relations. Let $\psi(x + N) = x' + N'$. Then by Remark 4.4

$$\{t_i^v, w_0^k, b_l \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$$

and

$$\{\varphi t_i^v, {w'}_0^k, {b'}_l \mid i \in \mathbb{N}, v \in I_i, k \in I, l \in P\}$$

are bases of V and thus the map Γ defined by $\Gamma t_i^v = \varphi t_i^v$, $\Gamma w_0^k = {w'}_0^k$, $\Gamma b_l = b_l'$ for $i \in \mathbb{N}, v \in I_i$, $k \in I, l \in P$ is an automorphism of V. Note that Γ reduces to the given isomorphism ψ. Moreover, by Theorem 2.3 we have $\Gamma N = \varphi N \subset N'$. Let $\bar{\Gamma}$ be the induced homomorphism of V/N to V/N'. In particular,

$$\begin{aligned}
p^{h-1}\bar{\Gamma}(w_{h-1}^k + N) &= \bar{\Gamma}(p^{h-1}w_{h-1}^k + N)\\
&= \bar{\Gamma}\left(w_0^k + \sum_{l\in P}\left(\sum_{i=1}^{h-1}\eta_{i-1,l}^k p^{i-1}\right)b_l + \sum_{j\in\mathbb{N}}\sum_{u\in I_j}\left(\sum_{i=1}^{h-1}\alpha_{i-1,j}^{k,u}p^{i-1}\right)t_j^u + N\right)\\
&= {w'}_0^k + \sum_{l\in P}\left(\sum_{i=1}^{h-1}\eta_{i-1,l}^k p^{i-1}\right)b_l' + \sum_{j\in\mathbb{N}}\sum_{u\in I_j}\left(\sum_{i=1}^{h-1}\alpha_{i-1,j}^{k,u}p^{i-1}\right)\varphi t_j^u + N'\\
&= p^{h-1}({w'}_{h-1}^k + N'),
\end{aligned}$$

as a consequence of Equation (1). Since $T/N' \subset H'/N'$ and ${w'}_{h-1}^k + N' \in H'/N'$ it follows that $\bar{\Gamma}(w_{h-1}^k + N) \in H'/N'$ for all $h \in \mathbb{N}$. Thus the image of $\bar{\Gamma}$ contains a basic generating system of H'/N' and so $\bar{\Gamma}(H/N) = H'/N'$ and $\Gamma H + N' = H'$.

It remains to show $\varphi T \cap N' \subset \varphi N$, since the other inclusion is obvious. By induction we assume $\varphi T_k \cap N' \subset \varphi N$ for all $k < i$. Let $t \in T_i \setminus T_{i-1}$ such that $\varphi t \in N'$. Since φ stabilizes a decomposition of T, $\varphi t \in T_i \cap N' + T_{i-1} = pT_i + T_{i-1}$ by Lemma 2.2. Now, by Theorem 2.3, φ induces an automorphism on $T_i/(pT_i + T_{i-1})$, hence $t \in pT_i + T_{i-1} = (T_i \cap N) + T_{i-1}$, by Lemma 2.2. Thus $t = n + t'$ where $n \in T_i \cap N$ and $t' \in T_{i-1}$. Consequently $\varphi t' = \varphi t - \varphi n \in N'$. By the induction hypothesis and the injectivity of φ we have $t' \in N$, thus $t \in N$ to complete the induction. $\square$

It is worth noting that the automorphism Γ in the if statement of the theorem respects T in much the same way as H, i.e. $\Gamma T + N' = T$. This is a consequence of the facts that H and H' belong to $\mathcal{M}(\mathbf{t}, r, d)$, i.e. $\mathbf{t}(H/N) = T/N \cong T/N' = \mathbf{t}(H'/N')$, and that H/N and H'/N' are isomorphic by the proof of the first part of the theorem.

We remark that Lemma 4.1 and Theorem 4.3 did not make use of the finiteness of the quotient p-rank. It was needed in the Construction 4.2, Corollary 4.5 and Theorem 4.6 in order to uniquely determine a hull for G. This is due to the fact that when the quotient p-rank is infinite the relative divisibility dimension can vary and hence the vector space is not uniquely determined. This is in much the same way that the rank of a quotient of a torsion-free module modulo a basic submodule is not invariant.

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