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THE BOCHNER LAPLACIAN, RIEMANNIAN SUBMERSIONS, HEAT CONTENT ASYMPTOTICS, AND HEAT EQUATION ASYMPTOTICS

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0. Introduction

Let M be a compact Riemannian manifold of dimension m with smooth boundary ∂M. Let $\Gamma(V)$ denote the space of smooth sections to a vector bundle V over M. We assume V is equipped with a pointwise fiber metric $(\cdot,\cdot)$ and a Riemannian connection $\nabla\colon \Gamma(V)\to\Gamma(T^*M\otimes V)$. We shall adopt the Einstein convention and sum over repeated indices. We use that connection ∇ on V and the Levi-Civita connection on the cotangent bundle T^*M to define the second covariant derivative $\nabla^2 f = \mathrm{d}x^i\otimes \mathrm{d}x^j\otimes f_{;ij}$. Let

$$D(f) := -\operatorname{Tr}(\nabla^2 f) = -g^{ij} f_{;ij}$$

be the Bochner or reduced Laplacian; this operator arises in many contexts.

We impose Dirichlet ($\mathscr{B}=\mathscr{B}_D$) or Neumann ($\mathscr{B}=\mathscr{B}_N$) boundary conditions to define a self-adjoint operator $D_{\mathscr{B}}$ of Laplace type. Let $E(\lambda,D,\mathscr{B})\subset\Gamma(V)$ be the eigenspaces of $D_{\mathscr{B}}$; there is an orthogonal direct $\Sigma L^2(V)=\bigoplus_\lambda E(\lambda,D,\mathscr{B})$. Let $k(t,x_1,x_2,D,\mathscr{B})$ be the fundamental solution of the heat equation and let

$$\begin{aligned} a(D,\mathscr{B})(t) &:= \operatorname{Tr}_{L^2}(\mathrm{e}^{-tD_{\mathscr{B}}}) = \sum_\lambda \mathrm{e}^{-t\lambda}\dim(E(\lambda,D,\mathscr{B}))\\ &= \int_M \operatorname{Tr}_{V_x} K(t,x,x,D,\mathscr{B})\,\mathrm{d}x. \end{aligned}$$

As $t\downarrow 0^+$, there is an asymptotic expansion

$$a(D,\mathscr{B})(t)\sim\sum_{n\geqslant 0} a_n(D,\mathscr{B})t^{(n-m)/2}.$$

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The asymptotics of the heat equation, $a_n(D,\mathscr{B})$, are locally computable. If we take the trivial connection on the trivial line bundle over M, the Bochner Laplacian is the usual scalar Laplacian $\Delta^0 = \delta_0 d_0$. We consider the following heat conduction problem. Suppose M has initial temperature 1 at time $t = 0$ and suppose that the boundary is kept at temperature 0 for all $t > 0$. Then the temperature function is $h(t,x) := \int_M K(t,x,y,\Delta^0,\mathscr{B}_D)\,\mathrm{d}y$ and the total heat energy content is

$$\beta_M(t) := \int_{M\times M} K(t,x_1,x_2,\Delta^0,\mathscr{B}_D)\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

Again, as $t \downarrow 0^+$, there is an asymptotic expansion

$$\beta_M(t) \sim \sum_{n\geqslant 0} \beta_n(M) t^{n/2}.$$

The heat content asymptotics, $\beta_n(M)$, are locally computable.

Let $\pi\colon Z \to Y$ be a Riemannian submersion with closed fibers $F(y) := \pi^{-1}(y)$ where Y is a compact manifold with smooth boundary ∂Y. Let V_Y be a vector bundle over Y with a fiber metric and a Riemannian connection ∇_Y. Give the pull back bundle $V_Z := \pi^*(V_Y)$ over Z the pull back fiber metric and pull back Riemannian connection $\nabla_Z := \pi^*\nabla_Y$. Pull-back induces a natural map $\pi^*\colon \Gamma(V_Y) \to \Gamma(V_Z)$ such that $\pi^*\nabla_Y = \nabla_Z\pi^*$. Let D_Y be the Bochner Laplacian on Y and let D_Z be the Bochner Laplacian on Z. In §1, we generalize a theorem of Watson [9] and show that $\pi^* D_Y = D_Z \pi^*$ if and only if the fibers of π are minimal.

If the fibers of π are minimal, then $\mathrm{vol}(F(y)) := \mathrm{vol}(F)$ is independent of y; see [1, 1.10]. We will show in Lemma 2.1 that if we average the heat kernel of D_Z over the fibers we recover the heat kernel of D_Y, i.e.

$$K(t,y_1,y_2,D_Y,\mathscr{B}) = \mathrm{vol}(F)^{-2}\int_{(f_1,f_2)\in F(y_1)\times F(y_2)} K(t,f_1,f_2,D_Z,\mathscr{B})\,\mathrm{d}f_1\,\mathrm{d}f_2.$$

We take $D = \Delta^0$ and $\mathscr{B} = \mathscr{B}_D$ to show that

$$\beta_Z(t) = \mathrm{vol}(F)\beta_Y(t) \quad \text{and} \quad \beta_n(Z) = \mathrm{vol}(F)\beta_n(Y).$$

Principal bundles form a particularly natural family of examples. We shall assume the structure group G is compact and choose a bivariant metric on G. If we choose a G equivariant connection on Z to split $TZ = \mathscr{H} \oplus \mathscr{V}$ into horizontal and vertical fibers, we define a G invariant metric on Z and π becomes a Riemannian submersion with totally geodesic fibers; see Besse [3, 9.59] for details. Let $\pi(z_i) = y_i$. Then

$$K(t,y_1,y_2,D_Y,\mathscr{B}) = \mathrm{vol}(G)^{-2}\int_{(g_1,g_2)\in G\times G} K(t,g_1 z_1,g_2 z_2,D_Z,\mathscr{B})\,\mathrm{d}g_1\,\mathrm{d}g_2.$$

Even in this special situation, these does not seem to be a simple relationship between $a(D_Y, \mathscr{B})(t)$ and $a(D_Z, \mathscr{B})(t)$; the curvature enters in a non-trivial fashion.

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1. The Bochner Laplacian

We refer [6] for the proof of:

Theorem 1.1. *Let $\pi\colon Z \to Y$ be a Riemannian submersion where Z and Y are closed manifolds.*

(a) *If $\varphi \in E(\lambda, \Delta_Y^0)$ and if $\pi^*\varphi \in E(\mu, \Delta_Z^0)$, then $\mu = \lambda$.*

(b) *The following assertions are equivalent:*

  (i) $\pi^*\Delta_Y^0 = \Delta_Z^0\pi^*$.

  (ii) *The fibers of π are minimal submanifolds of Z.*

Remark. See [6,7] for related results about the form valued Laplacian Δ^p; assertion (b) was first proved by Watson [9].

In this section, we generalize Theorem 1.1 to:

Theorem 1.2. *Let $\pi\colon Z \to Y$ be a Riemannian submersion where the fibers of π are compact and where Y is a compact manifold with smooth boundary. Impose Dirichlet or Neumann boundary conditions.*

(a) *If $\varphi \in E(\lambda, D_Y, \mathscr{B})$ and if $\pi^*\varphi \in E(\mu, D_Z, \mathscr{B})$, then $\mu = \lambda$.*

(b) *The following assertions are equivalent:*

  (i) $\pi^* D_Y = D_Z\pi^*$.

  (ii) *The fibers of π are minimal submanifolds of Z.*

Remark. If the fibers of π are minimal submanifolds of Z, then $\operatorname{vol}(F)^{-1/2}\pi^*$ is a partial isometry;

$$(\pi^*\varphi_1, \pi^*\varphi_2)_{L^2(Z)} = \operatorname{vol}(F)(\varphi_1, \varphi_2)_{L^2(Y)} \quad \forall \varphi_1, \varphi_2 \in \Gamma(V_Y).$$

We decompose the tangent bundle $TZ = \mathscr{V} \oplus \mathscr{H}$ into the vertical and horizontal distributions; let $\varrho_{\mathscr{V}}$ and $\varrho_{\mathscr{H}}$ be the corresponding projection operators. We use the metric to identify the tangent and cotangent spaces. Let indices $\{a, b\}$ range from 1 to $\dim(Y)$ and index local orthonormal frames $\{F_a\}$ for TY. Let $f_a = \pi^* F_a$ be the corresponding local orthonormal frames for the horizontal distribution $\mathscr{H}$. Let indices $\{i, j\}$ range from $\dim(Y) + 1$ to $\dim(Z)$ and index local orthonormal frames

$\{e_i\}$ for the vertical distribution $\mathscr{V}$. Let Γ denote the Christoffel symbols of the Levi-Civita connection. The mean curvature vector is defined by

$$\theta := \varrho_{\mathscr{H}}((\nabla z)_{e_i} e_i);$$

$\theta \equiv 0$ if and only if the fibers of π are minimal. Let int denote interior multiplication; if $\widetilde{f} \in \Gamma(V_Z)$, $\operatorname{int}(\theta)(\nabla_z \widetilde{f}) \in \Gamma(V_Z)$. We begin the proof of Theorem 1.2 by establishing a fundamental identity:

Lemma 1.3. $D_Z\pi^* - \pi^* D_Y = \operatorname{int} Z(\theta)\pi^*\nabla_Y$.

Proof of Lemma 1.3. Since the calculations are local, we may assume the vector bundle V is trivial. Let ω be the connection 1-form of the connection ∇_Y. If $\varphi = (\varphi_1, \ldots, \varphi_\nu) \in \Gamma(V_Y)$, let $F_a(\varphi) = (F_a(\varphi_1), \ldots, F_a(\varphi_\nu))$. We expand

$$\nabla_Y \varphi = F^a \otimes \varphi_{;a} \quad \text{and} \quad \nabla_Y^2 \varphi = F^a \otimes F^b \otimes \varphi_{;ab}$$

where $\varphi_{;a} = F_a(\varphi) + \omega_a(\varphi)$ and $\varphi_{;ba} = F_a(\varphi_{;b}) + w_a(\varphi_{;b}) + \Gamma^Y_{acb}\varphi_{;c}$. Thus

$$D_Y\varphi = -(F_a(\varphi_{;a}) + \omega_a(\varphi_{;a}) + \Gamma^Y_{aca}\varphi_{;c}).$$

Let $\widetilde{\varphi} = \pi^*\varphi$. Since $\widetilde{\omega} = \pi^*\omega$, we have that $\nabla_Z\widetilde{\varphi} = \pi^*(\nabla_Y\varphi)$. Thus $\widetilde{\varphi}_{;i} = 0$ so

$$D_Z\widetilde{\varphi} = -(f_a(\widetilde{\varphi}_{;a}) + \omega_a(\widetilde{\varphi}_{;a}) + \Gamma^Z_{aca}\widetilde{\varphi}_{;c}) - \Gamma^Z_{ici}\widetilde{\varphi}_{;c}.$$

Since $\Gamma^Z_{ab}{}^c = \pi^*(\Gamma^Y_{ab}{}^c)$,

$$D_Z\pi^* - \pi^* D_Y = -\Gamma^Z_{ici}\widetilde{\varphi}_{;c} = \operatorname{int}(\theta)(\nabla_Z\widetilde{\varphi}) = \operatorname{int}(\theta)\pi^*(\nabla_Y\varphi).$$

□

Proof of Theorem 1.2. Suppose that $0 \neq \varphi \in E(\lambda, D_Y, \mathscr{B})$ and $\widetilde{\varphi} \in E(\mu, D_Z, \mathscr{B})$ for $\lambda \neq \mu$. Then $(\mu - \lambda)\widetilde{\varphi} = \operatorname{int}(\theta)\pi^*\nabla_Y\varphi$. This implies that

$$(\mu - \lambda)|\widetilde{\varphi}|^2 = \operatorname{int}(\theta)\pi^*(\nabla_Y\varphi, \varphi) = \tfrac{1}{2}\operatorname{int}(\theta)\pi^* d(\varphi, \varphi) = \tfrac{1}{2}(\theta, \pi^* d(|\varphi|^2)).$$

We argue for a contradiction. Choose $y \in Y$ so $|\varphi|^2$ is maximal. If $\widetilde{y} \in \pi^{-1}(y)$, then $|\widetilde{\varphi}|^2$ is maximal at $\widetilde{y}$. If y is in the interior of Y, then $|\varphi|^2$ has an interior local maximum so $\mathrm{d}|\varphi|^2(y) = 0$. Consequently $\frac{1}{2}(\theta, \pi^* d(|\varphi|^2))$ vanishes at $\widetilde{y}$ and $(\lambda - \mu)|\widetilde{\varphi}|^2(\widetilde{y}) = 0$. Since $\lambda \neq \mu$, $|\varphi|^2(y) = 0$. This shows that the maximum value of $|\varphi|^2$ is zero so $\varphi \equiv 0$. This contradiction shows that y belongs to the boundary of Y. If $\mathscr{B} = \mathscr{B}_D$, then φ vanishes on the boundary so $|\varphi|^2$ can not attain its maximum on the boundary and this is impossible. If $\mathscr{B} = \mathscr{B}_N$, the normal derivative of φ vanishes on the boundary. Since $|\varphi|^2$ attains its maximum on the boundary, the tangential derivatives of $|\varphi|^2$ vanish at y so again $d(|\varphi|^2)(y) = 0$ which is impossible. This contradiction proves assertion (a).

The implication (ii) $\Rightarrow$ (i) is an immediate consequence of Lemma 1.3. Conversely, suppose (i) holds. Then $\operatorname{int}(\theta)\pi^*(\nabla_Y\varphi)$ vanishes identically for all $\varphi \in \Gamma(V_Y)$. Since θ is a horizontal differential form, $\theta \equiv 0$ so the fibers are minimal. □

2. Heat kernel

Let $K(t, x_1, x_2, D, \mathscr{B})$ denote the kernel of the fundamental solution of the heat equation for a Bochner Laplacian D with Dirichlet or Neumann boundary conditions $\mathscr{B}$. If $\{\lambda_\nu, \varphi_\nu\}$ is a spectral resolution of D, then

$$K(t, x_1, x_2, D, \mathscr{B}) := \sum_\nu \mathrm{e}^{-t\lambda_\nu}\varphi_\nu(x_1) \otimes \varphi_\nu(x_2).$$

Suppose the fibers of π are minimal. We give the fibers the induced Riemannian metric to define integration over the fibers.

Lemma 2.1. *Let $\pi\colon Z \to Y$ be a Riemannian submersion with minimal fibers.*

$$K(t, y_1, y_2, D_Y, \mathscr{B}) = \operatorname{vol}(F)^{-2} \int_{(f_1,f_2)\in F(y_1)\times F(y_2)} K(t, f_1, f_2, D_Z, \mathscr{B})\,\mathrm{d}f_1\,\mathrm{d}f_2.$$

P r o o f. Since the fibers are minimal, $\Delta_Z\pi^*\varphi_\nu = \lambda_\nu\pi^*\varphi_\nu$. Thus we may take a spectral resolution for Δ_Z of the form

$$\{\{\lambda_\nu, \pi^*\varphi_\nu\}, \{\mu_\sigma, \psi_\sigma\}\}$$

where $\{\mu_\sigma, \psi_\sigma\}$ are spectral resolution of Δ_Z acting on $\pi^*(L^2(V_Y))^\perp$;

$$\int_Z (\psi_\sigma(z), (\pi^*\psi_\nu))(z)\,\mathrm{d}z = 0\ \forall\nu.$$

Let $\Psi_\sigma(y)\colon \int_{z\in F(y)} \psi_\sigma(z)\,\mathrm{d}z = \pi^*(\chi_\sigma)$ for $\chi_\sigma \in \Gamma(V_Y)$. We use Fubini's theorem to express the integral over Z as an iterated integral by first integration over the fibers and then integrating over the base. Since the fibers have constant volume,

$$\operatorname{vol}(F)(\chi_\sigma, \varphi_\nu)_{L^2(Y)} = \int_Z (\psi_\sigma(z), \varphi_\nu(\pi z))\,\mathrm{d}z = 0 \quad \forall\nu.$$

This shows that $\chi_\sigma = 0$; the Lemma now follows. □

The following is now an immediate consequence of Lemma 2.1:

Lemma 2.2. *Let $\pi\colon\ Z \to Y$ be a Riemannian submersion with minimal fibers.*

$$\beta_Z(t) = \mathrm{vol}(F)\beta_Y(t) \text{ and } \beta_n(Z) = \mathrm{vol}(F)\beta_n(Y).$$

Remark. We refer to Theorem A.2 below. In the formula for β_3, there is no term R_{abab}; such a term would spoil this relationship since the fiber variable could enter. Similarly, in the formula for β_5, the are no terms involving $R_{ammb}R_{accb}$ or $L_{aa}L_{bc}R_{bddc}$ as again such terms would spoil this formula.

The analogue of Lemma 2.2 fails for the heat asymptotics $a(\cdot)$ since we must restrict to the diagonal; first averaging over the product of pairs of fibers and then restricting to the diagonal is not easily related to first restricting to the diagonal and then averaging over a single fiber. This is most easily illustrated with a pair of examples

Example 2.3. Let $F = S^1, Y = S^2$, and $Z = S^1 \times S^2$ define the trivial principal bundle

$$S^1 \to S^1 \times S^2 \to S^2.$$

If $\{\lambda_\nu, \varphi_\nu\}$ is a spectral resolution of Δ_Y^0 and $\{\mu_\sigma, \psi_\sigma\}$ is a spectral resolution of Δ_F^0, then $\{\lambda_\nu + \mu_\sigma, \varphi_\nu\psi_\sigma\}$ is a spectral resolution of Δ_Z^0. Thus

$$\begin{aligned} a(\Delta_Z^0)(t) &= a(\Delta_F^0)(t) \cdot a(\Delta_Y^0)(t), \\ a_n(\Delta_Z^0) &= \sum_{p+q=n} a_p(\Delta_F^0) a_q(\Delta_Y^0). \end{aligned}$$

Since $a_p(\Delta_F^0) = 0$ for $p > 0$, we see the invariants rescale;

$$a_n(\Delta_Z^0) = a_0(\Delta_F^0) \cdot a_n(\Delta_Y^0).$$

Example 2.4. Let $F = S_1$, $Y = S^2$, and $Z = S^3$ be the Hopf fibration

$$S^1 \to S^3 \to S^2.$$

The formulas of Theorem A.1 show

$$a_n(Z) \neq a_0(F)a_n(Y);$$

the curvature of the bundle enters. We refer to [8] for an explicit calculation of $a(S^2)(t)$ and $a(S^3)(t)$.

Appendix A

We recall for the convenience of the reader some well known formulas concerning the heat equation and the heat content asymptotics. We impose Dirichlet boundary conditions. Let R be the Riemann curvature tensor and L the second fundamental form. Let τ and ϱ be the scalar curvature and the Ricci tensor. We adopt the following notational conventions that differ from those established in §1. Let $\{e_i\}$ for $1 \leqslant i \leqslant m$ be a local orthonormal frame for the tangent bundle. Near the boundary we let e_m be the inward unit geodesic normal and let indices $a, b, \dots$ range from 1 to $m-1$. See [2, 4, 5] for the proofs of the following results:

Theorem A.1. *Let* $a_n = a_n(M, \Delta_0, \mathscr{B}_D)$.

(a) $a_0 = (4\pi)^{-m/2}\operatorname{vol}(M)$.

(b) $a_1 = -4^{-1}(4\pi)^{-(m-1)/2}\operatorname{vol}(\partial M)$.

(c) $a_2 = (4\pi)^{-m/2}6^{-1}\{\int_M \tau + \int_{\partial M} 2L_{aa}\}$

(d) $a_3(M, \Delta, \mathscr{B}_D) = -(384)^{-1}(4\pi)^{-(m-1)/2}96^{-1}\int_{\partial M}(16\tau + 8R_{amam} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab})$.

(e) $$a_4 = (4\pi)^{-m/2}360^{-1}\Big\{\int_M (12\tau_{;kk} + 5\tau^2 - 2\varrho^2 + 2R^2) + \int_{\partial M}(18\tau_{;m} + 20\tau L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab} + 4R_{abcb}L_{ac} + 24L_{aa:bb} + 40/21L_{aa}L_{bb}L_{cc} - 88/7L_{ab}L_{ab}L_{cc} + 320/21L_{ab}L_{bc}L_{ac})\Big\}.$$

(f) *In the special case that the boundary is totally geodesic, we have*

$$a_5 = -5760^{-1}(4\pi)^{(m-1)/2}\int_{\partial M}(48\tau_{;ii} + 20\tau^2 - 8\varrho^2 + 8R^2 - 20\varrho_{;mm}\tau + 12\tau_{;mm} + 15\varrho_{mm;mm} + 16R_{ammb}\varrho_{ab} - 17\varrho_{mm}\varrho_{mm} - 10R_{ammb}R_{ammb}).$$

Theorem A.2.

(0) $\beta_0 = \operatorname{vol}(M)$.

(1) $\beta_1 = -\frac{2}{\sqrt{\pi}}\operatorname{vol}(\partial M)$.

(2) $\beta_2 = \frac{1}{2}\int_{\partial M} L_{aa}$.

(3) $\beta_3 = -\frac{1}{6\sqrt{\pi}}\int_{\partial M}(L_{aa}L_{bb} - 2L_{ab}L_{ab} - 2\varrho_{mm})$.

(4) $\beta_4 = \frac{1}{32}\int_{\partial M}(-2L_{ab}L_{ab}L_{cc} + 4L_{ab}L_{ac}L_{bc} - 2R_{ambm}L_{ab} + 2R_{abcb}L_{ac} + \tau_{;m})$.

$$
(5)\quad \beta_5 = \frac{1}{240\sqrt{\pi}} \int_{\partial M} (8\varrho_{mm;mm} - 8L_{aa}\varrho_{mm;m} + 16L_{ab}R_{ammb;m} - 4\varrho_{mm}^2 + 16R_{ammb}R_{ammb} - 4L_{aa}L_{bb}\varrho_{mm} - 8L_{ab}L_{ab}\varrho_{mm} + 64L_{ab}L_{ac}R_{mbcm} - 16L_{aa}L_{bc}R_{mbcm} - 8L_{ab}L_{ac}R_{bddc} - 8L_{ab}L_{cd}R_{acbd} + 4R_{abcm}R_{abcm} + 8R_{abbm}R_{accm} - 16L_{aa:b}R_{bccm} - 8L_{ab:c}L_{ab:c} + L_{aa}L_{bb}L_{cc}L_{dd} - 4L_{aa}L_{bb}L_{cd} + 4L_{ab}L_{ab}L_{cd}L_{cd} - 24L_{aa}L_{bc}L_{cd}L_{db} + 48L_{ab}L_{bc}L_{cd}L_{da}).
$$

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