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ON FOUR-POINT BOUNDARY VALUE PROBLEM WITHOUT GROWTH CONDITIONS

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Abstract. We prove the existence of solutions of four-point boundary value problems under the assumption that f fulfils various combinations of sign conditions and no growth restrictions are imposed on f. In contrast to earlier works all our results are proved for the Carathéodory case.

1. Introduction

The paper deals with the four-point boundary value problem

$$x'' = f(t, x, x'), \tag{1}$$

$$x(a) = x(c),\ x(d) = x(b), \tag{2}$$

where $a, b, c, d \in \mathbb{R}$, $a < c \leqslant d < b$, $J = [a, b]$ and $f\colon J \times \mathbb{R}^2 \to \mathbb{R}$ is a function satisfying the Carathéodory conditions. We prove the existence of solutions of (1), (2) provided f fulfils various combinations of sign conditions. We need no growth restrictions for f. The results presented here complete our earlier existence theorems for problem (1), (2) which contained various linear or Nagumo-type growth restrictions, see [2], [3] or [4]. Our method of proofs was partially motivated by [1], where some two-point BVPs were considered. The results of [1] were generalized in several directions in [6] and [7]. In contrast to the papers mentioned all our results here are

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proved for f satisfying the Carathéodory conditions, i.e.

$$f(\cdot,x,y)\colon\ J\to\mathbb{R} \text{ is measurable for all } (x,y)\in\mathbb{R}^2,$$
$$f(t,\cdot,\cdot)\colon\ \mathbb{R}^2\to\mathbb{R} \text{ is continuous for a.e. } t\in J,$$
$$\sup\{|f(\cdot,x,y)|\colon\ |x|+|y|<\varrho\}\in\mathbb{L}_1(J) \text{ for any } \varrho\in\mathbb{R}.$$

In what follows we denote by $\mathbb{C}(J)$ the Banach space of all continuous functions on J with the norm $\|x\| = \{|x(t)|\colon\ t\in J\}$, $\mathbb{X}=\mathbb{C}^1(J)$ the Banach space of all functions having continuous first derivatives on J with the norm $\|x\|^1 = \|x\|+\|x'\|$, $\mathbb{Y}=\mathbb{L}_1(J)$ the Banach space of all Lebesgue integrable functions on J with the norm $\|x\|_1 = \int_a^b |x(t)|\,\mathrm{d}t$, $\mathbb{L}_\infty(J)$ the Banach space of all totally bounded functions on J with the norm $\|x\|_\infty = \operatorname{esssup}\{|x(t)|\ :\ t\in J\}$, $\mathbb{AC}^1(J)$ the set of all functions having absolutely continuous first derivatives on J.

2. Main results

Theorem 1. *Let there exist real numbers R_1, R_2, R_3, R_4, r_1, r_2 such that $r_1\leqslant r_2$, $R_1\neq R_3$, $R_2\neq R_4$, $R_1\leqslant 0\leqslant R_2$, $R_3\leqslant 0\leqslant R_4$, and for a.e. $t\in J$ let*

$$f(t,r_1,0)\leqslant 0,\ \ f(t,r_2,0)\geqslant 0, \tag{3}$$
$$f(t,x,R_2)\geqslant 0,\ \ f(t,x,R_1)\leqslant 0 \text{ for all } x\in[r_1,r_2]. \tag{4}$$

Further, for a.e. $t\in[d,b]$ and all $x\in[r_1,r_2]$ let

$$f(t,x,R_3)\geqslant 0,\ \ f(t,x,R_4)\leqslant 0. \tag{5}$$

Then problem (1), (2) *has at least one solution u which for all $t\in J$ fulfils the inequalities*

$$r_1\leqslant u(t)\leqslant r_2, \tag{6}$$
$$\min\{R_1,R_3\}\leqslant u'(t)\leqslant \max\{R_2,R_4\}. \tag{7}$$

Example 2. Function f fulfilling the conditions of Theorem 1 can quickly grow in x and y on J, but on the other hand it cannot be monotonous in y on $[d,b]$. Suppose that $h\in[1,\infty)$, $h_1\in\mathbb{L}_1(J)$, $h_1(t)>0$ for a.e. $t\in J$, $h_2\in\mathbb{L}_\infty(J)$, $\|h_2\|_\infty<h$, $n,k\in\mathbb{N}$, $n>k$. Then the function

$$f(t,x,y)=h_1(t)(-x^{2k+1}+y^{2n+1}+h_2(t))(y^2-h^2)$$

satisfies Theorem 1 for $r_1=-h$, $r_2=h$, $R_1=-2h$, $R_2=2h$, $R_3=-h$, $R_4=h$.

Theorem 3. *Let there exist real numbers R_1, R_2, R_3, R_4, r_1, r_2 such that $r_1 \leqslant r_2$, $R_1 \neq R_3$, $R_2 \neq R_4$, $R_1 \leqslant 0 \leqslant R_2$, $R_3 \leqslant 0 \leqslant R_4$, and for a.e. $t \in J$ let*

$$f(t,x,0) \geqslant 0 \text{ for all } x \in [r_1 + L_1(b-a), r_1], \tag{8}$$

$$f(t,x,0) \leqslant 0 \text{ for all } x \in [r_2, r_2 + L_2(b-a)], \tag{9}$$

where $L_1 = \min\{R_1, R_3\}$, $L_2 = \max\{R_2, R_4\}$. Further, for all $x \in [r_1 + L_1(b-a), r_2 + L_2(b-a)]$ let

$$f(t,x,R_2) \geqslant 0, \ f(t,x,R_1) \leqslant 0 \text{ for a.e. } t \in J, \tag{10}$$

$$f(t,x,R_3) \geqslant 0, \ f(t,x,R_4) \leqslant 0 \text{ for a.e. } t \in [d,b]. \tag{11}$$

Then problem (1), (2) *has at least one solution u which for all $t \in J$ fulfils the inequalities*

$$r_1 + L_1(b-a) \leqslant u(t) \leqslant r_2 + L_2(b-a), \quad L_1 \leqslant u'(t) \leqslant L_2. \tag{12}$$

Example 4. A function f fulfilling the conditions of Theorem 2 can have the form

$$f(t,x,y) = h_1(t)(-x + \sin 2\pi t + 7\sin y),$$

where $r_1 = -1$, $r_2 = 1$, $R_1 = -\pi/2$, $R_2 = \pi/2$, $R_3 = -3\pi/2$, $R_4 = 3\pi/2$ and $h_1 \in \mathbb{L}_1(J)$ is strictly positive, $J = [0,1]$.

3. Proofs

We will work with a one-parameter system

$$x'' = \lambda f^*(t,x,x',\lambda), \quad \lambda \in [0,1] \tag{13}$$

where $f^*\colon J \times (\mathbb{R}^2 \times [0,1]) \to \mathbb{R}$ satisfies the Carathéodory conditions and

$$f^*(t,x,y,1) = f(t,x,y) \ \text{ on } J \times \mathbb{R}^2.$$

Put

$$f_0(x) = \frac{1}{b-d}\int_d^b \int_a^s f^*(t,x,0,0)\,\mathrm{d}t\,\mathrm{d}s - \frac{1}{c-a}\int_a^c \int_a^s f^*(t,x,0,0)\,\mathrm{d}t\,\mathrm{d}s. \tag{14}$$

Our proofs are based on the following lemma.

Lemma 5. *Let there exist an open bounded set $\Omega \subset \mathbb{X}$ such that*

(a) *for any $\lambda \in (0,1)$, each solution u of problem* (13), (2) *satisfies $u \notin \partial\Omega$;*

(b) *for any root $x_0 \in \mathbb{R}$ of the equation $f_0(x) = 0$, the condition $x_0 \notin \partial\Omega$ is fulfilled, where x_0 is considered a constant function on J;*

(c) *the Brouwer degree $d[f_0, D, 0] \neq 0$, where $D \subset \mathbb{R}$ is the set of constants c such that the functions $u(t) \equiv c$ belong to Ω.*

Then problem (1), (2) *has at least one solutions in $\overline{\Omega}$.*

P r o o f. See [5]. □

Lemma 6. *Let there exist $r_1, r_2 \in \mathbb{R}$, $K \in (0,\infty)$ such that $r_1 \leqslant r_2$ and for a.e. $t \in J$ the inequalities* (3) *and*

$$\int_a^b |f(t,x,y)| \, \mathrm{d}t \leqslant K \text{ for all } x \in [r_1, r_2], y \in \mathbb{R} \tag{15}$$

are satisfied. Then problem (1), (2) *has at least one solution u with the property* (6).

P r o o f. Choose an arbitrary fixed $m \in \mathbb{N}$, $m > 1$. For $(t,x,y) \in D$ put

$$f_m(t,x,y) = \begin{cases} f(t,r_2,0) & \text{for } x \geqslant r_2 + \frac{1}{m}, \\ f(t,r_2,y) + [f(t,r_2,0) - f(t,r_2,y)]m(x-r_2) & \text{for } r_2 < x < r_2 + \frac{1}{m}, \\ f(t,x,y) & \text{for } r_1 \leqslant x \leqslant r_2, \\ f(t,r_1,y) - [f(t,r_1,0) - f(t,r_1,y)]m(x-r_1) & \text{for } r_1 - \frac{1}{m} < x < r_1, \\ f(t,r_1,0) & \text{for } x \leqslant r_1 - \frac{1}{m} \end{cases}$$

and consider system (13), where

$$f^*(t,x,y,\lambda) = \lambda f_m(t,x,y) + (1-\lambda)\left[\frac{x-r_1}{r_2-r_1+1}\right].$$

Put $r = 1 + \max\{|r_1|, |r_2|\}$ and define a set

$$\Omega = \{x \in \mathbb{X} \colon \ \|x\| < r, \|x'\| < K + (b-a)\}. \tag{16}$$

Let us check that problem (13), (2) fulfils the conditions of Lemma 1 on Ω.

(a): Let us prove that for any $\lambda \in (0,1)$ no solution of (13), (2) belongs to $\partial\Omega$. Let u be a solution of this problem for some $\lambda \in (0,1)$. Put $v(t) = u(t) - r_2 - \frac{1}{m}$ and suppose that $\max\{v(t) \colon \ t \in J\} = v(t_0) > 0$. Since $v(a) = v(c)$ and $v(b) = v(d)$, we can suppose that $t_0 \in (a,b)$. Thus there exists an interval $(\alpha,\beta) \subset (a,b)$ containing t_0 with $v(t) \geqslant 0$ for each $t \in (\alpha,\beta)$, $v'(\alpha) \geqslant 0$, $v'(\beta) \leqslant 0$. Hence we get for a.e. $t \in (\alpha,\beta)$

$$v''(t) = u''(t) = \lambda \left(\lambda f_m(t,u,u') + (1-\lambda) \left[\frac{u-r_1}{r_2-r_1+1} \right] \right) > 0.$$

Integrating the last inequality, we obtain a contradiction

$$0 \geqslant v'(\beta) - v'(\alpha) > 0.$$

Thus $v(t) \leqslant 0$ on J, which means that $u(t) \leqslant r_2 + \frac{1}{m}$ for all $t \in J$. By an analogous argument we prove that $u(t) \geqslant r_1 - \frac{1}{m}$ for all $t \in J$. Conditions (2) guarantee the existence of at least one zero of u' on J, so integrating (13) and using (15) we get $\|u'\| < K + (b-a)$. Therefore $u \notin \partial\Omega$.

(b): In view of (14)

$$f_0(x) = \frac{b+d-a-c}{2} \cdot \frac{x-r_1}{r_2-r_1+1},$$

thus the equation $f_0(x) = 0$ has the unique root $x_0 = r_1$, and the constant function $u_0(t) \equiv r_1$ does not belong to $\partial\Omega$.

(c): Since $D = (-r,r)$ and $f_0(-r) < 0$, $f_0(r) > 0$, the Brouwer degree $d[f_0,D,0] \neq 0$. Therefore Lemma 1 implies that the problem

$$x'' = f_m(t,x,x'), \ (2) \tag{17}$$

has at least one solution in $\overline{\Omega}$. Repeating this argument for each $m \in \mathbb{N}$, we obtain a sequence $(u_m)_1^\infty$ of solutions of problems (17). We can see that the sequence is bounded and equi-continuous in $\mathbb{X}$ and so, by the Arzelà-Ascoli Theorem it is possible to choose a subsequence converging in $\mathbb{X}$ to a function u_0. Since $r_1 - \frac{1}{m} \leqslant u_m(t) \leqslant r_2 + \frac{1}{m}$, u_0 satisfies (6) and thus it is a solution of (1), (2). □

Lemma 7. *Let there exist $r_1, r_2 \in \mathbb{R}$, $K \in (0,\infty)$ such that $r_1 \leqslant r_2$ and for a.e. $t \in J$ the inequalities*

$$f(t,x,0) \geqslant 0 \text{ for all } x \leqslant r_1, \tag{18}$$

$$f(t,x,0) \leqslant 0 \text{ for all } x \geqslant r_2, \tag{19}$$

and

$$\int_a^b |f(t,x,y)| \, \mathrm{d}t \leqslant K \text{ for all } x,y \in \mathbb{R} \tag{20}$$

are satisfied. Then problem (1), (2) has at least one solution u with the property

$$r_1 \leqslant u(t_u) \leqslant r_2, \tag{21}$$

where t_u is a point in (a,b).

P r o o f. For $t \in J$, $x, y \in \mathbb{R}$, $m \in \mathbb{N}$ and $\lambda \in [0,1]$ put

$$f_m(t,x,y) = \begin{cases} f(t,x,y) & \text{for } |y| > \frac{2}{m}, \\ f(t,x,y) + [f(t,x,0) - f(t,x,y)]\, m(\frac{2}{m} - |y|) & \text{for } \frac{1}{m} < |y| \leqslant \frac{2}{m}, \\ f(t,x,0) & \text{for } |y| \leqslant \frac{1}{m} \end{cases}$$

and consider system (13), where

$$f^*(t,x,y,\lambda) = \lambda f_m(t,x,y) + (1-\lambda)\frac{r_2 - x}{|r_2| + |x|}.$$

Put $r = 1 + \max\{|r_1|, |r_2|\} + (b-a)K + (b-a)^2$ and define a set Ω by (16). Now we can follow the proof of Lemma 2. The only difference is that we prove $\min\{u(t)\colon t \in J\} \leqslant r_2$ and $\max\{u(t) : t \in J\} \geqslant r_1$, which implies (21). Then by Lemma 1 and a limiting proces we get a solution u of (1), (2) with property (21). □

P r o o f o f T h e o r e m 1. Suppose that $R_3 < R_1$ and $R_4 > R_2$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geqslant n_0$ the inequalities $R_2 + \frac{2}{n} < R_4$, $R_1 - \frac{2}{n} > R_3$ are satisfied. For $n \geqslant n_0$ put

$$h_n(t,x,y) = \begin{cases} f(t,x,R_4) & \text{for } R_4 < y, \\ f(t,x,y) & \text{for } R_2 + \frac{2}{n} \leqslant y \leqslant R_4, \\ f(t,x,R_2 + \frac{2}{n}) + w_2 & \text{for } \frac{1}{n} + R_2 < y < R_2 + \frac{2}{n}, \\ f(t,x,R_2) & \text{for } R_2 < y \leqslant R_2 + \frac{1}{n}, \\ f(t,x,y) & \text{for } R_1 \leqslant y \leqslant R_2, \\ f(t,x,R_1) & \text{for } -\frac{1}{n} + R_1 \leqslant y < R_1, \\ f(t,x,R_1 - \frac{2}{n}) - w_1 & \text{for } R_1 - \frac{2}{n} < y < R_1 - \frac{1}{n}, \\ f(t,x,y) & \text{for } R_3 \leqslant y \leqslant R_1 - \frac{2}{n}, \\ f(t,x,R_3) & \text{for } R_3 > y \end{cases}$$

where

$$w_2 = \left[f\left(t,x,R_2 + \frac{2}{n}\right) - f\left(t,x,R_2\right)\right] n\left(y - R_2 - \frac{2}{n}\right),$$
$$w_1 = \left[f\left(t,x,R_1 - \frac{2}{n}\right) - f\left(t,x,R_1\right)\right] n\left(y - R_1 + \frac{2}{n}\right).$$

Then h_n fulfils (15) with K given by

$$K = \int_a^b \left(\sup\{|h_n(t,x,y)| : x \in [r_1, r_2], y \in [R_3, R_4]\}\right) \mathrm{d}t.$$

Since h_n fulfils (3), we get by Lemma 2 that the problem

$$x'' = h_n(t, x, x'), (2) \tag{22}$$

has a solution u_n satisfying (6). Let us prove a priori estimates for u_n' which are independent of u_n. It follows from (2) that there exist points $a_0 \in (a, c)$, $b_0 \in (d, b)$ with $u_n'(a_0) = u_n'(b_0) = 0$. Suppose that $\max\{u_n'(t) : t \in [a, b_0]\} = u_n'(z_0) > R_2 + \frac{1}{n}$. Then $z_0 \neq b_0$ and there exists $(\alpha, \beta) \subset (a, b_0)$ such that $u_n'(\beta) = R_2$, $u_n'(\alpha) = R_2 + \frac{1}{n}$ and $R_2 \leqslant u_n'(t) \leqslant R_2 + \frac{1}{n}$ for all $t \in (\alpha, \beta)$. Thus

$$0 > \int_\alpha^\beta u_n''(t)\,\mathrm{d}t = \int_\alpha^\beta f(t, u_n, R_2)\,\mathrm{d}t \geqslant 0,$$

a contradiction. A similar contradiction occurs provided $\min\{u_n'(t) : t \in [a, b_0]\} < R_1 - \frac{1}{n}$. Thus we have proved the estimate on $[a, b_0]$. Now, suppose that $\max\{u_n'(t) : t \in [b_0, b]\} = u_n'(z_1) > R_4 + \frac{1}{n}$. Then $z_1 \in (b_0, b]$ and there exists $(\alpha, \beta) \subset (b_0, b)$ such that $u_n'(\alpha) = R_4$, $u_n'(\beta) = R_4 + \frac{1}{n}$ and $R_4 \leqslant u_n'(t) \leqslant R_4 + \frac{1}{n}$ for all $t \in (\alpha, \beta)$. Thus

$$0 < \int_\alpha^\beta u_n''(t)\,\mathrm{d}t = \int_\alpha^\beta f(t, u_n, R_4)\,\mathrm{d}t \leqslant 0,$$

a contradiction. Similarly for $\min\{u_n'(t) : t \in [b_0, b]\} < R_3 - \frac{1}{n}$. So, we have proved the estimate on $[b_0, b]$, and therefore

$$R_3 - \frac{1}{n} \leqslant u_n'(t) \leqslant R_4 + \frac{1}{n} \text{ for all } t \in J. \tag{23}$$

From (6) and (23) it follows that the sequence of solutions $(u_n)_{n_0}^\infty$ to problems (22) is bounded and equi-continuous in $\mathbb{X}$ and thus by a limiting process we can get a function u which is a solution of problem

$$x'' = h(t, x, x'), (2) \tag{24}$$

where

$$h(t, x, y) = \begin{cases} f(t, x, R_4) & \text{for } y > R_4, \\ f(t, x, y) & \text{for } R_3 \leqslant y \leqslant R_4, \\ f(t, x, R_3) & \text{for } y < R_3. \end{cases}$$

By (23), u fulfils the inequality $R_3 \leqslant u'(t) \leqslant R_4$ for all $t \in J$, and thus it is a solution of (1), (2) with the properties (6) and (7).

In the case of $R_3 > R_1$, $R_2 < R_4$ we replace R_1 by R_3 in the formula for h_n and prove the existence of a solution u by the same argument. Similarly in the case of $R_4 < R_2$. □

P r o o f o f T h e o r e m 2. Using Lemma 3 instead of Lemma 2, we can argue similarly as in the proof of Theorem 1, only in the formula for the auxiliary function h_n we use a function g instead of f, where

$$g(t,x,y) = \begin{cases} f(t, r_2 + R_4(b-a), y) & \text{for } x > r_2 + R_4(b-a), \\ f(t,x,y) & \text{for } r_1 + R_3(b-a) \leqslant x \leqslant r_2 + R_4(b-a), \\ f(t, r_1 + R_3(b-a), y) & \text{for } x < r_1 + R_3(b-a). \end{cases}$$

□

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