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WHEN DOES THE INVERSE HAVE THE SAME SIGN PATTERN AS THE TRANSPOSE?

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Abstract. By a sign pattern (matrix) we mean an array whose entries are from the set $\{+,-,0\}$. The sign patterns A for which *every* real matrix with sign pattern A has the property that its inverse has sign pattern A^T are characterized. Sign patterns A for which *some* real matrix with sign pattern A has that property are investigated. Some fundamental results as well as constructions concerning such sign pattern matrices are provided. The relation between these sign patterns and the sign patterns of orthogonal matrices is examined.

0. Introduction

A *sign pattern matrix* $A = (a_{ij})$ is a matrix whose entries are from the set $\{+,-,0\}$. The set of all $m \times n$ sign pattern matrices is denoted by $Q_{m,n}$, while the set of all $n \times n$ sign pattern matrices is denoted by Q_n. A matrix $A \in Q_n$ is frequently referred to as a *sign pattern* or simply a *pattern*. Given a real matrix $B = (b_{ij})$, the sign pattern of B is $\operatorname{sgn} B = (\operatorname{sgn}(b_{ij}))$, where $\operatorname{sgn}(b_{ij})$ is $+$, $-$, or 0, whenever b_{ij} is positive, negative, or zero, respectively. The set of all real $n \times n$ matrices B for which A is the sign pattern of B is known as the *sign pattern class* of A and is given by

$$Q(A) = \{B \in M_n(\mathbb{R}) \mid \operatorname{sgn} B = A\}.$$

In recent years, researchers have been interested in predicting the sign patterns of the inverses of real invertible matrices in a sign pattern class $Q(A)$. In the paper [JLR], the authors ask the following question:

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When does the inverse of a real matrix B have the same sign pattern as B^T?

We refer to this question as the Inverse-Transpose problem, and we wish to characterize the sign patterns A of all real invertible matrices B such that $\mathrm{sgn}(B^{-1}) = \mathrm{sgn}(B^T)$.

Let P be a property referring to a real matrix. We say a sign pattern matrix A *requires* P if every matrix in $Q(A)$ has property P, and A *allows* P if some matrix in $Q(A)$ has property P. In this paper we are interested in the Inverse-Transpose property T, which is defined by "$\mathrm{sgn}(B^{-1}) = \mathrm{sgn}(B^T)$."

In section 1, we review relevant elementary qualitative concepts, and we introduce definitions and fundamental results. The next section is devoted to characterizing sign patterns that require T. The discussion of sign pattern matrices that allow T begins in section 3 with several necessary conditions. In addition, necessary and sufficient conditions are given for two special classes of $n \times n$ sign pattern matrices that allow T, where $n \leqslant 4$, or where $n = 5$ and A is entrywise nonzero. The goal of section 4 is to construct large classes of sign patterns that allow T. Section 5 concludes this paper with several remarks, and some interesting open questions.

1. Preliminaries

We use the symbol $\#$ to denote an *ambiguous* quantity, namely, $\# = (+) + (-)$. We define a *generalized sign pattern matrix* $A = (a_{ij})$ as a $(+, -, 0, \#)$ matrix, and the sign pattern class of such an $n \times n$ matrix is given by

$$Q(A) = \{B = (b_{ij}) \in M_n(\mathbb{R}) \mid a_{ij} = \# \text{ or } a_{ij} = \mathrm{sgn}(b_{ij})\}.$$

Note that every sign pattern matrix is also a generalized sign pattern matrix. We denote the set of $n \times n$ generalized sign pattern matrices by $\overline{Q}_n$. Generalized sign pattern matrices frequently occur as a result of sign pattern matrix multiplication. For example, if $A = \begin{pmatrix} + & - \\ + & + \end{pmatrix}$, then $AA^T = \begin{pmatrix} + & \# \\ \# & + \end{pmatrix} \in \overline{Q}_2$. If an operation results in a sign pattern $A \in Q_n$, then the operation is said to be *unambiguously defined.* For example, AA^T is unambiguously defined when $A = \begin{pmatrix} + & - \\ - & + \end{pmatrix}$, but not when $A = \begin{pmatrix} + & - \\ + & + \end{pmatrix}$.

The determinant of a sign pattern matrix $A \in Q_n$ is computed in the usual way as a sum of $n!$ terms. It is easy to see that every $B \in Q(A)$ is nonsingular if and only if $\det(A) = +$ or $\det(A) = -$. Such sign patterns are said to be *sign nonsingular*. Similarly, if $\det(A) = 0$, then every $B \in Q(A)$ is singular and A is said to be *sign singular*. For a square sign pattern A that is not sign singular, we define $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$. In general, $A^{-1} \in \overline{Q}_n$.

We say two patterns $A, A' \in \overline{Q}_n$ are *compatible* if, for all $i, j \in \{1, 2, \ldots, n\}$, either $a_{ij} = a'_{ij}$, or one of a_{ij} and a'_{ij} is #. Equivalently, A and A' are compatible if and only if $Q(A) \cap Q(A') \neq \emptyset$. We write $A \overset{c}{\longleftrightarrow} A'$ when A and A' are compatible. For example,

$$\begin{pmatrix} \# & 0 \\ + & - \end{pmatrix} \overset{c}{\longleftrightarrow} \begin{pmatrix} - & \# \\ + & \# \end{pmatrix}.$$

We now define some important special sign pattern matrices. A *signature pattern* $S \in Q_n$ is a diagonal sign pattern matrix whose diagonal entries are either $+$ or $-$. We note that as a consequence of the definition of S, S^2 is a diagonal sign pattern whose diagonal entries are all $+$. We call such an $n \times n$ pattern an *identity pattern* and denote it by I. It is easy to see that if A is a square sign pattern that is not sign singular, then $AA^{-1} \overset{c}{\longleftrightarrow} I$ and $A^{-1}A \overset{c}{\longleftrightarrow} I$. In addition, we see that if $B \in Q(A)$ is invertible, then $\operatorname{sgn}(B^{-1}) \overset{c}{\longleftrightarrow} \operatorname{adj}(A)$ or $\operatorname{sgn}(B^{-1}) \overset{c}{\longleftrightarrow} -\operatorname{adj}(A)$. A *permutation pattern* (*generalized permutation pattern*) P is obtained by substituting a $+$ ($+$ or $-$) sign into a real permutation matrix wherever a 1 appears. We note that any generalized permutation pattern can be written as the product of a permutation pattern and a signature pattern.

The set of all sign patterns that allow property T will be denoted by $\mathcal{T}$. More specifically, the set of all $n \times n$ patterns that allow T will be denoted by $\mathcal{T}_n$. Frequently, the set of patterns that allow a property is quite large and difficult to analyze. It turns out that the general characterization of $\mathcal{T}$ is formidable.

We first give basic necessary conditions for a pattern A to be in $\mathcal{T}$.

Proposition 1.1. *If $A \in \mathcal{T}$, then A is not sign singular.*

Proposition 1.2. *If $A \in \mathcal{T}$, then $AA^T \overset{c}{\longleftrightarrow} I$ and $A^TA \overset{c}{\longleftrightarrow} I$.*

Proof. Assume A allows T. Then there exists some invertible $B \in Q(A)$ such that $B^{-1} \in Q(A^T)$. It is easily verified that $BB^{-1} \in Q(AA^T)$ and $B^{-1}B \in Q(A^TA)$, and it follows that $AA^T \overset{c}{\longleftrightarrow} I$ and $A^TA \overset{c}{\longleftrightarrow} I$. $\square$

We note that the converse of Proposition 1.2 is not true in general.

Example 1.3. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & + & + & + & + \\ 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & + & - & + & - \\ + & + & + & 0 & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 & 0 \\ + & + & - & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_1 \\ A_1^T & 0 \end{pmatrix}.$$

Then clearly, $AA^T \stackrel{c}{\longleftrightarrow} I$ and $A^TA \stackrel{c}{\longleftrightarrow} I$. However, for every $B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix} \in Q(A)$, $\operatorname{rank}(B) = \operatorname{rank}(B_1) + \operatorname{rank}(B_2) \leqslant 3 + 3 = 6$. Thus A cannot allow T.

Example 1.3 shows that $AA^T \stackrel{c}{\longleftrightarrow} I$ and $A^TA \stackrel{c}{\longleftrightarrow} I$ do not imply A allows T for all 7×7 patterns. The question naturally arises as to what is the largest n for which the converse of Proposition 1.2 is true, which is investigated in section 3.

A square sign pattern matrix satisfying the compatibility conditions given in Proposition 1.2 is said to be *sign potentially orthogonal*, and the class of all $(n \times n)$ sign potentially orthogonal patterns is denoted by $\mathcal{SPO}$ ($\mathcal{SPO}_n$). Hence, the conditions $AA^T \stackrel{c}{\longleftrightarrow} I$ and $A^TA \stackrel{c}{\longleftrightarrow} I$ are called the $\mathcal{SPO}$ conditions.

Which sign pattern matrices allow orthogonality? In other words, given $A \in Q_n$, is there a $B \in Q(A)$ such that $BB^T = I$? This question was originally raised by M. Fiedler (see [F, problem 12, p. 160]), and was also posed in 1991 by C. Johnson at a conference at Georgia State University. Such patterns for which the answer to the question is "yes" are said to be *potentially orthogonal*. The class of all potentially orthogonal patterns is denoted by $\mathcal{PO}$, and the class of all $n \times n$ potentially orthogonal patterns is denoted by $\mathcal{PO}_n$.

Clearly, if B is a real orthogonal matrix, then $\operatorname{sgn} B \in \mathcal{T}$, so that $\mathcal{PO} \subseteq \mathcal{T}$. Consequently, we have the following relationships.

Proposition 1.4. $\mathcal{PO} \subseteq \mathcal{T} \subset \mathcal{SPO}$.

It is not yet known if the first inclusion is proper. This open question will be studied in more detail in section 3.

Since the closure properties of $\mathcal{T}$ are straightforward, we state Lemma 1.5 without proof.

Lemma 1.5. *$\mathcal{T}$ is closed under the following operations:*

(i) *transposition,*

(ii) *generalized permutation equivalence (multiplication on the left and right by two generalized permutation patterns),*

(iii) *generalized permutation and signature similarity (multiplication on the left and right by a generalized permutation pattern and its inverse, or by a signature pattern and its inverse),*
(iv) *Kronecker (tensor) product,*
(v) *direct sum.*

It can be easily verified that the classes $\mathcal{PO}$ and $\mathcal{SPO}$ are also closed under the operations given in Lemma 1.5. Further, it is easily shown that if $A_1, A_2 \in \mathcal{T}_n$, then $A_1A_2 \in \mathcal{T}_n$ whenever A_1A_2 is unambiguously defined. However, for $\mathcal{PO}_n$, we have the following stronger result.

Proposition 1.6. *If $A_1, A_2 \in \mathcal{PO}_n$, then there is some $A \in Q_n$, with $A \stackrel{c}{\longleftrightarrow} A_1A_2$, such that $A \in \mathcal{PO}_n$.*

Proof. $A_1, A_2 \in \mathcal{PO}_n$ implies that there exist $B_1 \in Q(A_1)$, $B_2 \in Q(A_2)$ such that $B_1B_1^T = I$, $B_2B_2^T = I$. Note that $B_1B_2 \in Q(A_1A_2)$ where $A_1A_2 \in \overline{Q}_n$. Let $B = B_1B_2$ and let $A = \operatorname{sgn}(B)$. Then $B_1B_2 \in Q(A)$ and $A \stackrel{c}{\longleftrightarrow} A_1A_2$. It follows that $BB^T = (B_1B_2)(B_1B_2)^T = I$ and $A \in \mathcal{PO}_n$. □

2. Characterization of patterns that require T

We say $A = (a_{ij}) \in Q_n$ is a *nonnegative sign pattern matrix* if $a_{ij} \in \{0, +\}$ for all $i, j \in \{1, 2, \ldots, n\}$, and we write $A \geqslant 0$. We begin this section with a characterization of nonnegative patterns that require T. Since the proof is clear, it is omitted.

Proposition 2.0. *Let $A \in Q_n$ be a nonnegative sign pattern matrix. Then the following are equivalent:*
i) *A allows T,*
ii) *A is a permutation pattern,*
iii) *A requires T,*
iv) *A is potentially orthogonal.*

We say a square pattern A is *partly decomposable* if there exist permutation patterns P and Q such that $PAQ = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, where A_1 and A_3 are square sign pattern matrices. A is *fully indecomposable* if it is not partly decomposable. The definition of a partly decomposable real matrix is completely analogous.

It is easy to show that if B is a partly decomposable real orthogonal matrix, such that $B = P\begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}Q$, where P and Q are permutation matrices, and B_1 and

B_3 are square, then B_2 is a zero matrix. Thus $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \in \mathcal{PO}$ (where A_1 and A_3 are square) implies A_2 is a zero pattern.

The next lemma provides a more general result for partly decomposable patterns in $\mathcal{T}$.

Lemma 2.1. *Let A_1, A_3 be square sign pattern matrices. Then*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \in \mathcal{T} \textit{ if and only if } A_1, A_3 \in \mathcal{T} \textit{ and } A_2 = 0.$$

Proof. ($\Rightarrow$) If $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \in \mathcal{T}$, then there exists $B \in Q(A)$ such that

$$B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} B_1^{-1} & -B_1^{-1} B_2 B_3^{-1} \\ 0 & B_3^{-1} \end{pmatrix} \in Q(A^T).$$

It follows that $A_2^T = 0$ and, hence, $A_2 = 0$. Also, $B_1^{-1} \in Q(A_1^T)$ and $B_3^{-1} \in Q(A_3^T)$ imply $A_1, A_3 \in \mathcal{T}$.

($\Leftarrow$) Closure of $\mathcal{T}$ under direct sum. $\square$

Theorem 2.2. *$A \in \mathcal{T}$ if and only if all fully indecomposable components of A allow T, and A is permutation equivalent to the direct sum of its fully indecomposable components.*

Proof. ($\Rightarrow$) Assume $A \in \mathcal{T}$. If A is fully indecomposable, then there is nothing to prove. Assume A is partly decomposable. Then by Lemma 2.1, there exist permutation patterns P and Q such that

$$PAQ = \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix} \quad \text{and} \quad A_1, A_3 \in \mathcal{T}.$$

Repeat the same argument for A_1 and A_3, and if necessary, continue to see that all fully-indecomposable components of A allow T. We then also see that A is permutation equivalent to the direct sum of its fully indecomposable components.

($\Leftarrow$) Closure of T under direct sum and permutation equivalence. $\square$

Statements similar to Theorem 2.2 hold for the class of patterns that require T and the class $\mathcal{PO}$. Thus it suffices to study only fully indecomposable patterns in order to describe all patterns that require T.

It is interesting to note that Fiedler conjectured that a fully indecomposable orthogonal matrix of order n has at least $4n - 4$ nonzero entries. This was recently proved in [BBS],and their proof shows that this holds for $\mathcal{T}$.

According to [BCS], A is said to be *strong sign nonsingular* if A is sign nonsingular and the inverses of the matrices in $Q(A)$ are in the same sign pattern class. We note that A is strong sign nonsingular if and only if A is sign nonsingular, and each $(n-1)\times(n-1)$ submatrix of A is sign nonsingular or sign singular (i.e., A^{-1} is unambiguously defined). Clearly, if A requires T, then A is strong sign nonsingular.

Proposition 2.5 is an immediate consequence of the following results from [BCS] and [T], respectively.

Lemma 2.3. *Given a fully indecomposable strong sign nonsingular pattern A, the inverse of every real matrix in $Q(A)$ is entrywise nonzero.*

Lemma 2.4. *If $n \geqslant 3$ and $A \in Q_n$ is sign nonsingular, then A has at least $\binom{n-1}{2}$ zero entries.*

Proposition 2.5. *If $A \in Q_n$ requires T and A is fully indecomposable, then $n \leqslant 2$.*

Lemma 2.6. *The following sign pattern matrices are all the fully indecomposable patterns that require T:*

$$(+), \quad (-),$$
$$\begin{pmatrix} + & + \\ + & - \end{pmatrix}, \begin{pmatrix} + & + \\ - & + \end{pmatrix}, \begin{pmatrix} + & - \\ + & + \end{pmatrix}, \begin{pmatrix} + & - \\ - & - \end{pmatrix},$$
$$\begin{pmatrix} - & + \\ + & + \end{pmatrix}, \begin{pmatrix} - & + \\ - & - \end{pmatrix}, \begin{pmatrix} - & - \\ - & + \end{pmatrix}, \begin{pmatrix} - & - \\ + & - \end{pmatrix}.$$

Corollary 2.7. *Let $A \in Q_2$ be entrywise nonzero. Then A requires T if and only if $\det(A) = +$ or $\det(A) = -$.*

We are now able to give a characterization of all $n \times n$ sign patterns that require the Inverse-Transpose property T.

Theorem 2.8. *Let $A \in Q_n$. Then A requires T if and only if A is equivalent, under left/right multiplication by generalized permutation patterns, to a sign pattern of the form $\bigoplus\limits_{i=1}^{k} A_i$, where for each i, $A_i = (+)$ or $A_i = \begin{pmatrix} + & + \\ + & - \end{pmatrix}$.*

Proof. ($\Rightarrow$) Assume $A \in Q_n$ requires T. Then by Theorem 2.2, A is permutation equivalent to the direct sum of its fully indecomposable components, all of which require T. Let $A_1, \ldots, A_k$ be the fully indecomposable components of A.

Then by Lemma 2.6, clearly A_j is generalized permutation equivalent to either $(+)$ or $\begin{pmatrix} + & + \\ + & - \end{pmatrix}$, for every $j \in \{1, 2, \ldots, k\}$.

($\Leftarrow$) This implication holds since the set of all patterns that require T is closed under direct sum and generalized permutation equivalence. $\square$

Corollary 2.9. *If A requires T, then $A \in \mathcal{PO}$.*

Proof. It is easy to verify that $(+)$ and $\begin{pmatrix} + & + \\ + & - \end{pmatrix}$ are potentially orthogonal, and since the class $\mathcal{PO}$ is closed under generalized permutation equivalence and direct sum, it follows that all $n \times n$ patterns that require T allow orthogonality. $\square$

To see that the converse of Corollary 2.9 is not true, consider $A = \begin{pmatrix} - & + & + \\ + & - & + \\ + & + & - \end{pmatrix}$. Then $B = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \in Q(A)$ is orthogonal. However, it follows from Theorem 2.8 that A does not require T.

Corollary 2.10. *For all $A \in Q_2$, A requires T if and only if $A \in \mathcal{T}$.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let $A \in \mathcal{T}_2$. If A is partly decomposable, then by Lemma 2.1, $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix}$, where $a_1, a_2 \in \{+, -\}$. It follows from Theorem 2.8 that A requires T. If A is fully indecomposable, then A is entrywise nonzero. It is easily verified that the only 2×2 entrywise nonzero patterns which are sign potentially orthogonal are the eight 2×2 patterns listed in Lemma 2.6. $\square$

3. Properties of patterns that allow T

We now return to the more general case of sign pattern matrices that allow the Inverse-Transpose property T. There are several necessary conditions that are useful since they rule out many patterns that are not in $\mathcal{T}$.

The most basic necessary condition for a pattern A to allow T is that A must allow nonsingularity. Although $\det(A)$ is not identically zero, it could be $\#$. For example, $A = \begin{pmatrix} + & + & + \\ + & + & - \\ + & - & - \end{pmatrix} \in \mathcal{T}$ and $\det(A) = \#$.

The next, perhaps most important, necessary conditions are the $\mathcal{SPO}$ conditions. We have already seen that every pattern which allows T must be sign potentially orthogonal.

There are several necessary conditions related to the determinants of principal submatrices of sign patterns. These conditions are listed in the following proposition, where $A(i,j)$ is the submatrix of A obtained by deleting the i^{th} row and j^{th} column.

Proposition 3.1. *If $A \in \mathcal{T}_n$, then*
(i) $\det(A(i,j)) \overset{c}{\longleftrightarrow} (-)^{i+j} a_{ij} \det(A)$, *for every* $i,j \in \{1,2,\ldots,n\}$,
(ii) $\det(A(i,j)) \det(A(k,l)) \overset{c}{\longleftrightarrow} (-)^{i+j+k+l} a_{ij} a_{kl}$, *for every* $i,j,k,l \in \{1,2,\ldots,n\}$, *and*
(iii) $A^T \overset{c}{\longleftrightarrow} \operatorname{adj}(A)$ *or* $A^T \overset{c}{\longleftrightarrow} -\operatorname{adj}(A)$.

P r o o f. Let $i,j,k,l \in \{1,2,\ldots,n\}$ be arbitrary. From the definition of A^{-1}, we have
$$(A^{-1})_{ji} = (-)^{i+j} \frac{\det(A(i,j))}{\det(A)}.$$
Now, if A allows T, then $(A^{-1})_{ji} \overset{c}{\longleftrightarrow} a_{ij}$, so that $(-)^{i+j} a_{ij} \det(A) \overset{c}{\longleftrightarrow} \det(A(i,j))$. This proves (i). From (i) we get
$$\det(A(i,j)) \det(A(k,l)) \overset{c}{\longleftrightarrow} (-)^{i+j+k+l} a_{ij} a_{kl},$$
since $(\det(B))^2 > 0$ for all invertible $B \in Q(A)$. Thus (ii) follows. Finally, (iii) is clear since for every invertible $B \in Q(A)$, we have $B^{-1} \in Q(\operatorname{adj}(A))$ or $B^{-1} \in Q(-\operatorname{adj}(A))$. $\square$

In the above proposition, the conditions depend on the determinants of both A and of principal submatrices of A. Because the determinant is frequently ambiguous for patterns which have few zero entries, the conditions are significant only for sparse sign pattern matrices.

A final necessary condition is given in the next proposition.

Proposition 3.2. *Let*
$$A_1 = \begin{pmatrix} v_1^T & + & + \\ v_2^T & + & - \\ v_3^T & - & + \\ v_4^T & - & - \end{pmatrix} \in Q_{4,n},$$
where $v_i \in Q_{(n-2),1}$, for $1 \leqslant i \leqslant 4$, and $v_1^T v_2 = v_1^T v_3 = v_4^T v_2 = v_4^T v_3 = +$. Suppose $A = \begin{pmatrix} A_1 \\ * \end{pmatrix} \in Q_n$. Then $A \notin \mathcal{T}$.

P r o o f. Suppose, for contradiction, that $A \in \mathcal{T}$. Let $B \in Q(A)$, $C \in Q(A^T)$ such that $BC = I$,

$$B = \begin{pmatrix} w_1^T & a & b \\ w_2^T & c & -d \\ w_3^T & -e & f \\ w_4^T & -g & -h \\ * & * & * \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & * \\ a_1 & c_1 & -e_1 & -g_1 & * \\ b_1 & -d_1 & f_1 & -h_1 & * \end{pmatrix},$$

where $a, b, \ldots, h, a_1, b_1, \ldots, h_1$ are positive numbers, and $\operatorname{sgn}(w_i) = \operatorname{sgn}(u_i) = v_i$, for $1 \leqslant i \leqslant 4$. Then $BC = I$ yields the following inequalities:

$$w_1^T u_2 = bd_1 - ac_1 > 0 \Longrightarrow bd_1 > ac_1, \tag{1}$$

$$w_1^T u_3 = ae_1 - bf_1 > 0 \Longrightarrow ae_1 > bf_1, \tag{2}$$

$$w_4^T u_2 = gc_1 - hd_1 > 0 \Longrightarrow gc_1 > hd_1, \tag{3}$$

$$w_4^T u_3 = hf_1 - ge_1 > 0 \Longrightarrow hf_1 > ge_1. \tag{4}$$

Multiplying (1) and (2) together, (3) and (4) together, then dividing the resulting inequalities by common factors yields

$$d_1e_1 > c_1f_1 \quad \text{and} \quad c_1f_1 > d_1e_1,$$

which is clearly a contradiction. Thus $A \notin \mathcal{T}$. □

We note that A may or may not be sign potentially orthogonal depending on how the $*$ entries are specified.

In order to give sufficient conditions for the allows question, we consider some classes of patterns. The first class, consisting of all square nonnegative patterns, was mentioned in section 2. Two additional classes of patterns are now discussed. The first of these is the class consisting of all sign pattern matrices of order less than or equal to 4, and all entrywise nonzero patterns of order equal to 5.

We say an $m \times n$ sign pattern matrix A is *sign potentially row orthogonal* if $AA^T \overset{c}{\longleftrightarrow} I$. Similarly, A is *sign potentially column orthogonal* if $A^TA \overset{c}{\longleftrightarrow} I$. The class of all sign potentially row (column) orthogonal sign patterns is denoted by $\mathcal{SPRO}$ ($\mathcal{SPCO}$), and clearly, $\mathcal{SPO} = \mathcal{SPRO} \cap \mathcal{SPCO}$.

The following theorem is found in [BS].

Theorem 3.3. *Let $A \in Q_n$ with $n \leqslant 3$. Then the following are equivalent:*

i) *A is potentially orthogonal,*

ii) *A is sign potentially orthogonal,*

iii) *A is sign potentially row orthogonal,*

iv) *A is sign potentially column orthogonal.*

Thus we conclude that an $n \times n$ $(n \leqslant 3)$ pattern A allows T if and only if A has sign potentially orthogonal rows or columns. The next example shows that iii) and iv) in Theorem 3.3 are not equivalent for $n > 3$.

Example 3.4. Let $A = \begin{pmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & + \\ + & + & - & - \end{pmatrix}$. Then $A \in \mathcal{SPRO}$, but $A \notin \mathcal{SPCO}$ (and, hence, $A \notin \mathcal{SPO}$).

As indicated in [W], it can be shown that for arbitrary patterns of order $n = 4$, the equivalence of i) and ii) in Theorem 3.3 still holds. In [W], it is further reported that in [J], it was proved that for entrywise nonzero sign patterns of order $\leqslant 5$, the same equivalence holds.

Theorem 3.5. *Let $A \in Q_n$ be a sign pattern matrix where either $n \leqslant 4$, or $n = 5$ and A is entrywise nonzero. Then $A \in \mathcal{PO}$ if and only if $A \in \mathcal{SPO}$.*

Next, we consider the case where A is an $n \times n$ sign pattern matrix of order equal to 5 that contains at least one zero entry, or where A is an arbitrary pattern of order greater than or equal to 6. For $n \geqslant 5$, it is known that $\mathcal{PO}_n \neq \mathcal{SPO}_n$. In [BS], the authors gave an example to show that sign potential orthogonality does not imply potential orthogonality. In [W], a fully indecomposable example is given. Other examples are given below.

Example 3.6.

$$A = \begin{pmatrix} - & + & + & + & + & + \\ + & - & + & + & + & + \\ + & + & - & + & + & + \\ 0 & 0 & 0 & - & + & + \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & + & + & - \end{pmatrix} \in \mathcal{SPO}.$$

But by Lemma 2.1, $A \notin \mathcal{T}$. Therefore, $A \notin \mathcal{PO}$.

For higher orders, this example can be extended in the following manner.

Example 3.7. Assume $n \geqslant 6$, and let

$$A = \begin{pmatrix} - & + & + & & & \\ + & - & + & & J & \\ + & + & - & & & \\ & & & - & & + \\ & 0 & & & \ddots & \\ & & & + & & - \end{pmatrix} = \begin{pmatrix} A_1 & J \\ 0 & A_2 \end{pmatrix},$$

where $J \in Q_{3,(n-3)}$ is entrywise positive, and $A_2 \in Q_{n-3}$. Again, $A \in \mathcal{SPO}$, but $A \notin \mathcal{T}$.

Example 3.8. Let

$$A = \begin{pmatrix} + & + & + & + & + & + \\ + & + & + & + & + & - \\ + & + & + & + & - & + \\ + & + & + & + & - & - \\ + & + & - & - & * & * \\ + & - & + & - & * & * \end{pmatrix}.$$

Clearly $A \in \mathcal{SPO}$, and by Proposition 3.2, $A \notin \mathcal{T}$.

We note that Examples 3.7 and 3.8 can be used to construct higher order entrywise nonzero $\mathcal{SPO}$ patterns not in $\mathcal{T}$ as follows.

Example 3.9. Let $A = \begin{pmatrix} A_1 & + \\ + & A_2 \end{pmatrix} \in Q_n$, $n \geqslant 7$, where $A_1 \in Q_6$ is given in Example 3.8, and $A_2 \in Q_{(n-6)}$ is of the same form as A_2 in Example 3.7. Then $A \in \mathcal{SPO}$, and by Proposition 3.2, $A \notin \mathcal{T}$.

Finally, for the 5×5 case, we obtained the following example that shows $\mathcal{PO}_5 \neq \mathcal{SPO}_5$.

Example 3.10. Let

$$A = \begin{pmatrix} 0 & + & + & + & + \\ + & + & + & + & - \\ - & + & + & - & + \\ 0 & + & + & - & - \\ + & + & - & - & + \end{pmatrix}.$$

It is readily checked that $A \in \mathcal{SPO}_5$. However, by Proposition 3.2, $A \notin \mathcal{PO}_5$.

4. Constructions to obtain patterns in $\mathcal{T}$

For $n \geqslant 6$ and $n = 5$ (some zero entries) it is not yet known whether all patterns that allow T also allow real orthogonality (i.e., is $\mathcal{T} \subseteq \mathcal{PO}$?). The search for the answer to this question partially motivates the construction of various patterns that allow T. We first note that the simplest patterns that naturally allow T are the generalized permutation patterns. Also, nontrivial sign pattern matrices in $\mathcal{T}$ can easily be constructed from potentially orthogonal patterns by one of several procedures. Examples of such constructions are given below.

(1) Factor an arbitrary real matrix $B \in M_n(\mathbb{R})$ into its Q-R factorization. Then Q is a real orthogonal matrix. Thus $\operatorname{sgn}(Q) \in \mathcal{PO}$, and $\operatorname{sgn}(Q)$ allows T.

(2) Let $A \in Q_n$ be a pattern that is generalized permutation equivalent to $A_1 \oplus A_2 \oplus \ldots \oplus A_k$ where $A_1, A_2, \ldots, A_k$ are patterns listed in Lemma 2.6. Then by Theorem 2.8, A requires T, so that A allows T.

(3) Recall that a matrix $B \in M_n(\mathbb{R})$ whose entries are from the set $\{+1, -1\}$ is called a *Hadamard matrix* if and only if $BB^T = nI$ ($B^T B = nI$). Let $B \in M_n(\mathbb{R})$ be any Hadamard matrix. From the above definition, we get $(\frac{1}{\sqrt{n}}B)(\frac{1}{\sqrt{n}}B)^T = I$. Clearly $\operatorname{sgn}(\frac{1}{\sqrt{n}}B) = \operatorname{sgn}(B)$. It follows that $\operatorname{sgn}(B) \in \mathcal{PO}$ and, hence, allows T.

(4) Multiply $A \in \mathcal{T}$ on the left and right by generalized permutation patterns P and Q.

(5) Compute the Kronecker product of $A, A' \in \mathcal{T}$.

(6) We say a sign pattern A has the Inverse-Pair property if there exists $B \in Q(A)$ with $B^{-1} \in Q(A)$. The class of all patterns which allow the Inverse-Pair property is denoted by $\mathcal{IP}$ (see [EHL]), and symmetric patterns in $\mathcal{IP}$ allow T.

Example 4.1. Let $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $y = \frac{1}{3}x$, $P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, and $B = (xy^T - P)$. It is easily verified that $B^2 = I$ or $B = B^{-1}$. Thus

$$A = \operatorname{sgn}(B) = \begin{pmatrix} + & + & - & + & + & + \\ + & - & + & + & + & + \\ - & + & + & + & + & + \\ + & + & + & - & + & + \\ + & + & + & + & + & - \\ + & + & + & + & - & + \end{pmatrix} \in \mathcal{IP}.$$

Since A is symmetric, it follows that $A \in \mathcal{T}$. In fact, $A \in \mathcal{PO}$.

The next construction gives rise to a large class of patterns that are potentially orthogonal. The proof requires the following two technical lemmas.

Lemma 4.2. *If $x_1 > 0$ and $x_{k+1} = \frac{1}{2}(x_k + \frac{1}{x_k})$ for $k \geqslant 1$, then $x_2 \geqslant x_3 \geqslant \ldots$, and $\lim\limits_{k\to\infty} x_k = 1$.*

P r o o f. For all $k \geqslant 1$, it is clear that $x_k > 0$ and

$$x_{k+1} = \frac{1}{2}(x_k + \frac{1}{x_k}) \geqslant \sqrt{(x_k)\Big(\frac{1}{x_k}\Big)} = 1.$$

Also,

$$x_{k+1} - x_k = \frac{1}{2}\Big(x_k + \frac{1}{x_k}\Big) - x_k = \frac{1}{2}\Big(\frac{1}{x_k} - x_k\Big) \leqslant 0, \quad \text{for} \quad k \geqslant 2.$$

Thus $x_2 \geqslant x_3 \geqslant \ldots$. Finally, since $x_{k+1} \geqslant 1$ for all $k \geqslant 1$ and the sequence $(x_k)_{k\geqslant 2}$ is nonincreasing, it follows that $\lim\limits_{k\to\infty} x_k$ exists. Let $x = \lim\limits_{k\to\infty} x_k$. Then x must satisfy the equation $x = \frac{1}{2}(x + \frac{1}{x})$. It follows that $x = 1$ since $x_k \geqslant 1$ for all $k \geqslant 2$. $\square$

If $x_1 \geqslant 1$ in the above sequence, then $x_1 \geqslant x_2 \geqslant x_3 \geqslant \ldots$.

We note that if B is nonsingular, then $\frac{1}{2}[B + (B^{-1})^*] = \frac{1}{2}B[I + B^{-1}(B^{-1})^*]$ is also nonsingular, since $I + B^{-1}(B^{-1})^*$ is positive definite.

Lemma 4.3. *Let B_1 be a nonsingular complex matrix, and let $B_{k+1} = \frac{1}{2}[B_k + (B_k^{-1})^*]$ for $k \geqslant 1$. Then $\lim\limits_{k\to\infty} B_k = B_\infty$, where B_∞ is a unitary matrix.*

P r o o f. Let $B_1 = U\Sigma_1 V$ be the singular value decomposition of B_1, where

$$\Sigma_1 = \begin{pmatrix} \sigma_1^{(1)} & & & 0 \\ & \sigma_2^{(1)} & & \\ & & \ddots & \\ 0 & & & \sigma_n^{(1)} \end{pmatrix}.$$

Define $\sigma_j^{(k+1)} = \frac{1}{2}[\sigma_j^{(k)} + (\sigma_j^{(k)})^{-1}]$. Then $B_k = U\Sigma_k V$, where

$$\Sigma_k = \begin{pmatrix} \sigma_1^{(k)} & & & 0 \\ & \sigma_2^{(k)} & & \\ & & \ddots & \\ 0 & & & \sigma_n^{(k)} \end{pmatrix} \quad \text{for all integers } k \geqslant 2.$$

By Lemma 4.2 it follows that $\lim\limits_{k\to\infty} \sigma_j^{(k)} = 1$. Thus $\lim\limits_{k\to\infty} B_k = UIV = B_\infty$, which is clearly unitary. $\square$

Corollary 4.4. *Let B_1 be a real nonsingular matrix and let $B_{k+1} = \frac{1}{2}[B_k + (B_k^{-1})^T]$, for $k \geqslant 1$. Then $\lim_{k\to\infty} B_k = B_\infty$, where B_∞ is a real orthogonal matrix.*

Theorem 4.5. *Let $S \in Q_n$ $(n \geqslant 2)$ be a skew-symmetric pattern all of whose off-diagonal entries are nonzero. Then $I + S \in \mathcal{PO}$.*

P r o o f. Assume $n \geqslant 2$ is fixed. Let $S \in Q_n$ be a skew-symmetric pattern all of whose off-diagonal entries are nonzero. Consider $B_1 = (I + \frac{S_1}{n^3}) \in Q(I+S)$, where S_1 is the skew-symmetric matrix in Q(S) with all off-diagonal entries of absolute value 1. Clearly S_1 is normal, and all the eigenvalues of S_1 are pure imaginary (including 0). From the entrywise inequality $0 \leqslant |\frac{S_1}{n^3}| \leqslant \frac{1}{n^3}J$, it follows that $\varrho(\frac{S_1}{n^3}) \leqslant \varrho(\frac{1}{n^3}J) = \frac{1}{n^2}$ (see [HJ1], page 491). Thus for each $\lambda \in \sigma(B_1)$, we have $\lambda = 1 \pm \beta i$, where $0 \leqslant \beta \leqslant \frac{1}{n^2}$. Hence, B_1 is a real nonsingular unitarily diagonalizable matrix, and the sequence defined by $B_{k+1} = \frac{1}{2}[B_k + (B_k^{-1})^T]$ converges to a real orthogonal matrix B by Corollary 4.4. It remains to be shown that $B \in Q(I + S)$.

Write $B_1 = UDU^*$, where U is a unitary matrix and

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

with $\lambda_j = \sigma_j^{(1)} e^{i\theta_j}$, $\sigma_j^{(1)} > 0$, for all $j = 1, 2, \ldots, n$.

$$\text{Let } \Sigma_1 = \begin{pmatrix} \sigma_1^{(1)} & & & 0 \\ & \sigma_2^{(1)} & & \\ & & \ddots & \\ 0 & & & \sigma_n^{(1)} \end{pmatrix} \text{ and let } V = \begin{pmatrix} e^{i\theta_1} & & & 0 \\ & e^{i\theta_2} & & \\ & & \ddots & \\ 0 & & & e^{i\theta_n} \end{pmatrix} U^*.$$

Then $B_1 = U\Sigma_1 V$ is a singular value decomposition of B_1.

Just as in the proof of Lemma 4.3, we have $B_k = U\Sigma_k V$, for all $k \geqslant 2$. Since $\sigma_j^{(1)} = |\lambda_j| \geqslant 1$, it follows from Lemma 4.2 that for all $p, q \geqslant 1$,

$$|\sigma_j^{(p)} - \sigma_j^{(q)}| \leqslant \sigma_j^{(1)} - 1 = |\lambda_j| - 1 = |1 + \beta_j i| - 1 \leqslant \sqrt{1 + \left(\frac{1}{n^2}\right)^2} - 1.$$

Note that for all $x \geqslant 0$, $\sqrt{1+x} \leqslant 1 + \frac{x}{2}$. Hence,

$$|\sigma_j^{(p)} - \sigma_j^{(q)}| \leqslant \sqrt{1 + \left(\frac{1}{n^2}\right)^2} - 1 \leqslant 1 + \frac{(\frac{1}{n^4})}{2} - 1 = \frac{1}{2n^4}.$$

Recall that the Frobenius norm $\|\cdot\|_F$ is invariant under left or right multiplication by a unitary matrix. Therefore, for every $p, q \geqslant 1$,

$$\begin{aligned}
\|B_p - B_q\|_F &= \|U(\Sigma_p - \Sigma_q)V\|_F \\
&= \|\Sigma_p - \Sigma_q\|_F \\
&= \sqrt{\sum_{j=1}^{n} |\sigma_j^{(p)} - \sigma_j^{(q)}|^2} \\
&\leqslant \sqrt{\sum_{j=1}^{n} \Big(\frac{1}{2n^4}\Big)^2} \\
&= \sqrt{n}\Big(\frac{1}{2n^4}\Big) \\
&\leqslant \frac{1}{2n^3}.
\end{aligned}$$

By letting $p \to \infty$ and $q = 1$, we see that $\|B_\infty - B_1\|_F = \|B - B_1\|_F \leqslant \frac{1}{2n^3}$. It follows that corresponding entries in B and B_1 can differ by at most $\frac{1}{2n^3}$, but each entry of $B_1 = I + \frac{S_1}{n^3}$ is of absolute value at least $\frac{1}{n^3}$. This implies $\operatorname{sgn}(B) = \operatorname{sgn}(B_1)$. Thus B is a real orthogonal matrix in $Q(I + S)$ as desired. $\square$

Next we consider another set of potentially orthogonal patterns that are generated by Householder transformations. Recall that an $n \times n$ matrix of the form $I - 2uu^T$, where u is a unit vector in $\mathbb{R}^n$, is called a Householder transformation. Clearly, the sign pattern of any Householder transformation is potentially orthogonal since $(I - 2uu^T)$ is a symmetric orthogonal matrix. We use the term Householder patterns to refer to the sign patterns of irreducible Householder transformations.

Proposition 4.6. *Up to signature and permutation similarity, the sign patterns of all $n \times n$ $(n \geqslant 3)$ irreducible Householder transformations are given by*

$$\begin{pmatrix} + & & - \\ & \ddots & \\ - & & + \end{pmatrix}, \quad \begin{pmatrix} - & & & - \\ & + & & \\ & & \ddots & \\ - & & & + \end{pmatrix}, \quad \textit{and} \quad \begin{pmatrix} 0 & & & - \\ & + & & \\ & & \ddots & \\ - & & & + \end{pmatrix}.$$

Proof. Let $B = I - 2uu^T$ be any irreducible Householder transformation. Clearly, u must be an entrywise nonzero unit vector in $\mathbb{R}^n$ in order for B to be irreducible. Performing a signature similarity on B, if necessary, we may assume that all of the components of u are positive. Thus $\operatorname{sgn}(uu^T) = J_n$, and $\operatorname{sgn}(B) =$

$\begin{pmatrix} * & & - \\ & \ddots & \\ - & & * \end{pmatrix}$. Note that for all $j = 1, 2, \dots, n$, the j^{th} diagonal entry of B is given by $b_{j,j} = 1 - 2u_j^2$. There are three cases.

(1) If $u_j < \frac{1}{\sqrt{2}}$, for all $j = 1, 2, \dots, n$, we get the first Householder pattern.

Now note that if $u_i, u_j \geqslant \frac{1}{\sqrt{2}}$, for some $i \neq j$, then $u_i^2 + u_j^2 \geqslant 1$ which contradicts the assumption that u is an entrywise nonzero unit vector. Hence, at most one component of u may be greater than or equal to $\frac{1}{\sqrt{2}}$. Assume, without loss of generality, that u_1 is that component.

(2) If $u_1 > \frac{1}{\sqrt{2}}$, then $b_{1,1} < 0$, which yields the second Householder pattern.

(3) If $u_1 = \frac{1}{\sqrt{2}}$, then $b_{1,1} = 0$, which yields the final Householder pattern. □

It is easy to verify that the sign patterns of the 2×2 irreducible Householder transformations are $\begin{pmatrix} 0 & - \\ - & 0 \end{pmatrix}$, $\begin{pmatrix} + & + \\ + & - \end{pmatrix}$, and $\begin{pmatrix} - & - \\ - & + \end{pmatrix}$, up to signature similarity.

The symmetric pattern A given in Example 4.1 is generalized permutation equivalent to the first Householder pattern in Proposition 4.6. In fact, suppose that $B = xy^T - P$, where P is a symmetric permutation matrix, $Px = x$, $Py = y$, and $x^T y = 2$. Since B is required to be symmetric in the construction in Example 4.1, we have $xy^T = yx^T$. This implies that $y = \frac{2}{a}x$, where $a = x^T x$. It follows that $P(xy^T - P) = xy^T - I = \frac{2}{a}xx^T - I = 2uu^T - I$, where $u = \frac{x}{\sqrt{a}}$. Therefore, for all real matrices B satisfying the given conditions, Proposition 4.6 and Example 4.1 generate equivalent sign patterns.

We conclude this section with an examination of several bordering constructions that give rise to additional patterns in $\mathcal{PO}$.

If an $m \times n$ real matrix B is given, and if $C = \begin{pmatrix} B & * \\ * & * \end{pmatrix}$ is any larger matrix, where B is the (1,1) block of C, then C is said to be a *dilation* of B. An analogous definition can be made for a dilation of a sign pattern matrix A given by $\begin{pmatrix} A & * \\ * & * \end{pmatrix}$. We say B is a *contraction* if $\|B\|_2 \leqslant 1$. An important idea using contractions and dilations from [HJ2] is given in the following lemma.

Lemma 4.7. *If $B \in M_n(\mathbb{R})$ is a contraction and $B = PU$ is a polar decomposition of B, then*

$$Z = \begin{pmatrix} B & (I - P^2)^{1/2}U \\ -(I - P^2)^{1/2}U & B \end{pmatrix} \in M_{2n}(\mathbb{R})$$

is a real orthogonal dilation of B.

The next proposition follows from Lemma 4.7 and the fact that, for any sign pattern matrix $A \in Q_n$, there exists a contraction $B \in Q(A)$.

Proposition 4.8. *For every $A \in Q_n$, there exists an $X \in Q_n$ such that*

$$\begin{pmatrix} A & X \\ -X & A \end{pmatrix} \in \mathcal{PO}_{2n}.$$

Thus every $n \times n$ sign pattern matrix A is a principal submatrix of a $2n \times 2n$ pattern in $\mathcal{PO}$. If A itself is potentially orthogonal, there are simpler block matrix constructions that contain A as a principal submatrix.

Propostion 4.9. *If $A \in \mathcal{PO}_n$, then*

$$\hat{A} = \begin{pmatrix} A & A \\ -A^T & A^T \end{pmatrix} \in \mathcal{PO}_{2n}.$$

P r o o f. Assume $A \in \mathcal{PO}_n$. Then there exists $B \in Q(A)$ with $BB^T = I$ and $B^TB = I$. Let $\hat{B} = \begin{pmatrix} B & B \\ -B^T & B^T \end{pmatrix} \in Q(\hat{A})$. It is easily seen that $\frac{1}{\sqrt{2}}\hat{B}$ is real orthogonal. Thus $\hat{A} \in \mathcal{PO}$. $\square$

In addition, it is reasonable to ask whether there exists a dilation in $\mathcal{PO}$ of size smaller than $2n \times 2n$ for every sign pattern matrix $A \in Q_n$. Before answering this question, we examine the restrictions placed on the size of a dilation of a sign pattern matrix A by the quantity $\operatorname{rank}(I - BB^T)$, where $B \in Q(A)$.

Lemma 4.10. *Let $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \in M_m(\mathbb{R})$, be any orthogonal matrix, where B_1 is a submatrix of order $n \leqslant m$. Then $\operatorname{rank}(I - B_1B_1^T) \leqslant m - n$.*

P r o o f. Since $BB^T = I$, we have $B_1B_1^T + B_2B_2^T = I_n$. This implies $\operatorname{rank}(I - B_1B_1^T) = \operatorname{rank}(B_2B_2^T) \leqslant \min\{n, m-n\} \leqslant m - n$. $\square$

For $A \in Q_n$, we define $d(A)$ to be

$$d(A) = \min_{\substack{B \in Q(A) \\ \|B\|_2 \leqslant 1}} \{\operatorname{rank}(I - BB^T)\}.$$

The quantity $d(A)$ measures how far any pattern A is from being in $\mathcal{PO}_n$, as illustrated in the following results. Further, it can be seen that $d(A) = n - s$, where s is the maximum multiplicity of the largest eigenvalue of BB^T, over all $B \in Q(A)$.

Theorem 4.11. *If $A \in \mathcal{PO}_m$, and $A_1 \in Q_n$ is any square submatrix of A, then $d(A_1) \leqslant m - n$.*

P r o o f. Let $A \in \mathcal{PO}_m$, and let $A_1 \in Q_n$ be any square submatrix of A. We may assume that A_1 is the upper left block of A. Then there exists $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \in Q(A)$ that is orthogonal. This implies that $\|B_1\|_2 \leqslant 1$. It follows from Lemma 4.10 that $d(A_1) \leqslant \operatorname{rank}(I - B_1 B_1^T) \leqslant m - n$. □

The corollaries to Theorem 4.11 provide examples of how the quantity $d(A_1)$ controls the sizes of potentially orthogonal dilations of a given sign pattern matrix A_1.

Corollary 4.12. *If $A \in \mathcal{PO}_m$ has a zero submatrix of order n, then $m \geqslant 2n$.*

P r o o f. Assume $A \in \mathcal{PO}_m$ has a zero submatrix A_1 of order n. Then $d(A_1) = n$, and by Theorem 4.11 we have $n \leqslant m - n$, or $m \geqslant 2n$. □

Corollary 4.13. *If $A \in \mathcal{PO}_m$ has a positive submatrix of order n, then $m \geqslant 2n - 1$.*

P r o o f. Assume $A \in \mathcal{PO}_m$ has a positive submatrix A_1 of order n. Then $d(A_1) = n-1$, since the Perron root of BB^T for any $B \in Q(A_1)$ is a simple eigenvalue. Thus $d(A_1) = n - 1 \leqslant m - n$, by Theorem 4.11. It follows that $m \geqslant 2n - 1$. □

We note that Corollaries 4.12 and 4.13 provide necessary conditions for a sign pattern to allow orthogonality. In fact, Corollary 4.13 implies that $A \notin \mathcal{PO}_6$ in Example 3.8. Even though Proposition 3.2 shows that this sign pattern does not allow T, it is worth asking whether or not there exists a sign pattern A such that $A \notin \mathcal{PO}$ and yet $A \in \mathcal{T}$. One possible example is given below.

Example 4.14.

$$A = \left(\begin{array}{ccccc|ccc} & & & & & + & + & + \\ & & & & & + & + & - \\ & & J_5 & & & + & - & + \\ & & & & & - & - & - \\ & & & & & - & + & + \\ + & + & + & - & - & & & \\ + & + & - & - & + & & J_3 & \\ + & - & + & - & + & & & \end{array}\right) \notin \mathcal{PO} \quad \text{by Corollary 4.13,}$$

where J_5 and J_3 are entrywise positive patterns. How can we show either $A \in \mathcal{T}$ or $A \notin \mathcal{T}$?

Finally, we use the following result from [HJ2] to prove the existence of an $m \times m$ ($m \leqslant 2n$) dilation in $\mathcal{PO}$ for any sign pattern $A \in Q_n$.

Lemma 4.15. *Let $B \in M_n(\mathbb{R})$ be a contraction and let $\delta = \operatorname{rank}(I - BB^T)$. Then there exist nonsingular matrices $X, Y \in M_n(\mathbb{R})$ such that, for*

$$C = X \begin{pmatrix} I_\delta \\ 0 \end{pmatrix}, \quad D = -\begin{pmatrix} 0 & I_\delta \end{pmatrix}, \quad \textit{and} \quad E = \begin{pmatrix} 0 & I_\delta \end{pmatrix} Y B^T (X^T)^{-1} \begin{pmatrix} I_\delta \\ 0 \end{pmatrix},$$

the matrix $Z = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \in M_{n+\delta}(\mathbb{R})$ is orthogonal.

Theorem 4.16. *For every $A \in Q_n$, there exists a dilation $\hat{A}$ of order $m = n + d(A)$ such that $\hat{A} \in \mathcal{PO}_m$.*

Proof. Let $A \in Q_n$ be arbitrary. Then there exists a $B \in Q(A)$ with $\|B\|_2 \leqslant 1$ such that $\operatorname{rank}(I - BB^T) = d(A)$. It follows from Lemma 4.15 that there exists a real orthogonal matrix $Z = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ of order $m = n + d(A)$. Thus $\hat{A} = \operatorname{sgn}(Z) \in \mathcal{PO}_m$ is a desired dilation of A. □

5. Remarks and open questions

There are many related questions still open for investigation. As with many allows questions, it is not an easy task to characterize all patterns which allow the Inverse-Transpose property T. For $n \geqslant 5$, there is no effective way to determine if a pattern $A \in Q_n$ allows T.

We have also seen that $\mathcal{PO} \subseteq \mathcal{T}$, but it is not yet known if these classes are, in fact, equal. We conjecture that $A \in \mathcal{T}$ implies $A \in \mathcal{PO}$. Perhaps the answers to the following questions will prove or disprove the conjecture.

Given a pattern $A \in \mathcal{T}$, how can we border A in order to produce patterns that allow T. Is there an example of such a dilation of A that is not potentially orthogonal? (see Example 4.14).

In addition, recall that in the proof of Theorem 4.5, limits of sequences of real matrices were used to show a certain pattern allows orthogonality. For an arbitrary $A \in \mathcal{T}$, where $B_1 \in Q(A)$, $B_1^{-1} \in Q(A^T)$ and $B_{k+1} = \frac{1}{2}[B_k + (B_k^{-1})^T]$, for $k \geqslant 1$, is there a way to show that the sequence $(B_k)_{k\geqslant 1}$ converges to a real orthogonal matrix in $Q(A)$?

Another interesting problem is that of characterizing the intersection $\mathcal{T} \cap \mathcal{IP}$. It is clear that $A = A^T$ and $A \in \mathcal{T} \cup \mathcal{IP}$ imply $A \in \mathcal{T} \cap \mathcal{IP}$. We also know that $\begin{pmatrix} + & + & + & - \\ + & - & - & + \\ + & + & - & - \\ + & + & - & + \end{pmatrix}$ is in $\mathcal{T} \cap \mathcal{IP}$, which shows that there are non-symmetric patterns in $\mathcal{T} \cap \mathcal{IP}$. In general, how can we describe the patterns in this intersection, and is there a nonsymmetric pattern in the intersection which does not allow orthogonality?

Finally, for an arbitrary $A \in \mathcal{T}_n$, the maximum number of mutually orthogonal rows (or columns) of any $B \in Q(A)$ is not yet known. If it can be shown that this maximum is n, then there exists a $B \in Q(A)$ with n orthonormal rows. Thus B is orthogonal and $A \in \mathcal{PO}$.

We conjecture that if $A \in Q_{3,n}$ satisfies $AA^T \overset{c}{\longleftrightarrow} I$, then there is a real matrix $B \in Q(A)$ such that $BB^T = I$.

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