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AN EXTENSION OF INVERTIBILITY OF HAMMERSTEIN-TYPE  
OPERATORS

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*Abstract.* My aim is to show that some properties, proved to be true for the square matrices, are true for some not necessarily linear operators on a linear space, in particular, for Hammerstein-type operators.

If  $A$  and  $B$  are operators satisfying angleboundedness assumption (for details see [1]), or the strong monotonicity condition following [2], there are sufficient conditions for invertibility of a Hammerstein-type operator  $I + AB$  as described in [3].

In this paper we shall provide some generalizations of the results from [3]. We will mark the end of a proof with Halmos' square  $\square$ .

**Proposition 1.** *Let  $K_1$  and  $K_2$  be two linear spaces and  $A: K_1 \rightarrow K_2$ ,  $B: K_2 \rightarrow K_1$  two operators such that the operator  $ABA$  is linear.*

1. *If  $I_2 - ABAB: K_2 \rightarrow K_2$  has a left inverse  $U_l$  then  $I_1 - BABA: K_1 \rightarrow K_1$  has a left inverse  $V_l = I_1 + BU_lABA$ .*
2. *If  $I_2 - ABAB: K_2 \rightarrow K_2$  has a right inverse  $U_d$  then  $I_1 - BABA: K_1 \rightarrow K_1$  has a right inverse  $V_d = I_1 + BU_dABA$ .*

Proof. 1. We know that  $U_l(I_2 - ABAB) = I_2$ . Then we can write

$$\begin{aligned}
 V_l(I_1 - BABA) &= (I_1 + BU_lABA)(I_1 - BABA) \\
 &= I_1 - BABA + BU_lABA(I_1 - BABA) \\
 &= I_1 - BABA + BU_l(ABA - ABABABA) \\
 &= I_1 - BABA + BU_l(I_2 - ABAB)ABA \\
 &= I_1 - BABA + BI_2ABA \\
 &= I_1 - BABA + BABA \\
 &= I_1.
 \end{aligned}$$

So  $V_l$  is a left inverse for  $I_1 - BABA$ .

2. Because  $(I_2 - ABAB)U_d = I_2$  we can write

$$\begin{aligned}
 (I_1 - BABA)(I_1 + BU_dABA) &= I_1 + BU_dABA - BABA(I_1 + BU_dABA) \\
 &= I_1 + BU_dABA - B(ABA + ABABU_dABA) \\
 &= I_1 + BU_dABA - B(I_2 + ABABU_d)ABA.
 \end{aligned}$$

But

$$\begin{aligned}
 (I_2 - ABAB)U_d &= I_2, \\
 U_d - ABABU_d &= I_2, \\
 I_2 + ABABU_d &= U_d.
 \end{aligned}$$

Therefore  $(I_1 - BABA)V_d = I_1 + BU_dABA - BU_dABA = I_1$  so we can say that  $V_d$  is a right inverse of  $I_1 - BABA$ .  $\square$

**Observation 1.** For the above conditions we get that

$$ABAV_d = U_dABA.$$

Proof.

$$\begin{aligned}
 ABAV_d &= ABA(I_1 + BU_dABA) = ABA + ABABU_dABA \\
 &= (I_2 + ABABU_d)ABA = U_dABA.
 \end{aligned}$$

$\square$

**Corollary 1.** Let  $ABA$  be linear and  $I_2 - ABAB$  invertible. Then  $I_1 - BABA$  is invertible, too.

**P r o o f.** This is obvious because  $U_l = U_d = U$  implies that

$$V_l = V_d = I_1 + BUABA.$$

□

**Remark 1.** We cannot make a weaker assumption than  $ABA$  to be linear because there are operators  $A$  and  $B$  such that  $I_2 - ABAB$  is invertible and  $I_1 - BABA$  is not invertible.

**Example.**  $A: \mathbb{R} \rightarrow \mathbb{R}$ ,  $Ax = \log \sqrt{x}$  if  $x > 0$  and  $Ax = 0$  otherwise, and  $B: \mathbb{R} \rightarrow \mathbb{R}$ ,  $Bx = \exp \frac{2x}{3}$ . Then  $(I - ABAB)x = \frac{8}{9}x$  which is invertible. At the same time  $(I - BABA)x = x - \sqrt[3]{x}$  for  $x > 0$  and  $(I - BABA)x = x - 1$  for  $x \leq 0$ . As can be easily seen this is not invertible.

**Proposition 2.** Let  $K$  be a linear complex space and let operators  $A, B: K \rightarrow K$  be given,  $A$  being linear. If there is any nonzero complex number  $\lambda$  such that

$$\lambda AB + A + B = 0$$

then  $\lambda A + I$  has a right inverse and  $\lambda B + I$  has a left inverse.

**P r o o f.**

$$\begin{aligned} \lambda AB + A + B &= 0, \quad \lambda \neq 0, \\ \lambda^2 AB + \lambda A + \lambda B &= 0. \end{aligned}$$

Adding the identity operator to both sides we get

$$\begin{aligned} \lambda^2 AB + \lambda A + \lambda B + I &= I, \\ \lambda(\lambda AB + B) + \lambda A + I &= I, \\ \lambda(\lambda A + I)B + \lambda A + I &= I, \\ (\lambda A + I)(\lambda B + I) &= I. \end{aligned}$$

Hence  $(\lambda A + I)$  is a left inverse for  $(\lambda B + I)$  and  $(\lambda B + I)$  is a right inverse for  $(\lambda A + I)$ . □

**Corollary 2.** Let  $A, B: K \rightarrow K$  be operators and let  $A$  be linear. Let  $\lambda$  be a nonzero complex number such that  $\lambda AB + A + B = 0$ . Then

1. If  $\lambda A + I$  is left invertible then  $\lambda A + I$  is invertible and its inverse is  $\lambda B + I$ .
2. If  $\lambda B + I$  is right invertible then  $\lambda B + I$  is invertible and its inverse is  $\lambda A + I$ .

**Observation 2.** There are examples when  $A, B$  are not linear and there is a nonzero complex number  $\lambda$  for which  $(\lambda A + I)(\lambda B + I) = I$ .

Let  $Ax = \frac{\tan x - x}{\lambda}$  for  $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$  and  $Ax = 0$  otherwise.

Let  $Bx = \frac{\arctan x - x}{\lambda}$  for every  $x \in \mathbb{R}$ .

Then  $(\lambda A + I)(\lambda B + I)x = x$  and it is obvious that  $A, B$  are not linear.

**Proposition 3.** Let  $K_1, K_2$  be two complex linear spaces and let  $A: K_1 \rightarrow K_2$  be linear. Consider  $B: K_2 \rightarrow K_1$  and  $m$  a positive integer.

1. If  $I_2 + (AB)^m$  has a left inverse  $U_l$  then  $I_1 + (BA)^m$  has a left inverse  $V_l$ , where  $V_l = I_1 - B(AB)^{m-1}U_lA$ .
2. If  $I_2 + (AB)^m$  has a right inverse  $U_d$  then  $I_1 + (BA)^m$  has a right inverse  $V_d$ , where  $V_d = I_1 - B(AB)^{m-1}U_dA$ .

Moreover,  $AV_d = U_dA$ .

*P r o o f.* For  $m = 1$  this result was proved in [3].

1. For  $m > 1$  we observe that  $I_2 + (AB)^m = I_2 + AC$ , where  $C = B(AB)^{m-1}$ . Therefore

$$I_1 + (BA)^m = I_1 + CA,$$

$$U_l[I_2 + (AB)^m] = I_2,$$

$$U_l[I_2 + AC] = I_2.$$

We know that the result is true for  $m = 1$ , hence there is  $V_l = I_1 - CU_lA$  which is a left inverse for  $I_1 + CA$ . This means that  $V_l = I_1 - B(AB)^{m-1}U_lA$  is a left inverse for  $I_1 + (BA)^m$ .

2. As above  $I_1 + CA$  has a right inverse  $V_d = I_1 - CU_dA$ . So  $I_1 + (BA)^m$  has a right inverse  $V_d = I_1 - B(AB)^{m-1}U_dA$ .

$$AV_d = A [I_1 - B(AB)^{m-1}U_dA]$$

$$= A - (AB)^mU_dA$$

$$= [I_2 - (AB)^mU_d]A.$$

However,  $I_2 - (AB)^mU_d = U_d$  from where we conclude that  $AV_d = U_dA$ . □

**Corollary 3.** Let  $m$  be a positive integer. If  $I_2 + (AB)^m$  is invertible then  $I_1 + (BA)^m$  is invertible.

*P r o o f.* Let  $U$  be the inverse of  $I_2 + (AB)^m$ . Then for  $C$  as above

$$[I_1 + (BA)^m]^{-1} = I_1 - CUA = I_1 - B(AB)^{m-1}UA.$$

□

**Proposition 4.** Let  $K_1, K_2$  be two complex linear spaces and  $m$  a nonnegative integer. Let  $A: K_1 \rightarrow K_2, B: K_2 \rightarrow K_1$  be two operators such that the operator  $B(AB)^{m-1}$  is invertible and linear. Then

1. If  $I_2 + (AB)^m$  has  $U_d$  as a right inverse then  $I_1 + (BA)^m$  has  $V_d$  as a right inverse, where  $V_d = B(AB)^{m-1}U_d [B(AB)^{m-1}]^{-1}$ ;
2. If  $I_2 + (AB)^m$  has  $U_l$  as a left inverse then  $I_1 + (BA)^m$  has  $V_l$  as a left inverse, where  $V_l = B(AB)^{m-1}U_l [B(AB)^{m-1}]^{-1}$ .

*Proof.* Let  $C = B(AB)^{m-1}, C: K_2 \rightarrow K_1$ , which is linear and invertible.

1. We know that  $[I_2 + (AB)^m]U_d = I_2$ . Hence we can write

$$\begin{aligned} [I_1 + (BA)^m]V_d &= [I_1 + (BA)^m](CU_dC^{-1}) \\ &= CU_dC^{-1} + (BA)^mCU_dC^{-1} \\ &= [C + (BA)^mC]U_dC^{-1} \\ &= C[I_2 + (AB)^m]U_dC^{-1} \\ &= CI_2C^{-1} = CC^{-1} = I_1, \end{aligned}$$

- 2.

$$\begin{aligned} V_l[I_1 + (BA)^m] &= B(AB)^{m-1}U_l [B(AB)^{m-1}]^{-1}[I_1 + (BA)^m] \\ &= CU_lC^{-1}[I_1 + (BA)^m] \\ &= CU_l[C^{-1} + C^{-1}(BA)^m] \\ &= CU_l[I_2 + (AB)^m]C^{-1}. \end{aligned}$$

However, we know that  $U_l[I_2 + (AB)^m] = I_2$ . Therefore we obtain that

$$V_l[I_1 + (BA)^m] = CC^{-1} = I_1.$$

□

**Corollary 4.** Let  $A, B$  be two operators as in Proposition 4. If  $I_2 + (AB)^m$  is invertible then  $I_1 + (BA)^m$  is invertible and

$$[I_1 + (BA)^m]^{-1} = B(AB)^{m-1}[I_2 + (AB)^m]^{-1}[B(AB)^{m-1}]^{-1}.$$

**Observation 3.** For  $m = 1$  we get the principal results from [3]. Let  $K_1$  and  $K_2$  be Banach spaces such that the hypotheses of the last corollary are satisfied and  $B(AB)^{m-1}$  is bounded. Then using the Open Map Theorem we can say that  $[I_1 + (BA)^m]^{-1}$  is (Lipschitz) continuous if  $[I_2 + (AB)^m]^{-1}$  is (Lipschitz) continuous.

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